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Research Article

Extremal Solutions of Periodic Boundary Value Problems for First-Order Impulsive Integrodifferential Equations of Mixed-Type on Time Scales

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We consider the existence of minimal and maximal solutions of periodic boundary value problems for first-order impulsive integrodifferential equations of mixed-type on time scales by establishing a comparison result and using the monotone iterative technique.

1. Introduction

The theory of calculus on time scales (see [1, 2] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1990 [3] in order to unify continuous and discrete analyses, and it has a tremendous potential for applications and has recently received much attention since his foundational work. In this paper, we will study the periodic boundary value problem for the first-order impulsive integrodifferential equations of mixed-type (PBVP):

$$\begin{gathered} u^{\Delta}(t) = f\big(t, u(t), [Tu](t), [Su](t)\big), \quad t \neq t_k,\ t \in J_{\mathbb{T}}, \\ u(t_k^+) - u(t_k^-) = I_k\big(u(t_k^-)\big), \quad k = 1, 2, \dots, p, \\ u(0) = u(T), \end{gathered} \tag{1.1}$$

where $\mathbb{T}$ is a time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. For each interval J of $\mathbb{R}$, we denote by $J_{\mathbb{T}} = J \cap \mathbb{T}$, $f \in C[J_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $J = [0, T]$, $I_k \in C[\mathbb{R}, \mathbb{R}]$, where $u(t_k^+)$ and $u(t_k^-)$ represent right and left limits of $u(t)$ at $t = t_k\,(k = 1, 2, \dots, p)$ in the sense of time scales, and in addition, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, whereas if t_k is left scattered, then $y(t_k^-) = y(t_k)$,

$0 < t_1 < t_2 < \cdots < t_k < \cdots < t_p < T$,

$$[Tu](t) = \int_0^t k(t,s)u(s)\Delta s, \qquad [Su](t) = \int_0^T h(t,s)u(s)\Delta s, \tag{1.2}$$

$k(t,s) \in C[D,\mathbb{R}^+]$, $D = \{(t,s) \in J_{\mathbb{T}} \times J_{\mathbb{T}} : t \geq s\}$, $h(t,s) \in C[J_{\mathbb{T}} \times J_{\mathbb{T}},\mathbb{R}^+]$, $\mathbb{R}^+ = [0,+\infty)$, $k_0 = \max\{k(t,s) : (t,s) \in D\}$, $h_0 = \max\{h(t,s) : (t,s) \in J_{\mathbb{T}} \times J_{\mathbb{T}}\}$.

The study of impulsive dynamic equations on time scales has been initiated by Henderson [4], Benchohra et al. [5], and Atici and Biles [6]. Extremal solutions of PBVP for impulsive differential equations and difference equations has been studied by some authors (see [7, 8]). In this paper, we will obtain an inequality on time scales. And then, using this inequality, a comparison result is obtained. At last, we obtain an existence theorem of minimal and maximal solutions of PBVP (1.1) by using monotone iterative technique (see [7–9]).

2. Preliminaries and comparison principle

In this section, we will first recall some basic definitions and lemmas, which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t. \tag{2.1}$$

A point $t \in \mathbb{T}$ is called left dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$, left scattered if $\rho(t) < t$, right dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, and right scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$ to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s| \tag{2.2}$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Lemma 2.1 (see [1]). *Assume that $f,g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then,*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)). \tag{2.3}$$

Let y be right-dense continuous. If $Y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a). \tag{2.4}$$

A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0 \tag{2.5}$$

for all $t \in \mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\} \quad \text{for } s,t \in \mathbb{T} \tag{2.6}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \tag{2.7}$$

Let $p,q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus q). \tag{2.8}$$

Then, the generalized exponential function has the following properties.

Lemma 2.2 (see [1]). *Assume that $p,q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $e_p(t,\sigma(s)) = e_p(t,s)/(1+\mu(s)p(s))$;
(iv) $1/e_p(t,s) = e_{\ominus p}(t,s)$;
(v) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;
(vi) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vii) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(viii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$.

Lemma 2.3 [1]. *Let $r : \mathbb{T} \to \mathbb{R}$ be right-dense continuous and regressive, $a \in \mathbb{T}$, and $y_a \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^{\Delta}(t) = r(t)y(t) + h(t), \quad y(a) = y_a, \tag{2.9}$$

is given by

$$y(t) = e_r(t,a)y_a + \int_a^t e_r(t,\sigma(s))h(s)\Delta s. \tag{2.10}$$

Throughout this paper, we assume that, for each $k = 1,\dots,p$, the points of impulse t_k are right dense. For convenience, we introduce the notation $PC[J_{\mathbb{T}},\mathbb{R}] = \{u : J_{\mathbb{T}} \to \mathbb{R}, u(t)$

is continuous everywhere except some t_k at which $u(t_k^-)$ and $u(t_k^+)$ exist and $u(t_k^-) = u(t_k)\}$. Evidently, $PC[J_{\mathbb{T}},\mathbb{R}]$ is a Banach space with norm $\|u\|_{PC} = \sup_{t\in J_{\mathbb{T}}} |u(t)|$. Let $J'_{\mathbb{T}} = J_{\mathbb{T}} \setminus \{t_1,t_2,\ldots,t_p\}$, $C^1[J'_{\mathbb{T}},\mathbb{R}] = \{u^{\Delta}(t)$ is continuous on $J'_{\mathbb{T}}\}$, $\Omega = PC[J_{\mathbb{T}},\mathbb{R}] \cap C^1[J'_{\mathbb{T}},\mathbb{R}]$, $\mathbb{T}^+ = \mathbb{T}\cap\mathbb{R}^+$, $PC^1[\mathbb{T}^+,\mathbb{R}] = PC[\mathbb{T}^+,\mathbb{R}] \cap C^1[\mathbb{T}^+,\mathbb{R}]$. A function $u \in \Omega$ is called a solution of PBVP (1.1) if it satisfies (1.1).

