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Research Article

Positive Solutions of Boundary Value Problems for System of Nonlinear Fourth-Order Differential Equations

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Some existence theorems of the positive solutions and the multiple positive solutions for singular and nonsingular systems of nonlinear fourth-order boundary value problems are proved by using topological degree theory and cone theory.

1. Introduction and preliminary

Fourth-order nonlinear differential equations have many applications such as balancing condition of an elastic beam which may be described by nonlinear fourth-order ordinary differential equations. Concerning the studies for singular and nonsingular case, one can refer to [1–10]. However, there are not many results on the system for nonlinear fourth-order differential equations. In this paper, by using topological degree theory and cone theory, we study the existence of the positive solutions and the multiple positive solutions for singular and nonsingular system of nonlinear fourth-order boundary value problems. Our conclusions and conditions are different from the ones used in [1–10] for single equations.

This paper is divided into three sections: in Section 2, we prove the existence of the positive solutions and the multiple positive solutions for systems of nonlinear fourth-order boundary value problems with nonlinear singular terms $f_i(t,u)$ which may be singular at $t=0$, $t=1$. In Section 3, we prove some existence theorems of the positive solutions and the multiple positive solutions for nonsingular system of nonlinear fourth-order boundary value problems.

Let $(E,\|\cdot\|)$ be a real Banach space and $P\subset E$ a cone, $B_\rho=\{u\in E:\|u\|<\rho\}$ $(\rho>0)$.

Lemma 1.1 [11]. *Assume that* $A : \overline{B}_\rho \cap P \to P$ *is a completely continuous operator. If there exists* $x_0 \in P \backslash \{\theta\}$ *such that*

$$x - Ax \neq \lambda x_0, \quad \forall \lambda \geq 0,\ x \in \partial B_\rho \cap P, \tag{1.1}$$

then $i(A, B_\rho \cap P, P) = 0$.

Lemma 1.2 [12]. *Assume that* $A : \overline{B}_\rho \cap P \to P$ *is a completely continuous operator and has no fixed point at* $\partial B_\rho \cap P$.

(1) *If* $\|Au\| \leq \|u\|$ *for any* $u \in \partial B_\rho \cap P$, *then* $i(A, B_\rho \cap P, P) = 1$.

(2) *If* $\|Au\| \geq \|u\|$ *for any* $u \in \partial B_\rho \cap P$, *then* $i(A, B_\rho \cap P, P) = 0$.

2. Singular case

We consider boundary value problems of singular system for nonlinear four order ordinary differential equations (SBVP)

$$\begin{gathered} x^{(4)} = f_1(t, y), \quad t \in (0,1), \\ x(0) = x(1) = x''(0) = x''(1) = 0, \\ -y'' = f_2(t, x), \quad t \in (0,1), \\ y(0) = y(1) = 0, \end{gathered} \tag{2.1}$$

where $f_i \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$ $(i = 1,2)$, $\mathbb{R}^+ = [0, +\infty)$, $f_i(t, 0) \equiv 0$, $f_i(t, u)$ are singular at $t = 0$ and $t = 1$. $(x, y) \in C^4(0,1) \cap C^2[0,1] \times C^2(0,1) \cap C[0,1]$ are a solution of SBVP (2.1) if (x, y) satisfies (2.1). Moreover, we call that (x, y) is a positive solution of SBVP (2.1) if $x(t) > 0$, $y(t) > 0$, $t \in (0,1)$.

First, we list the following assumptions.

(H_1) There exist $q_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $p_i \in C((0,1), [0, +\infty))$ such that $f_i(t, u) \leq p_i(t) q_i(u)$ and

$$0 < \int_0^1 t(1-t) p_i(t)\, dt < +\infty \quad (i = 1,2). \tag{2.2}$$

(H_2) There exist $\alpha \in (0,1]$, $0 < a < b < 1$ such that

$$\liminf_{u \to +\infty} \frac{f_1(t,u)}{u^\alpha} > 0, \qquad \liminf_{u \to +\infty} \frac{f_2(t,u)}{u^{1/\alpha}} = +\infty \tag{2.3}$$

uniformly on $t \in [a, b]$.

(H_3) There exists $\beta \in (0, +\infty)$ such that

$$\limsup_{u \to 0^+} \frac{f_1(t,u)}{u^\beta} < +\infty, \qquad \limsup_{u \to 0^+} \frac{f_2(t,u)}{u^{1/\beta}} = 0 \tag{2.4}$$

uniformly on $t \in (0,1)$.