Next, we combine [10, 11] to obtain an inequality as follows.

Lemma 2.4. *Assume that*

(A_0) *the sequence* $\{t_k\}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ *with* $\lim_{k\to+\infty} t_k = +\infty$,
(A_1) $m \in PC^1[\mathbb{T}^+,\mathbb{R}]$ *is right-dense continuous at* t_k *for* $k = 1,2,\ldots$,
(A_2) $\inf_{t\in J_{\mathbb{T}}}\{\mu(t)p(t)\} > -1$. *For* $k = 1,2,\ldots, t \ge t_0$,

$$m^{\Delta}(t) \ge p(t)m(t) + q(t), \quad t \ne t_k, \quad m(t_k^+) \ge d_k m(t_k) + b_k, \tag{2.11}$$

where $p,q \in C(\mathbb{T}^+,\mathbb{R})$, $d_k \ge 0$, *and* b_k *are real constants. Then,*

$$\begin{aligned} m(t) \ge{}& m(t_0)\prod_{t_0<t_k<t} d_k e_p(t,t_0) + \int_{t_0}^{t}\prod_{s<t_k<t} d_k e_p(t,\sigma(s))q(s)\Delta s \\ &+ \sum_{t_0<t_k<t}\prod_{t_k<t_j<t} d_j e_p(t,t_k)b_k. \end{aligned} \tag{2.12}$$

Proof. By condition (A_2), we know that $e_{\ominus p}(\sigma(t),t_0) \ge 0$ for $t \in [t_0,+\infty)_{\mathbb{T}}$. For the following inequality:

$$m^{\Delta}(t) \ge p(t)m(t) + q(t), \tag{2.13}$$

on multiplying $e_{\ominus p}(\sigma(t),t_0)$ and arranging the terms, we obtain

$$e_{\ominus p}(\sigma(t),t_0)m^{\Delta}(t) - p(t)m(t)e_{\ominus p}(\sigma(t),t_0) \ge e_{\ominus p}(\sigma(t),t_0)q(t), \tag{2.14}$$

which is the same as

$$\left(e_{\ominus p}(t,t_0)m(t)\right)^{\Delta} \ge e_{\ominus p}(\sigma(t),t_0)q(t). \tag{2.15}$$

Integrating (2.15) from t_0 to t_1, then

$$e_{\ominus p}(t_1,t_0)m(t_1) \ge m(t_0) + \int_{t_0}^{t_1} e_{\ominus p}(\sigma(s),t_0)q(s)\Delta s. \tag{2.16}$$

Again integrating (2.15) from t_1 to t, where $t \in (t_1,t_2]$, then

$$\begin{aligned} e_{\ominus p}(t,t_0)m(t) &\ge e_{\ominus p}(t_1,t_0)m(t_1^+) + \int_{t_1}^{t} e_{\ominus p}(\sigma(s),t_0)q(s)\Delta s \\ &\ge e_{\ominus p}(t_1,t_0)\left(d_1 m(t_1) + b_1\right) + \int_{t_1}^{t} e_{\ominus p}(\sigma(s),t_0)q(s)\Delta s \\ &\ge d_1\left(m(t_0) + \int_{t_0}^{t_1} e_{\ominus p}(\sigma(s),t_0)q(s)\Delta s\right) + b_1 e_{\ominus p}(t_1,t_0) \\ &\quad + \int_{t_1}^{t} e_{\ominus p}(\sigma(s),t_0)q(s)\Delta s, \end{aligned} \tag{2.17}$$

that is,

$$m(t) \geq m(t_0)d_1 e_p(t,t_0) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k e_p(t,\sigma(s))q(s)\Delta s + b_1 e_p(t,t_1). \tag{2.18}$$

Repeating the above procession for $t \in [t_0,+\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} m(t) \geq m(t_0) \prod_{t_0<t_k<t} d_k e_p(t,t_0) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k e_p(t,\sigma(s))q(s)\Delta s \\ + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j e_p(t,t_k) b_k. \end{aligned} \tag{2.19}$$

Thus the proof of Lemma 2.4 is complete. □

The following comparison result plays an important role in this paper.

LEMMA 2.5. *Let $t_0 = 0$, $t_{p+1} = T$. Assume that $u \in \Omega$ satisfies*

$$\begin{gathered} u^{\Delta}(t) \geq -a(t)u(t) - b(t)[Tu](t) - c(t)[Su](t), \quad t \neq t_k,\ t \in J_{\mathbb{T}}, \\ u(t_k^+) - u(t_k) \geq -L_k u(t_k), \quad k = 1,2,\dots,p, \\ u(0) \geq u(T), \end{gathered} \tag{2.20}$$

where $a,b,c \in C[J_{\mathbb{T}},\mathbb{R}^+]$, a is not identically vanishing, and $\sup_{t\in J_{\mathbb{T}}}\{\mu(t)a(t)\} < 1$, $0 \leq L_k < 1$ $(k = 1,2,\dots,p)$. If

$$(Bk_0 + Ch_0)e_{\ominus(-a)}(T,0) \leq \frac{\left\{\prod_{0<t_k<T}(1-L_k)\right\}^2}{\int_0^T \prod_{s<t_k<T}(1-L_k)\Delta s} \tag{2.21}$$

with $B = \sup_{t\in J_{\mathbb{T}}}\{b(t)\int_0^t e_{\ominus(-a)}(\sigma(t),s)\Delta s\}$ and $C = \sup_{t\in J_{\mathbb{T}}}\{c(t)\int_0^T e_{\ominus(-a)}(\sigma(t),s)\Delta s\}$, then $u(t) \geq 0$ for $t \in J_{\mathbb{T}}$.