(H_4) There exists $\gamma \in (0,1]$, $0 < a < b < 1$ such that

$$\liminf_{u \to 0^+} \frac{f_1(t,u)}{u^\gamma} > 0, \qquad \liminf_{u \to 0^+} \frac{f_2(t,u)}{u^{1/\gamma}} = +\infty \tag{2.5}$$

uniformly on $t \in [a,b]$.

(H_5) There exists $R > 0$ such that $q_1[0,N] \int_0^1 t(1-t)p_1(t)dt < R$, where $N = q_2[0,R] \int_0^1 t(1-t)p_2(t)dt$, $q_i[0,d] = \sup\{q_i(u) : u \in [0,d]\}$ $(i = 1,2)$.

Lemma 2.1 [13]. *Assume that $p_i \in C((0,1),[0,+\infty))$ $(i = 1,2)$ satisfies (H_1), then*

$$\lim_{t \to 0^+} t \int_t^1 (1-s)p_i(s)ds = \lim_{t \to 1^-} (1-t) \int_t^1 s p_i(s)ds = 0. \tag{2.6}$$

By (H_1) and Lemma 2.1, we know that SBVP (2.1) is equivalent to the following system of nonlinear integral equations:

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) \int_0^1 G(s,r) f_1(r,y(r))\,dr\,ds, \\ y(t) &= \int_0^1 G(t,s) f_2(s,x(s))\,ds, \end{aligned} \tag{2.7}$$

where

$$G(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.8}$$

Clearly, (2.7) is equivalent to the following nonlinear integral equation:

$$x(t) = \int_0^1 G(t,s) \int_0^1 G(s,r) f_1\left(r, \int_0^1 G(r,\tau) f_2(\tau,x(\tau))\,d\tau\right) dr\,ds. \tag{2.9}$$

Let $J = [0,1]$, $J_0 = [a,b] \subset (0,1)$, $\varepsilon_0 = a(1-b)$, $E = C[0,1]$, $\|u\| = \max_{t \in J} |u(t)|$ for $u \in E$,

$$K = \{u \in C[0,1] : u(t) \ge 0,\ u(t) \ge t(1-t)\|u\|,\ t \in J\}. \tag{2.10}$$

It is easy to show that $(E, \|\cdot\|)$ is a real Banach pace, K is a cone in E and

$$\begin{gathered} G(t,s) \ge \varepsilon_0, \quad \forall (t,s) \in J_0 \times J_0, \\ t(1-t)G(r,s) \le G(t,s) \le G(s,s) = (1-s)s, \quad \forall t,s,r \in J. \end{gathered} \tag{2.11}$$

By virtue of (H_1), we can define $A : C[0,1] \to C[0,1]$ as follows:

$$(Ax)(t) = \int_0^1 G(t,s) \int_0^1 G(s,r) f_1(r, Tx(r))\,dr\,ds, \tag{2.12}$$

where

$$(Tx)(t) = \int_0^1 G(t,s) f_2(s,x(s))\,ds. \tag{2.13}$$

Then the positive solutions of SBVP (2.1) are equivalent to the positive fixed points of A.

Lemma 2.2. *Let (H_1) hold, then $A : K \to K$ is a completely continuous operator.*

Proof. Firstly, we show that $T : K \to K$ is uniformly bounded continuous operator. For any $x \in K$, it follows from (2.13) that $(Tx)(t) \ge 0$ and

$$(Tx)(t) \ge t(1-t)\int_0^1 G(r,s) f_2(s,x(s))\,ds, \quad t,r \in J. \tag{2.14}$$

From (2.14), we get that $(Tx)(t) \ge t(1-t)\|Tx\|$ for any $t \in J$. So $T(K) \subset K$.

Let $D \subset K$ be a bounded set, we assume that $\|x\| \le d$ for any $x \in D$. Equation (2.13) and (H_1) imply that

$$\|Tx\| \le q_2[0,d]\int_0^1 s(1-s)p_2(s)\,ds =: C_1, \tag{2.15}$$

from this we know that $T(D)$ is a bounded set.

Next, we show that $T : K \to K$ is a continuous operator. Let $x_n, x_0 \in K$, $\|x_n - x_0\| \to 0$ $(n \to \infty)$. Then $\{x_n\}$ is a bounded set, we assume that $\|x_n\| \le d$ $(n = 0,1,2,\dots)$. By (H_1), we have

$$\begin{gathered} f_2(t,x_n(t)) \le q_2[0,d]p_2(t), \quad t \in (0,1),\ n = 0,1,2,\dots, \\ |Tx_n(t) - Tx_0(t)| \le \int_0^1 s(1-s)\,|f_2(s,x_n(s)) - f_2(s,x_0(s))|\,ds, \quad t \in J. \end{gathered} \tag{2.16}$$

Now (2.16), (H_1), and Lebesgue control convergent theorem yield

$$\|Tx_n - Tx_0\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.17}$$

Thus $T : K \to K$ is a continuous operator. By $T \in C[K,K]$ and $f_1 \in C((0,1)\times\mathbb{R}^+,\mathbb{R}^+)$, similarly we can show that $A : K \to K$ is a uniformly bounded continuous operator.

We verify that A is equicontinuous on D. Since $G(t,s)$ is uniformly continuous in $J \times J$, for any $\varepsilon > 0$, $0 \le t_1 < t_2 \le 1$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta$ imply that $|G(t_1,s) - G(t_2,s)| < \varepsilon$, $s \in J$. Then

$$\begin{aligned} |(Ax)(t_1) - (Ax)(t_2)| &\le \int_0^1 |G(t_1,s) - G(t_2,s)| \int_0^1 G(s,r) f_1(r,(Tx)(r))\,dr\,ds \\ &\le q_1[0,C_1]\int_0^1 |G(t_1,s) - G(t_2,s)|\,ds \int_0^1 r(1-r)p_1(r)\,dr \\ &< \varepsilon q_1[0,C_1]\int_0^1 r(1-r)p_1(r)\,dr, \quad \forall x \in D. \end{aligned} \tag{2.18}$$

This implies that $A(D)$ is equicontinuous. So $A: K \to K$ is complete continuous. This completes the proof of Lemma 2.1. □

THEOREM 2.3. *Let* (H_1), (H_2), *and* (H_3) *hold, then SBVP (2.1) has at least one positive solution.*

Proof. From Lemma 2.2, we know that $A: K \to K$ is completely continuous. According to the first limit of (H_2), there are $\nu > 0$, $M_1 > 0$ such that

$$f_1(t,u) \geq \nu u^{\alpha}, \quad \forall (t,u) \in [a,b] \times (M_1, +\infty). \tag{2.19}$$

Let $R_1 \geq \max\{((b-a)\varepsilon_0^{(1+\alpha)/\alpha})^{-1}, ((\varepsilon_0^3(b-a)^2\nu/2)\max_{t\in J}\int_a^b G(t,s)ds)^{-1/\alpha}\}$. By the second limit of (H_2), there exists $M_2 > 0$ such that

$$f_2(t,u) \geq R_1 u^{1/\alpha}, \quad \forall (t,u) \in [a,b] \times (M_2, +\infty). \tag{2.20}$$

Taking $M \geq \max\{M_1, M_2\}$, $\overline{R} = (M+1)\varepsilon_0^{-1}$, $x_0(t) = \sin \pi t \in K \setminus \{\theta\}$, we affirm that

$$x - Ax \neq \lambda x_0, \quad \forall \lambda \geq 0,\ x \in \partial B_{\overline{R}} \cap K. \tag{2.21}$$

In fact, if there are $\lambda \geq 0$, $x \in \partial B_{\overline{R}} \cap K$ such that $x - Ax = \lambda x_0$, then for $t \in J$, we have

$$x(t) \geq (Ax)(t) \geq \int_a^b G(t,s) \int_a^b G(s,r) f_1\left(r, \int_0^1 G(r,\xi) f_2(\xi, x(\xi))\,d\xi\right) dr\,ds. \tag{2.22}$$

Owing to $\alpha \in (0,1]$ and $x(t) \geq \varepsilon_0 \overline{R} > M$, $t \in J_0$, (2.20) implies that

$$\begin{aligned}
\int_0^1 G(r,\xi) f_2(\xi, x(\xi))\,d\xi &\geq \int_a^b G(r,\xi) f_2(\xi, x(\xi))\,d\xi \\
&\geq R_1 \int_a^b G(r,\xi) x^{1/\alpha}(\xi)\,d\xi \geq R_1 \left(\varepsilon_0 \overline{R}\right)^{1/\alpha} \int_a^b G(r,\xi)\,d\xi \\
&\geq \overline{R} R_1 (b-a) \varepsilon_0^{1+1/\alpha} \geq \overline{R} > M, \quad r \in J_0.
\end{aligned} \tag{2.23}$$