Proof. Let $p(t) = u(t)e_{\ominus(-a)}(t,0)$ for $t \in J_{\mathbb{T}}$. Then $p \in \Omega$ satisfies

$$\begin{aligned} p^{\Delta}(t) &\geq -b(t)\int_0^t e_{\ominus(-a)}(\sigma(t),s)k(t,s)p(s)\Delta s \\ &\quad - c(t)\int_0^T e_{\ominus(-a)}(\sigma(t),s)h(t,s)p(s)\Delta s, \quad t \neq t_k,\ t \in J_{\mathbb{T}}, \\ p(t_k^+) - p(t_k) &\geq -L_k p(t_k), \quad k = 1,2,\dots,p, \\ p(0) &\geq e_{(-a)}(T,0)p(T). \end{aligned} \tag{2.22}$$

We now prove

$$p(t) \geq 0 \quad \text{for } t \in J_{\mathbb{T}}. \tag{2.23}$$

Assume that (2.23) is not true. Then, there are two cases:

(a) there exists $t_1^* \in J_{\mathbb{T}}$ such that $p(t_1^*) < 0$ and $p(t) \le 0$ for $t \in J_{\mathbb{T}}$;

(b) there exists $t_1^*, t_2^* \in J_{\mathbb{T}}$ such that $p(t_1^*) < 0$ and $p(t_2^*) > 0$.

In case (a), (2.22) implies that

$$\begin{aligned} p^{\Delta}(t) &\ge 0, \quad t \ne t_k,\ t \in J_{\mathbb{T}}, \\ p(t_k^+) - p(t_k) &\ge 0, \quad k = 1,2,\dots,p. \end{aligned} \tag{2.24}$$

This means that $p(t)$ is nondecreasing in $J_{\mathbb{T}}$; therefore,

$$\begin{aligned} p(0) &\le p(t_1^*) < 0, \\ p(0) &\le p(T) \le 0, \end{aligned} \tag{2.25}$$

which contradicts $p(T) \le e_{\ominus(-a)}(T,0)p(0) < 0$.

In case (b) let $\sup_{t\in J_{\mathbb{T}}} p(t) = \lambda$. Then, $\lambda > 0$ and there exists $t_i < t_0^* \le t_{i+1}$ for some i such that $p(t_0^*) = \lambda$ or $p(t_i^+) = \lambda$. We may assume that $p(t_0^*) = \lambda$ (since, in case of $p(t_i^+) = \lambda$, the proof is similar). From (2.22), we have

$$\begin{aligned} p^{\Delta}(t) &\ge -\lambda k_0 b(t) \int_0^t e_{\ominus(-a)}(\sigma(t),s)\Delta s - \lambda h_0 c(t) \int_0^T e_{\ominus(-a)}(\sigma(t),s)\Delta s \\ &\ge -\lambda(Bk_0 + Ch_0), \quad t \ne t_k,\ t \in J_{\mathbb{T}}. \end{aligned} \tag{2.26}$$

For $t \in [t_0^*, T]_{\mathbb{T}}$, $k = i+1, i+2, \dots, p$,

$$p^{\Delta}(t) \ge -\lambda(Bk_0 + Ch_0), \quad t \ne t_k, \quad p(t_k^+) \ge (1 - L_k)p(t_k). \tag{2.27}$$

By Lemma 2.4, we have

$$p(t) \ge p(t_0^*) \prod_{t_0^* < t_k < t} (1 - L_k) + \int_{t_0^*}^t \prod_{s < t_k < t} (1 - L_k)(-\lambda(Bk_0 + Ch_0))\Delta s. \tag{2.28}$$

Let $t = T$ in (2.28), then

$$p(T) \ge \lambda \prod_{t_0^* < t_k < T} (1 - L_k) - \lambda(Bk_0 + Ch_0) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k)\Delta s. \tag{2.29}$$

If $p(T) < 0$, then (2.29) gives

$$(Bk_0 + Ch_0) > \frac{\prod_{t_0^* < t_k < T} (1 - L_k)}{\int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k)\Delta s} \ge \frac{\prod_{0 < t_k < T} (1 - L_k)}{\int_0^T \prod_{s < t_k < T} (1 - L_k)\Delta s}, \tag{2.30}$$

which contradicts (2.21), so, we have $p(T) \ge 0$, and by (2.22), $p(0) \ge p(T)e_{-a}(T,0) \ge 0$. Hence, $0 < t_1^* < T$. Let $t_j < t_1^* \le t_{j+1}$ for some j. We first assume that $t_0^* < t_1^*$, so $i \le j$. Let $t = t_1^*$ in (2.28), we have

$$0 > p(t_1^*) \ge \lambda \prod_{t_0^* < t_k < t_1^*} (1 - L_k) + \int_{t_0^*}^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)[-\lambda(Bk_0 + Ch_0)]\Delta s, \tag{2.31}$$

which gives

$$(Bk_0 + Ch_0) > \frac{\prod_{t_0^* < t_k < t_1^*} (1 - L_k)}{\int_{t_0^*}^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)\Delta s} \geq \frac{\prod_{0 < t_k < T} (1 - L_k)}{\int_0^T \prod_{s < t_k < T} (1 - L_k)\Delta s}, \tag{2.32}$$

which contradicts (2.21).