By using $0 \leq G(t,s) \leq 1$, $\alpha \in (0,1]$ and Jensen inequality, it follows from (2.19)–(2.23) that

$$\begin{aligned}
x(t) &\geq \nu \int_a^b G(t,s) \int_a^b G(s,r) \left(\int_0^1 G(r,\xi) f_2(\xi, x(\xi))\,d\xi \right)^{\alpha} dr\,ds \\
&\geq \varepsilon_0 \nu \int_a^b G(t,s) \int_a^b \left(\int_0^1 G^{\alpha}(r,\xi) f_2^{\alpha}(\xi, x(\xi))\,d\xi \right) dr\,ds \\
&\geq \varepsilon_0 \nu \int_a^b G(t,s) \int_a^b \left(\int_a^b G(r,\xi) f_2^{\alpha}(\xi, x(\xi))\,d\xi \right) dr\,ds \\
&\geq \varepsilon_0 \nu R_1^{\alpha} \int_a^b G(t,s) \int_a^b \int_a^b G(r,\xi) x(\xi)\,d\xi\,dr\,ds \\
&\geq \overline{R} \varepsilon_0^3 (b-a)^2 \nu R_1^{\alpha} \int_a^b G(t,s)\,ds, \quad t \in J.
\end{aligned} \tag{2.24}$$

Thus

$$\overline{R} = \|x\| \geq \overline{R}\varepsilon_0^3(b-a)^2\nu R_1^{\alpha}\max_{t\in J}\int_a^b G(t,s)ds \geq 2\overline{R}. \tag{2.25}$$

This is a contradiction. By Lemma 1.1, we get

$$i(A, B_{\overline{R}} \cap K, K) = 0. \tag{2.26}$$

On the other hand, there exists $\rho_1 \in (0,1)$ according to the first limit of (H_3) such that

$$C_2 =: \sup\left\{\frac{f_1(t,u)}{u^{\beta}} : (t,u) \in (0,1) \times (0,\rho_1]\right\} < +\infty. \tag{2.27}$$

Taking $\varepsilon_1 = \min\{\rho_1, (1/2C_2)^{1/\beta}\} > 0$. By the second limit of (H_3), there exists $\rho_2 \in (0,1)$ such that

$$f_2(t,u) \leq \varepsilon_1 u^{1/\beta}, \quad \forall (t,u) \in (0,1) \times (0,\rho_2]. \tag{2.28}$$

Let $\rho = \min\{\rho_1, \rho_2\}$. Equations (2.27) and (2.28) imply that

$$\begin{aligned}
\int_0^1 G(r,\xi) f_2(\xi, x(\xi))d\xi &\leq \varepsilon_1 \int_0^1 G(r,\xi)x(\xi)^{1/\beta}d\xi \leq \rho_1\|x\|^{1/\beta} \\
&\leq \rho_1^{1+1/\beta} < \rho_1, \quad \forall x \in \overline{B}_{\rho} \cap K,\ r \in (0,1), \\
(Ax)(t) &\leq C_2\int_0^1 G(t,s)\int_0^1 G(s,r)\left(\int_0^1 G(r,\xi)f_2(\xi,x(\xi))d\xi\right)^{\beta} dr\, ds \\
&\leq C_2\varepsilon_1^{\beta}\|x\| \leq \frac{1}{2}\|x\|, \quad \forall x \in \overline{B}_{\rho} \cap K,\ t \in [0,1].
\end{aligned} \tag{2.29}$$

Then $\|Ax\| \leq (1/2)\|x\| < \|x\|$ for any $x \in \partial B_{\rho} \cap K$. Lemma 1.2 yields

$$i(A, B_{\rho} \cap K, K) = 1. \tag{2.30}$$

Equations (2.26) and (2.30) imply that

$$i(A, (B_{\overline{R}} \backslash \overline{B}_{\rho}) \cap K, K) = i(A, B_{\overline{R}} \cap K, K) - i(A, B_{\rho} \cap K, K) = -1. \tag{2.31}$$

So A has at least one fixed point $x \in (B_{\overline{R}} \backslash \overline{B}_{\rho}) \cap K$ which satisfies $0 < \rho < \|x\| \leq \overline{R}$. We know that $x(t) > 0$, $t \in (0,1)$ by definition of K. This shows that SBVP (2.1) has at least one positive solution $(x,y) \in C^4(0,1) \cap C^2[0,1] \times C^2(0,1) \cap C[0,1]$ by (2.7), and the solution (x,y) satisfies $x(t) > 0$, $y(t) > 0$ for any $t \in (0,1)$. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let (H_1), (H_4), and (H_5) hold, then SBVP (2.1) has at least one positive solution.*

Proof. By the first limit of (H_4), there exist $\eta > 0$ and $\delta_1 \in (0,R)$ such that

$$f_1(t,u) \geq \eta u^{\gamma}, \quad \forall (t,u) \in [a,b] \times [0,\delta_1]. \tag{2.32}$$

Let $m \geq 2[\varepsilon_0^3(b-a)^2\eta\int_a^b G(1/2,s)ds]^{-1}$, then according to the second limit of (H_4), there exists $\delta_2 \in (0,R)$ such that

$$f_2^{\gamma}(t,u) \geq mu, \quad \forall (t,u) \in [a,b] \times [0,\delta_2]. \tag{2.33}$$

Taking $\delta = \min\{\delta_1,\delta_2\}$. Since $f_2(t,0) \equiv 0$, $f_2 \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$, there exists small enough $\sigma \in (0,\delta)$ such that $f_2(t,x) \leq \delta$ for any $(t,x) \in (0,1) \times [0,\sigma]$. Then we have

$$\int_0^1 G(r,\tau)f_2(\tau,x(\tau))d\tau \leq \delta, \quad \forall x \in \overline{B}_\sigma \cap K,\ r \in (0,1). \tag{2.34}$$

By using Jensen inequality and $0 < \gamma \leq 1$, from (2.32)–(2.34) we can get that

$$\begin{aligned}(Ax)\left(\frac{1}{2}\right) &\geq \eta\int_a^b G\left(\frac{1}{2},s\right)\int_a^b G(s,r)\left(\int_0^1 G(r,\tau)f_2(\tau,x(\tau))d\tau\right)^{\gamma} dr\,ds\\ &\geq \varepsilon_0\eta\int_a^b G\left(\frac{1}{2},s\right)ds\int_a^b\left(\int_a^b G(r,\tau)f_2^{\gamma}(\tau,x(\tau))d\tau\right)dr\\ &\geq \varepsilon_0\eta m\int_a^b G\left(\frac{1}{2},s\right)ds\int_a^b\int_a^b G(r,\tau)x(\tau)d\tau\,dr\\ &\geq \varepsilon_0^3(b-a)^2\eta m\|x\|\int_a^b G\left(\frac{1}{2},s\right)ds \geq 2\|x\|, \quad \forall x \in \overline{B}_\sigma \cap K.\end{aligned} \tag{2.35}$$

From this we know that

$$\|Ax\| \geq 2\|x\| > \|x\|, \quad \forall x \in \partial B_\sigma \cap K. \tag{2.36}$$

Equation (2.36) and Lemma 1.2 imply that

$$i(A, B_\sigma \cap K, K) = 0. \tag{2.37}$$

On the other hand, for any $x \in \partial B_R \cap K$, $t \in [0,1]$, (H_1) and (H_5) imply that

$$\int_0^1 G(r,\tau)f_2(\tau,x(\tau))d\tau \leq q_2[0,R]\int_0^1 \tau(1-\tau)p_2(\tau)d\tau = N, \tag{2.38}$$

$$\|Ax\| \leq q_1[0,N]\int_0^1 r(1-r)p_1(r)dr < R = \|x\|, \quad \forall x \in \partial B_R \cap K. \tag{2.39}$$

By (2.39) and Lemma 1.2, we obtain that

$$i(A, B_R \cap K, K) = 1. \tag{2.40}$$

Now, (2.37) and (2.40) imply that

$$i(A,(B_R\backslash\overline{B}_\sigma) \cap K, K) = i(A, B_R \cap K, K) - i(A, B_\sigma \cap K, K) = 1. \tag{2.41}$$

So A has at least one fixed point $x \in (B_R\backslash\overline{B}_\sigma) \cap K$, then SBVP (2.1) has at least one positive solution (x,y) which satisfies $x(t) > 0$, $y(t) > 0$ for any $t \in (0,1)$. This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let (H_1), (H_2), (H_4), and (H_5) hold, then SBVP (2.1) has at least two positive solutions.*

Proof. We take $M > R > \sigma$ such that (2.26), (2.37), and (2.40) hold by the proof of Theorems 2.3 and 2.4. Then

$$\begin{aligned} i(A,(B_{\overline{R}}\backslash\overline{B}_R)\cap K,K) &= i(A,B_{\overline{R}}\cap K,K) - i(A,B_R\cap K,K) = -1, \\ i(A,(B_R\backslash\overline{B}_\sigma)\cap K,K) &= i(A,B_R\cap K,K) - i(A,B_\sigma\cap K,K) = 1. \end{aligned} \tag{2.42}$$

So A has at least two fixed points in $(B_{\overline{R}}\backslash\overline{B}_R)\cap K$ and $(B_R\backslash\overline{B}_\sigma)\cap K$, then SBVP (2.1) has at least two positive solutions (x_i,y_i) and satisfies $x_i(t) > 0$, $y_i(t) > 0$ $(i = 1,2)$ for any $t \in (0,1)$. This completes the proof of Theorem 2.5. □

In the following, we give some applied examples.