Next we assume that $t_1^* < t_0^*$. So $j \leq i$. For $t \in J_{\mathbb{T}}$, $k = 1, 2, \ldots, p$,

$$p^{\Delta}(t) \geq -\lambda(Bk_0 + Ch_0), \quad t \neq t_k, \quad p(t_k^+) \geq (1 - L_k)p(t_k). \tag{2.33}$$

By Lemma 2.4, we have

$$p(t) \geq p(0) \prod_{0 < t_k < t} (1 - L_k) + \int_0^t \prod_{s < t_k < t} (1 - L_k)(-\lambda(Bk_0 + Ch_0))\Delta s. \tag{2.34}$$

Let $t = t_1^*$ in (2.34), then

$$0 > p(t_1^*) \geq p(0) \prod_{0 < t_k < t_1^*} (1 - L_k) - \lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)\Delta s, \tag{2.35}$$

which implies

$$p(0) \prod_{0 < t_k < t_1^*} (1 - L_k) < \lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)\Delta s. \tag{2.36}$$

By (2.22), we obtain

$$\lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)\Delta s > e_{(-a)}(T, 0)p(T) \prod_{0 < t_k < t_1^*} (1 - L_k). \tag{2.37}$$

From (2.29), (2.37), we have

$$\begin{aligned} &\lambda(Bk_0 + Ch_0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)\Delta s \\ &\quad > e_{(-a)}(T, 0) \prod_{0 < t_k < t_1^*} (1 - L_k) \left\{\lambda \prod_{t_0^* < t_k < T} (1 - L_k) - \lambda(Bk_0 + Ch_0) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k)\Delta s\right\} \end{aligned} \tag{2.38}$$

or

$$\begin{aligned} \prod_{0 < t_k < t_1^*} (1 - L_k) \prod_{t_0^* < t_k < T} (1 - L_k) &< (Bk_0 + Ch_0) \prod_{0 < t_k < t_1^*} (1 - L_k) \int_{t_0^*}^T \prod_{s < t_k < T} (1 - L_k)\Delta s \\ &\quad + (Bk_0 + Ch_0)e_{\ominus(-a)}(T, 0) \int_0^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)\Delta s. \end{aligned} \tag{2.39}$$

Hence

$$\begin{aligned}\Big\{\prod_{0<t_k<T}(1-L_k)\Big\}^2 &\le \prod_{0<t_k<t_1^*}(1-L_k)\prod_{t_0^*<t_k<T}(1-L_k)\prod_{0<t_k<T}(1-L_k)\\ &< (Bk_0+Ch_0)\prod_{0<t_k<t_1^*}(1-L_k)\prod_{0<t_k<T}(1-L_k)\int_{t_0^*}^{T}\prod_{s<t_k<T}(1-L_k)\Delta s\\ &\quad + (Bk_0+Ch_0)e_{\ominus(-a)}(T,0)\prod_{0<t_k<T}(1-L_k)\int_0^{t_1^*}\prod_{s<t_k<t_1^*}(1-L_k)\Delta s\\ &< (Bk_0+Ch_0)e_{\ominus(-a)}(T,0)\int_0^T\prod_{s<t_k<T}(1-L_k)\Delta s,\end{aligned} \tag{2.40}$$

which contradicts (2.21).

Thus the proof of Lemma 2.5 is complete. □

For any $\delta(t)\in PC[J_{\mathbb{T}},\mathbb{R}]$ and $\eta\in\Omega$, $a,b,c\in C[J_{\mathbb{T}},\mathbb{R}^+]$, a is not identically vanishing, and $0\le L_k<1$ $(k=1,2,\dots,p)$, $I_k\in C[\mathbb{R},\mathbb{R}]$ $(k=1,2,\dots,p)$, we consider the linear periodic boundary value problem for a linear impulsive integrodifferential equation(PBVP):

$$\begin{gathered} u^{\Delta}(t)+a(t)u(t)=-b(t)[Tu](t)-c(t)[Su](t)+\delta(t),\quad t\neq t_k,\ t\in J_{\mathbb{T}},\\ u(t_k^+)-u(t_k)=-L_ku(t_k)+I_k(\eta(t_k))+L_k\eta(t_k),\quad k=1,2\dots,p,\\ u(0)=u(T).\end{gathered} \tag{2.41}$$

Lemma 2.6. *$u\in\Omega$ is a solution of PBVP (2.41) if and only if $u\in PC[J_{\mathbb{T}},\mathbb{R}]$ is a solution of the following impulsive integral equation:*

$$\begin{aligned}u(t)&=\int_0^T G(t,s)\{\delta(s)-b(s)[Tu](s)-c(s)[Su](s)\}\Delta s\\ &\quad+\sum_{0<t_k<T}G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)\big(-L_ku(t_k)+I_k(\eta(t_k))+L_k\eta(t_k)\big),\quad t\in J_{\mathbb{T}},\end{aligned} \tag{2.42}$$

where

$$G(t,s)=\frac{1}{1-e_{(-a)}(T,0)}\begin{cases} e_{(-a)}(t,\sigma(s)), & 0\le s<t\le T,\\ e_{(-a)}(T,0)e_{(-a)}(t,\sigma(s)), & 0\le t\le s\le T.\end{cases} \tag{2.43}$$

Proof. Assume that $u\in\Omega$ is a solution of (2.41). For the first equation of (2.41), using Lemma 2.3 on $t\in[0,t_1]_{\mathbb{T}}$, we have

$$u(t)=e_{(-a)}(t,0)u(0)+\int_0^t e_{(-a)}(t,\sigma(s))\{\delta(s)-b(s)[Tu](s)-c(s)[Su](s)\}\Delta s. \tag{2.44}$$

Then

$$u(t_1) = e_{(-a)}(t_1,0)u(0) + \int_0^{t_1} e_{(-a)}(t_1,\sigma(s))\{\delta(s) - b(s)[Tu](s) - c(s)[Su](s)\}\Delta s. \quad (2.45)$$

Again using Lemma 2.3 on $t \in (t_1,t_2]_{\mathbb{T}}$, then

$$\begin{aligned} u(t) &= u(t_1^+)e_{(-a)}(t,t_1) + \int_{t_1}^{t} e_{(-a)}(t,\sigma(s))\{\delta(s) - b(s)[Tu](s) - c(s)[Su](s)\}\Delta s \\ &= u(t_1)e_{(-a)}(t,t_1) + \int_{t_1}^{t} e_{(-a)}(t,\sigma(s))\{\delta(s) - b(s)[Tu](s) - c(s)[Su](s)\}\Delta s \\ &\quad + e_{(-a)}(t,t_1)(-L_1u(t_1) + I_1(\eta(t_1)) + L_1\eta(t_1)) \\ &= e_{(-a)}(t,0)u(0) + \int_0^{t} e_{(-a)}(t,\sigma(s))\{\delta(s) - b(s)[Tu](s) - c(s)[Su](s)\}\Delta s \\ &\quad + e_{(-a)}(t,t_1)(-L_1u(t_1) + I_1(\eta(t_1)) + L_1\eta(t_1)). \end{aligned} \quad (2.46)$$

Repeating the above procession for $t \in J_{\mathbb{T}}$, we have

$$\begin{aligned} u(t) &= u(0)e_{(-a)}(t,0) + \int_0^{t} e_{(-a)}(t,\sigma(s))\{\delta(s) - b(s)[Tu](s) - c(s)[Su](s)\}\Delta s \\ &\quad + \sum_{0<t_k<t} e_{(-a)}(t,t_k)(-L_ku(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)). \end{aligned} \quad (2.47)$$

Setting $t = T$ in (2.47) and using the boundary condition $u(0) = u(T)$, we obtain

$$\begin{aligned} u(0) = \frac{1}{1-e_{(-a)}(T,0)} \bigg\{ &\int_0^{T} e_{(-a)}(T,\sigma(s))(\delta(s) - b(s)[Tu](s) - c(s)[Su](s))\Delta s \\ &+ \sum_{0<t_k<T} e_{(-a)}(T,t_k)(-L_ku(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)) \bigg\}. \end{aligned} \quad (2.48)$$

Substituting (2.48) into (2.47), we see that $u \in PC[J_{\mathbb{T}},\mathbb{R}]$ satisfies (2.42).

If $u \in PC[J_{\mathbb{T}},\mathbb{R}]$ is a solution of (2.42), then $u \in C^1(J'_{\mathbb{T}},R)$ and

$$\begin{aligned} u^{\Delta}(t) + a(t)u(t) &= -b(t)[Tu](t) - c(t)[Su](t) + \delta(t), \quad t \neq t_k,\ t \in J_{\mathbb{T}}, \\ u(t_k^+) - u(t_k) &= -L_ku(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k), \quad k = 1,2\dots,p. \end{aligned} \quad (2.49)$$

Setting $t = 0,T$ in (2.42), respectively, we have

$$\begin{aligned} u(T) = \frac{1}{1-e_{(-a)}(T,0)} \bigg\{ &\int_0^{T} e_{(-a)}(T,\sigma(s))(\delta(s) - b(s)[Tu](s) - c(s)[Su](s))\Delta s \\ &+ \sum_{0<t_k<T} e_{(-a)}(T,t_k)(-L_ku(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)) \bigg\} = u(0). \end{aligned} \quad (2.50)$$

Therefore, $u \in \Omega$ is a solution of (2.41). Thus Lemma 2.6 is proved. □

Lemma 2.7. *Assume that $a,b,c \in C[J_{\mathbb{T}},\mathbb{R}^+]$ and $0 \le L_k < 1$ ($k = 1,2,\dots,p$), $I_k \in C[\mathbb{R},\mathbb{R}]$ ($k = 1,2\dots,p$), $\delta \in PC[J_{\mathbb{T}},\mathbb{R}]$, $\eta \in \Omega$, and the following inequality holds:*

$$\frac{1}{1-e_{(-a)}(T,0)}\left(\int_0^T (k_0 s b(s) + T h_0 c(s))\Delta s + \sum_{k=1}^{p} L_k\right) < 1. \tag{2.51}$$

Then PBVP (2.41) possesses a unique solution in Ω.

Proof. For any $u \in \Omega$, consider the operator F defined by the formula

$$\begin{aligned}(Fu)(t) = &\int_0^T G(t,s)\{\delta(s) - b(s)[Tu](s) - c(s)[Su](s)\}\Delta s \\ &+ \sum_{0<t_k<T} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)(-L_k u(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)), \quad t \in J_{\mathbb{T}}.\end{aligned} \tag{2.52}$$

Then $Fu \in \Omega$, that is, $F\Omega \subset \Omega$.