Example 2.6. Let $f_1(t,y) = y^2/t(1-t)$, $f_2(t,x) = x^3/t(1-t)$, $\alpha = \beta = 1/2$. From Theorem 2.3, we know that SBVP (2.1) has at least one positive solution, here, $f_1(t,y)$ and $f_2(t,x)$ are superliner on y, x, respectively.

Example 2.7. Let $f_1(t,y) = y^{1/2}/t(1-t)$, $f_2(t,x) = x^3/t(1-t)$, $\alpha = \beta = 1/2$. From Theorem 2.3, we know that SBVP (2.1) has at least one positive solution, here, $f_1(t,y)$ and $f_2(t,x)$ are sublinear and superliner on y, x, respectively.

Example 2.8. Let $f_1(t,y) = (y^2+y^{1/2})/\sqrt{t(1-t)}$, $f_2(t,x) = 4(x^3+x^{1/2})/\pi\sqrt{t(1-t)}$, $\alpha = \gamma = 1/2$. It is easy to examine that conditions (H_1), (H_2), and (H_4) of Theorem 2.5 are satisfied and $\int_0^1 (dt/\sqrt{t(1-t)}) = \pi$. In addition, taking $R = 1$, then $q_2[0,1] = \sup\{(x^3+x^{1/2})/2 : x \in [0,1]\} = 1$, $N = (8/\pi)q_2[0,1]\int_0^1 \sqrt{t(1-t)}\,dt = 1$, $q_1[0,1] = 2$, where $\int_0^1 \sqrt{t(1-t)}dt = \pi/8$. Then $q_1[0,1]\int_0^1 \sqrt{t(1-t)}dt = \pi/4 < 1$. Thus, the condition (H_5) of Theorem 2.5 is satisfied. From Theorem 2.5, we know that SBVP (2.1) has at least two positive solutions.

Remark 2.9. Balancing condition of a pair of elastic beams for fixed two ends may be described by boundary value problems for nonlinear fourth-order singular system (SBVP)

$$\begin{gathered} x^{(4)} = f_1(t,-y''), \quad t \in (0,1), \\ x(0) = x(1) = x''(0) = x''(1) = 0, \\ y^{(4)} = f_2(t,x), \quad t \in (0,1), \\ y(0) = y(1) = y''(0) = y''(1) = 0, \end{gathered} \tag{2.43}$$

where $f_i \in C((0,1)\times\mathbb{R}^+,\mathbb{R}^+)$ $(i = 1,2)$, $f_i(t,0) \equiv 0$, $f_i(t,u)$ are singular at $t = 0$ and $t = 1$. Let $-y''(t) = v(t)$, $t \in [0,1]$. Then $v(0) = v(1) = 0$, $y(t) = \int_0^1 G(t,s)v(s)ds$, where $G(t,s)$ is given by (2.8). SBVP (2.43) is changed into the form of SBVP (2.1)

$$\begin{gathered} x^{(4)} = f_1(t,v), \quad t \in (0,1), \\ x(0) = x(1) = x''(0) = x''(1) = 0, \\ -v'' = f_2(t,x), \quad t \in (0,1), \\ v(0) = v(1) = 0. \end{gathered} \tag{2.44}$$

Thus, from Theorems 2.3–2.5, we can get the existence of the positive solutions and multiple positive solutions of SBVP (2.43) under the conditions (H_1)–(H_5).

Remark 2.10. Balancing condition of a pair of bending elastic beams for fixed two ends may be described by boundary value problems for nonlinear fourth-order singular system (SBVP)

$$\begin{gathered} x^{(4)} = f_1(t, -y''), \quad t \in (0,1), \\ x(0) = x(1) = x''(0) = x''(1) = 0, \\ y^{(4)} = f_2(t, -x''), \quad t \in (0,1), \\ y(0) = y(1) = y''(0) = y''(1) = 0, \end{gathered} \tag{2.45}$$

where $f_i \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$ $(i = 1,2)$, $f_i(t,0) \equiv 0$, $f_i(t,u)$ are singular at $t = 0$ and $t = 1$. Let $-x''(t) = u(t)$, $-y''(t) = v(t)$, $t \in [0,1]$, then $u(0) = u(1) = 0$, $v(0) = v(1) = 0$ and the problem is equivalent to the following nonlinear integral equation system:

$$\begin{gathered} x(t) = \int_0^1 G(t,s)u(s)ds, \\ y(t) = \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1], \end{gathered} \tag{2.46}$$

where $G(t,s)$ is given by (2.8). SBVP (2.45) is changed into the following boundary value problems for nonlinear second-order singular system:

$$\begin{gathered} -u'' = f_1(t,v), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ -v'' = f_2(t,u), \quad t \in (0,1), \\ v(0) = v(1) = 0. \end{gathered} \tag{2.47}$$

For SBVP (2.47), under conditions (H_1)–(H_5), by using the similar methods of our proof, we can show that SBVP (2.47) has the similar conclusions of Theorems 2.3–2.5.

3. Continuous case

We consider boundary value problems of system for nonlinear fourth-order ordinary differential equations (BVP)

$$\begin{gathered} x^{(4)} = f_1(t,y), \quad t \in [0,1], \\ x(0) = x(1) = x''(0) = x''(1) = 0, \\ -y'' = f_2(t,x), \quad t \in [0,1], \\ y(0) = y(1) = 0, \end{gathered} \tag{3.1}$$

where $f_i \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $f_i(t,0) \equiv 0$ $(i = 1,2)$. $(x, y) \in C^4[0,1] \times C^2[0,1]$ is a solution of BVP (3.1) if (x, y) satisfies (3.1). Moreover, we call that (x, y) is a positive solution of BVP (3.1) if $x(t) > 0$, $y(t) > 0$, $t \in (0,1)$.

To prove our results, we list the following assumptions.

(Q_1) There exists $\tau \in (0,+\infty)$ such that

$$\limsup_{u\to+\infty}\frac{f_1(t,u)}{u^{\tau}} < +\infty, \qquad \limsup_{u\to+\infty}\frac{f_2(t,u)}{u^{1/\tau}} = 0 \tag{3.2}$$

uniformly on $t \in [0,1]$.

(Q_2) There exists $\beta \in (0,+\infty)$ such that

$$\limsup_{u\to0^+}\frac{f_1(t,u)}{u^{\beta}} < +\infty, \qquad \limsup_{u\to0^+}\frac{f_2(t,u)}{u^{1/\beta}} = 0 \tag{3.3}$$

uniformly on $t \in [0,1]$.

(Q_3) There exist $q_i \in C(\mathbb{R}^+,\mathbb{R}^+)$, $p_i \in C([0,1],\mathbb{R}^+)$ such that $f_i(t,u) \le p_i(t)q_i(u)$ and there exists $R > 0$ such that $q_1[0,N]\int_0^1 t(1-t)p_1(t)dt < R$, where $N = q_2[0,R]\int_0^1 t(1-t)p_2(t)dt$, $q_i[0,d] = \sup\{q_i(u) : u \in [0,d]\}$ $(i = 1,2)$.

Obviously, for continuous case, the integral operator $A : K \to K$ defined by (2.12) is complete continuous.

Theorem 3.1. *Let (Q_1) and (H_4) hold. Then BVP (3.1) has at least one positive solution.*

Proof. We know that (2.37) holds by (H_4). On the other hand, it follows from (Q_1) that there are $\omega > 0$, $C_3 > 0$, $C_4 > 0$ such that

$$\begin{gathered} f_1(t,u) \le \omega u^{\tau} + C_3, \quad \forall (t,u) \in [0,1]\times\mathbb{R}^+, \\ f_2(t,u) \le \left(\frac{u}{2\omega}\right)^{1/\tau} + C_4, \quad \forall (t,u) \in [0,1]\times\mathbb{R}^+. \end{gathered} \tag{3.4}$$

Noting that $0 \le G(t,s) \le 1$, (3.4) implies that

$$\begin{aligned} (Ax)(t) &\le \int_0^1 G(t,s)\int_0^1 G(s,r)\left[\omega\left(\int_0^1 G(r,\xi)f_2(\xi,x(\xi))d\xi\right)^{\tau} + C_3\right]dr\,ds \\ &\le \omega\left(\int_0^1\left[\left(\frac{x(\xi)}{2\omega}\right)^{1/\tau} + C_4\right]d\xi\right)^{\tau} + C_3 \le \omega\left[\left(\frac{\|x\|}{2\omega}\right)^{1/\tau} + C_4\right]^{\tau} + C_3. \end{aligned} \tag{3.5}$$