For every $u,v \in \Omega, t \in J_{\mathbb{T}}$, we have

$$\begin{aligned}|(Fu)(t) - (Fv)(t)| \le &\int_0^T G(t,s)\{b(s)|[Tu](s) - [Tv](s)| + c(s)|[Su](s) - [Sv](s)|\}\Delta s \\ &+ \sum_{0<t_k<T} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)L_k|u(t_k) - v(t_k)| \\ &< \frac{1}{1-e_{(-a)}(T,0)}\left(\int_0^T (k_0 s b(s) + T h_0 c(s))\Delta s + \sum_{k=1}^{p} L_k\right)\|u-v\|_{PC}.\end{aligned} \tag{2.53}$$

Hence

$$\|Fu - Fv\|_{PC} = \sup_{t\in J_{\mathbb{T}}}|(Fu)(t) - (Fv)(t)| \le \alpha\|u-v\|_{PC}, \tag{2.54}$$

where

$$\alpha = \frac{1}{1-e_{(-a)}(T,0)}\left(\int_0^T (k_0 s b(s) + T h_0 c(s))\Delta s + \sum_{k=1}^{p} L_k\right) < 1. \tag{2.55}$$

Thus the operator F is a contraction on Ω. That is, there is a unique element $u \in \Omega$ such that $u = Fu$. Therefore, u is the unique solution of PBVP (2.41). The proof of Lemma 2.7 is complete. □

Lemma 2.8. *$u \in \Omega$ is a solution of PBVP (1.1) if and only if $u \in PC[J_{\mathbb{T}},\mathbb{R}]$ is solution of the following integral equation:*

$$\begin{aligned}u(t) = &\int_0^T G(t,s)[f(s,u(s),[Tu](s),[Su](s)) + a(s)u(s)]\Delta s \\ &+ \sum_{0<t_k<1} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)I_k(u(t_k)),\end{aligned} \tag{2.56}$$

where

$$G(t,s) = \frac{1}{1-e_{(-a)}(T,0)} \begin{cases} e_{(-a)}(t,\sigma(s)), & 0 \le s < t \le T, \\ e_{(-a)}(T,0)e_{(-a)}(t,\sigma(s)), & 0 \le t \le s \le T. \end{cases} \tag{2.57}$$

The proof of Lemma 2.8 is similar to that of Lemma 2.6 and we will omit it here.

3. Main results

In this section, we will use the monotone iterative technique to prove the existence of minimal and maximal solutions of the PBVP (1.1).

Theorem 3.1. *Assume that the following conditions hold.*

(H_1) There exist functions $u_0, v_0 \in \Omega$, $u_0(t) \le v_0(t)$ for all $t \in J_{\mathbb{T}}$ such that

$$\begin{aligned} &u_0^{\Delta}(t) \le f(t,u_0(t),[Tu_0](t),[Su_0](t)), \quad t \ne t_k,\ t \in J_{\mathbb{T}}, \\ &u_0(t_k^+) - u_0(t_k) \le I_k(u_0(t_k)), \quad k = 1,2,\dots,p, \\ &u_0(0) \le u_0(T), \\ &v_0^{\Delta}(t) \ge f(t,v_0(t),[Tv_0](t),[Sv_0](t)), \quad t \ne t_k,\ t \in J_{\mathbb{T}}, \\ &v_0(t_k^+) - v_0(t_k) \ge I_k(v_0(t_k)), \quad k = 1,2,\dots,p, \\ &v_0(0) \ge v_0(T). \end{aligned} \tag{3.1}$$

(H_2) The function $f \in C[J_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ satisfies

$$f(t,u_2,v_2,w_2) - f(t,u_1,v_1,w_1) \ge -a(t)(u_2-u_1) - b(t)(v_2-v_1) - c(t)(w_2-w_1), \tag{3.2}$$

whenever $u_0(t) \le u_1 \le u_2 \le v_0(t), [Tu_0](t) \le v_1 \le v_2 \le [Tv_0](t), [Su_0](t) \le w_1 \le w_2 \le [Sv_0](t)$, $t \in J_{\mathbb{T}}$, where for $a,b,c \in C[J_{\mathbb{T}},\mathbb{R}^+]$, $\sup_{t\in J_{\mathbb{T}}}\{\mu(t)a(t)\} < 1$, a is not identically vanishing.

(H_3) The function $I_k \in C[\mathbb{R},\mathbb{R}]$ satisfies

$$I_k(x) - I_k(y) \ge -L_k(x-y), \tag{3.3}$$

whenever $u_0(t_k) \le y \le x \le v_0(t_k)$ $(k = 1,2,\dots,p)$, and $0 \le L_k < 1$ $(k = 1,2,\dots,p)$.

Further, assume that the inequalities (2.21) and (2.51) hold. Then PBVP (1.1) has the minimal solution u^* and maximal v^* in $[u_0,v_0]$. Moreover, there exist monotone iteration sequences $\{u_n(t)\},\{v_n(t)\} \subset [u_0,v_0]$ such that

$$u_n(t) \longrightarrow u^*(t), v_n(t) \longrightarrow v^*(t) \quad \text{as } n \longrightarrow \infty \text{ uniformly on } t \in J_{\mathbb{T}}, \tag{3.4}$$

where $\{u_n(t)\},\{v_n(t)\}$ *satisfy*

$$
\begin{aligned}
&u_n^{\Delta}(t)=f(t,u_{n-1}(t),[Tu_{n-1}](t),[Su_{n-1}](t))-a(t)(u_n-u_{n-1})(t)\\
&\qquad -b(t)[T(u_n-u_{n-1})](t)-c(t)[S(u_n-u_{n-1})](t),\quad t\neq t_k,\ t\in J_{\mathbb{T}},\\
&u_n(t_k^+)-u_n(t_k)=-L_ku_n(t_k)+I_k(u_{n-1}(t_k))+L_ku_{n-1}(t_k),\quad k=1,2,\dots,p,\\
&u_n(0)=u_n(T)\quad (n=1,2,3,\dots),\\
&v_n^{\Delta}(t)=f(t,v_{n-1}(t),[Tv_{n-1}](t),[Sv_{n-1}](t))-a(t)(v_n-v_{n-1})(t)\\
&\qquad -b(t)[T(v_n-v_{n-1})](t)-c(t)[S(v_n-v_{n-1})](t),\quad t\neq t_k,\ t\in J_{\mathbb{T}},\\
&v_n(t_k^+)-v_n(t_k)=-L_kv_n(t_k)+I_k(v_{n-1}(t_k))+L_kv_{n-1}(t_k),\quad k=1,2,\dots,p,\\
&v_n(0)=v_n(T)\quad (n=1,2,3,\dots),
\end{aligned}
\tag{3.5}
$$

$$
u_0\le u_1\le\cdots\le u_n\le\cdots\le u^*\le v^*\le\cdots\le v_n\le\cdots\le v_1\le v_0. \tag{3.6}
$$

Proof. For any $u_{n-1},v_{n-1}\in\Omega$, by Lemma 2.7, we know that (3.5) has unique solution u_n and v_n in Ω, respectively.