By simple calculating, we get that

$$\lim_{\|x\|\to+\infty}\frac{\omega\left[\left(\|x\|/2\omega\right)^{1/\tau} + C_4\right]^{\tau} + C_3}{\|x\|} = \frac{1}{2}. \tag{3.6}$$

Then there exists a number $G > 0$ such that $\|x\| \ge G$ implies that

$$\omega\left[\left(\frac{\|x\|}{2\omega}\right)^{1/\tau} + C_4\right]^{\tau} + C_3 < \frac{3}{4}\|x\|. \tag{3.7}$$

Thus, we have

$$\|Ax\| < \|x\|, \quad \forall x \in \partial B_G \cap K. \tag{3.8}$$

It follows from (3.8) and Lemma 1.2 that

$$i(A, B_G \cap K, K) = 1. \tag{3.9}$$

Now, (2.37) and (3.9) imply that

$$i(A, (B_G \backslash \overline{B}_\sigma) \cap K, K) = i(A, B_G \cap K, K) - i(A, B_\sigma \cap K, K) = 1. \tag{3.10}$$

So A has at least one fixed point $x \in (B_G \backslash \overline{B}_\sigma) \cap K$, then BVP (3.1) has at least one positive solution (x, y) which satisfies $x(t) > 0$, $y(t) > 0$, $t \in (0,1)$. This completes the proof of Theorem 3.1. □

Similar to the proof of Theorem 2.4, we can get the following theorems.

THEOREM 3.2. *Let* (H_2) *and* (Q_2) *hold. Then BVP (3.1) has at least one positive solution.*

THEOREM 3.3. *Let* (H_4) *and* (Q_3) *hold. Then BVP (3.1) has at least one positive solution.*

Proof. Similar to the proof of (2.37) and (2.40) in Section 2, we can prove that there exists $\sigma \in (0, R)$ such that

$$i(A, B_\sigma \cap K, K) = 0, \qquad i(A, B_R \cap K, K) = 1. \tag{3.11}$$

These imply that

$$i(A, (B_R \backslash \overline{B}_\sigma) \cap K, K) = i(A, B_R \cap K, K) - i(A, B_\sigma \cap K, K) = 1. \tag{3.12}$$

So A has at least one fixed point $x \in (B_{\overline{R}} \backslash \overline{B}_\sigma) \cap K$ and satisfies $0 < \sigma < \|x\| \leq \overline{R}$. It follows from the definition of K that $x(t) > 0$, $t \in (0,1)$. This shows that BVP (3.1) has at least one positive solution (x, y) which satisfies $x(t) > 0$, $y(t) > 0$ for any $t \in (0,1)$. This completes the proof of Theorem 3.3. □

From Theorems 3.2 and 3.3, we can get the following theorem.

THEOREM 3.4. *Let* (H_2), (H_4), *and* (Q_3) *hold. Then BVP (3.1) has at least two positive solutions.*

In the following, we give some applications of Theorems 3.1–3.4.

Example 3.5. Let $f_1(t, y) = y^{1/2}$, $f_2(t, x) = x^{1/2}$, $\tau = \gamma = 1/2$. From Theorem 3.1, we know that BVP (3.1) has at least one positive solution, here, $f_1(t, y)$ and $f_2(t, x)$ are sublinear on y, x, respectively.

Example 3.6. Let $f_1(t, y) = y^2$, $f_2(t, x) = x^3$, $\alpha = \beta = 1/2$. From Theorem 3.2, we know that BVP (3.1) has at least one positive solution, here, $f_1(t, y)$ and $f_2(t, x)$ are superliner on y, x, respectively.

Example 3.7. Let $f_1(t, y) = y^{1/2}$, $f_2(t, x) = x^3$, $\alpha = \beta = 1/2$. From Theorem 3.2, we know that BVP (3.1) has at least one positive solution, here, $f_1(t, y)$ and $f_2(t, x)$ are sublinear and superliner on y, x, respectively.

Example 3.8. Let $f_1(t,y) = (1/3)(y^2 + y^{1/2})$, $f_2(t,x) = t(x^3 + x^{1/2})$, $\alpha = \gamma = 1/2$, $R = 1$. From Theorem 3.4, we know that BVP (3.1) has at least two positive solutions.

Remark 3.9. From these examples we know that all conclusions in this paper are different from the ones in [1–10].

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