In the following, we will show by induction that

$$
u_{n-1}\le u_n\le v_n\le v_{n-1},\quad n=1,2,3,\dots. \tag{3.7}
$$

By (3.5) and the conditions (H_1), (H_2), and (H_3), we have

$$
\begin{aligned}
&(u_1-u_0)^{\Delta}(t)\ge -a(t)(u_1-u_0)(t)-b(t)[T(u_1-u_0)](t)\\
&\qquad -c(t)[S(u_1-u_0)](t),\quad t\neq t_k,\ t\in J_{\mathbb{T}},\\
&(u_1-u_0)(t_k^+)-(u_1-u_0)(t_k)\ge -L_k(u_1-u_0)(t_k),\quad k=1,2,\dots,p,\\
&(u_1-u_0)(0)\ge (u_1-u_0)(T),\\
&(v_0-v_1)^{\Delta}(t)\ge -a(t)(v_0-v_1)(t)-b(t)[T(v_0-v_1)](t)\\
&\qquad -c(t)[S(v_0-v_1)](t),\quad t\neq t_k,\ t\in J_{\mathbb{T}},\\
&(v_0-v_1)(t_k^+)-(v_0-v_1)(t_k)\ge -L_k(v_0-v_1)(t_k),\quad k=1,2,\dots,p,\\
&(v_0-v_1)(0)\ge (v_0-v_1)(T),\\
&(v_1-u_1)^{\Delta}(t)\ge -a(t)(v_1-u_1)(t)-b(t)[T(v_1-u_1)](t)\\
&\qquad -c(t)[S(v_1-u_1)](t),\quad t\neq t_k,\ t\in J_{\mathbb{T}},\\
&(v_1-u_1)(t_k^+)-(v_1-u_1)(t_k)\ge -L_k(v_1-u_1)(t_k),\quad k=1,2,\dots,p,\\
&(v_1-u_1)(0)=(v_1-u_1)(T).
\end{aligned}
\tag{3.8}
$$

Thus, by Lemma 2.5, we have $u_0\le u_1\le v_1\le v_0$.

Now we assume that (3.7) is true for $i > 1$, that is, $u_{i-1} \le u_i \le v_i \le v_{i-1}$, and we prove that (3.7) is true for $i+1$ too. In fact, by (3.5), and the conditions H_2 and H_3, we have that

$$\begin{aligned}
&(u_{i+1}-u_i)^{\Delta}(t) \ge -a(t)(u_{i+1}-u_i)(t) - b(t)[T(u_{i+1}-u_i)](t)\\
&\qquad\qquad - c(t)[S(u_{i+1}-u_i)](t), \quad t \ne t_k,\ t \in J_{\mathbb{T}},\\
&(u_{i+1}-u_i)(t_k^+) - (u_{i+1}-u_i)(t_k) \ge -L_k(u_{i+1}-u_i)(t_k), \quad k = 1,2,\dots,p,\\
&(u_{i+1}-u_i)(0) = (u_{i+1}-u_i)(T),\\
&(v_{i+1}-v_i)^{\Delta}(t) \ge -a(t)(v_{i+1}-v_i)(t) - b(t)[T(v_{i+1}-v_i)](t)\\
&\qquad\qquad - c(t)[S(v_{i+1}-v_i)](t), \quad t \ne t_k,\ t \in J_{\mathbb{T}},\\
&(v_{i+1}-v_i)(t_k^+) - (v_{i+1}-v_i)(t_k) \ge -L_k(v_{i+1}-v_i)(t_k), \quad k = 1,2,\dots,p,\\
&(v_{i+1}-v_i)(0) = (v_{i+1}-v_i)(T),\\
&(v_{i+1}-u_{i+1})^{\Delta}(t) \ge -a(t)(v_{i+1}-u_{i+1})(t) - b(t)[T(v_{i+1}-u_{i+1})](t)\\
&\qquad\qquad - c(t)[S(v_{i+1}-u_{i+1})](t), \quad t \ne t_k,\ t \in J_{\mathbb{T}},\\
&(v_{i+1}-u_{i+1})(t_k^+) - (v_{i+1}-u_{i+1})(t_k) \ge -L_k(v_{i+1}-u_{i+1})(t_k), \quad k = 1,2,\dots,p,\\
&(v_{i+1}-u_{i+1})(0) = (v_{i+1}-u_{i+1})(T).
\end{aligned} \tag{3.9}$$

Thus, by Lemma 2.5, we have that $u_i \le u_{i+1} \le v_{i+1} \le v_i$. So, by induction, (3.7) holds for any positive integer n.

It is easy to know by (3.7) that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{3.10}$$

Furthermore, by (3.5), and Lemma 2.6, we have

$$\begin{aligned}
u_n(t) = &\int_0^T G(t,s)\{f(s,u_{n-1}(s),[Tu_{n-1}](s),[Su_{n-1}](s)) + a(s)u_{n-1}(s)\\
&\qquad - b(s)[T(u_n-u_{n-1})](s) - c(s)[S(u_n-u_{n-1})](s)\}\Delta s\\
&+ \sum_{0<t_k<T} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)(-L_k u_n(t_k) + I_k(u_{n-1}(t_k)) + L_k u_{n-1}(t_k)), \quad t \in J_{\mathbb{T}},\\
v_n(t) = &\int_0^T G(t,s)\{f(s,v_{n-1}(s),[Tv_{n-1}](s),[Sv_{n-1}](s)) + a(s)v_{n-1}(s)\\
&\qquad - b(s)[T(v_n-v_{n-1})](s) - c(s)[S(v_n-v_{n-1})](s)\}\Delta s\\
&+ \sum_{0<t_k<T} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)(-L_k v_n(t_k) + I_k(v_{n-1}(t_k)) + L_k v_{n-1}(t_k)), \quad t \in J_{\mathbb{T}}.
\end{aligned} \tag{3.11}$$

By (3.5) and the condition (H_2), we have

$$\begin{aligned} & f(t,u_0(t),T[u_0](t),S[u_0](t)) - a(t)(v_0-u_0)(t) \\ &\qquad - b(t)T[(v_0-u_0)](t) - c(t)S[(v_0-u_0)](t) \\ &\quad \le u_n^{\Delta}(t) \le f(t,v_0(t),T[v_0](t),S[v_0](t)) \\ &\qquad + a(t)(v_0-u_0)(t) + b(t)T[(v_0-u_0)](t) + c(t)S[(v_0-u_0)](t). \end{aligned} \tag{3.12}$$

Thus, $\{u_n^{\Delta}(t)\}$ is uniformly bounded. Also, similarly to the above we can show that $\{v_n^{\Delta}(t)\}$ is uniformly bounded. Using Lemma 2.4 [12], we know that there exist u^*, v^* such that $\lim_{n\to\infty} u_n(t) = u^*(t), \lim_{n\to\infty} v_n(t) = v^*(t)$ uniformly on $J_{\mathbb{T}}$.

Taking limits as $n \to \infty$, by (3.11), we have that

$$\begin{aligned} u^*(t) &= \int_0^T G(t,s)[f(s,u^*(s),[Tu^*](s),[Su^*](s)) + a(s)u^*(s)]\Delta s \\ &\quad + \sum_{0<t_k<1} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)I_k(u^*(t_k)), \\ v^*(t) &= \int_0^T G(t,s)[f(s,v^*(s),[Tv^*](s),[Sv^*](s)) + a(s)v^*(s)]\Delta s \\ &\quad + \sum_{0<t_k<1} G(t,t_k)e_{(-a)}(\sigma(t_k),t_k)I_k(v^*(t_k)). \end{aligned} \tag{3.13}$$

From the above, by Lemma 2.8, we know that u^* and v^* are solutions of PBVP (1.1) in $[u_0, v_0]$.

Next we prove that u^* and v^* are the minimal and maximal solutions of PBVP (1.1) in $[u_0, v_0]$.

In fact, let $w \in [u_0, v_0]$ be a solution of PBVP(1.1), that is,

$$\begin{gathered} w^{\Delta}(t) = f(t,w(t),[Tw](t),[Sw](t)), \quad t \neq t_k,\ t \in J_{\mathbb{T}}, \\ w(t_k^+) - w(t_k) = I_k(w(t_k)), \quad k = 1,2,\ldots,p, \\ w(0) = w(T). \end{gathered} \tag{3.14}$$

Using induction, suppose that there exists a positive integer n such that $u_n(t) \le w(t) \le v_n(t)$ on $J_{\mathbb{T}}$. Then,

$$\begin{aligned} (w-u_{n+1})^{\Delta}(t) &= f(t,w(t),[Tw](t),[Sw](t)) \\ &\quad - \{f(t,u_n(t),[Tu_n](t),[Su_n](t)) - a(t)(u_n-u_{n+1})(t) \\ &\qquad - b(t)[T(u_n-u_{n+1})](t) - c(t)[S(u_n-u_{n+1})](t)\} \\ &\ge -a(t)(w(t)-u_{n+1}(t)) - b(t)[T(w-u_{n+1})](t) \\ &\quad - c(t)[S(w-u_{n+1})](t), \quad t \neq t_k,\ t \in J_{\mathbb{T}}, \end{aligned}$$

$$\begin{aligned}(w-u_{n+1})(t_k^+) &= (w-u_{n+1})(t_k)+I_k(w(t_k))-[-L_ku_{n+1}(t_k)+I_k(u_n(t_k))+L_ku_n(t_k)]\\ &\geq (1-L_k)(w-u_{n+1})(t_k), \quad k=1,2,\dots,p,\\ (w-u_{n+1})(0) &= (w-u_{n+1})(T).\end{aligned} \tag{3.15}$$

By Lemma 2.5, it follows that $w(t)\geq u_{n+1}(t)$ on $J_{\mathbb{T}}$. Similarly, we obtain $v_{n+1}(t)\geq w(t)$ on $J_{\mathbb{T}}$. Since $u_0(t)\leq w(t)\leq v_0(t)$ on $J_{\mathbb{T}}$, by induction we get

$$u_{n+1}(t)\leq w(t)\leq v_{n+1}(t), \quad n=1,2,3,\dots. \tag{3.16}$$

Thus, letting $n\to\infty$ in (3.16), we have that

$$u^*\leq w\leq v^*, \tag{3.17}$$

that is, u^* and v^* are the minimal and maximal solutions of the PBVP (1.1) in the interval $[u_0,v_0]$.

The proof of Theorem 3.1 is complete. □

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