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Research Article

A Boundary Harnack Principle for Infinity-Laplacian and Some Related Results

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We prove a boundary comparison principle for positive infinity-harmonic functions for smooth boundaries. As consequences, we obtain (a) a doubling property for certain positive infinity-harmonic functions in smooth bounded domains and the half-space, and (b) the optimality of blowup rates of Aronsson's examples of singular solutions in cones.

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1. Introduction

In this work, one of our main efforts is to prove a boundary Harnack principle for positive infinity-harmonic functions on domains with smooth boundaries. This will generalize the result in [1] proven for flat boundaries. In this connection, also see [2–5]. This result will also be applied to study some special positive infinity-harmonic functions defined on such domains. One could refer to these as infinity-harmonic measures, however, being solutions to a nonlinear equation, these are not true measures. We derive some properties of these functions and among these would be the doubling property. A decay rate and a halving property for such functions on the half-space will also be presented. Another application will be to show optimality of Aronsson's singular examples in cones, thus generalizing the result in [6, 7].

We now introduce notations for describing our results. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with boundary $\partial\Omega$. We say u is infinity-harmonic in Ω if u solves in the sense of viscosity

$$\Delta_\infty u = \sum_{i,j=1}^{n} D_i u(x) D_j u(x) D_{ij} u(x) = 0, \quad x \in \Omega. \tag{1.1}$$

For more discussion, see [8, 1, 9]. For a motivation for these problems, see [8, 10]. For

$r > 0$ and $x \in \mathbb{R}^n$, $B_r(x)$ will be the open ball centered at x and has radius r. Let $\widehat{A}$ denote the closure of the set A and let χ_A denote its characteristic function. Define $\Omega_r(x) = \Omega \cap B_r(x)$, $P_r(x) = \partial\Omega \cap B_r(x)$. We will assume throughout this work that $\partial\Omega \in C^2$. More precisely, we first define for every $x \in \partial\Omega$ R_x to be the radius of the largest interior ball tangential to Ω at x. We will assume that $R_y > 0$ for every $y \in \partial\Omega$ and $R_x \geq R_y/2 > 0$, $x \in P_{\delta_y}(y)$, for some $\delta_y > 0$. For every $x \in \partial\Omega$, set ν_x to be the inner unit normal at x and $x_s = x + s\nu_x$, $s > 0$. We will now state Theorem 1.1 which is the result about boundary Harnack principle [2, 1, 3, 4].

Theorem 1.1 (Boundary Harnack Principle). *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial\Omega$ satisfying the interior ball condition as stated above. Let u and v be infinity-harmonic in Ω. Suppose that $y \in \partial\Omega$, $0 < 4\delta \leq \inf(\delta_y, R_y/2)$, and $u, v > 0$ in $\Omega_{\delta_y}(y)$. Suppose that u, and v vanish continuously on $P_{\delta_y}(y)$, then there exist positive constants C, C_1, C_2 independent of u, v, and δ, such that for every $z \in \Omega_\delta(y)$,*

(i) $u(z) \leq cu(y_\delta)$,

(ii) $c_1 u(y_\delta)/v(y_\delta) \leq u(z)/v(z) \leq c_2 u(y_\delta)/v(y_\delta)$.

Inequality (i) is often referred to as the Carleson inequality. A proof is provided in Section 2. At this time, we are unable to determine if Theorem 1.1 also holds when Ω has Lipschitz continuous boundary. We will apply Theorem 1.1 to prove (a) the doubling property of solutions of (1.2), and (b) the optimality of blowup rates of the Aronsson singular functions in cones [6]. Let Ω be a bounded domain. Fix $y \in \partial\Omega$; for every $r > 0$, define $Q_r(y) = \partial\Omega \setminus \widehat{P}_r(y)$. Consider the problem

$$\Delta_\infty u(x) = 0, \quad x \in \Omega, \quad u(x) = 1, \quad x \in P_r(y), \quad u(x) = 0, \quad x \in Q_r(y). \tag{1.2}$$

By a solution u of (1.2), we mean that (i) u is infinity-harmonic, in the viscosity sense, in Ω, and (ii) u assumes the values 1 and 0 continuously on $P_r(y)$ and $Q_r(y)$. More precisely, if $x \in P_r(y)$ and $z \to x$, $z \in \Omega$, then $u(z) \to 1$, and analogously for $Q_r(y)$. We show the existence of bounded solutions of (1.2) in Lemma 3.1. One could refer to u as the nonlinear infinity-harmonic measure in Ω (although we have not shown uniqueness). Clearly, it is not a true measure. Our motivation for studying such quantities arises from the works [2–5]. In the context of boundary behavior, for instance the Fatou theorem, the works [4, 5] have studied such solutions for the linearized version of the p-Laplacian for finite p. We will show that requiring boundedness implies the maximum principle and comparison, see Lemma 3.1. Let $H = \{x \in \mathbb{R}^n : x_n > 0\}$ denote the half-space in $\mathbb{R}^n$. Set e_n to be the unit vector along the positive x_n-axis. Set $T = \{x \in \mathbb{R}^n : x_n = 0\}$; for $y \in T$, define $P_r(y) = T \cap B_r(y)$, $Q_r(y) = T \setminus \widehat{P}_r(y)$, and $M_y^u(\rho) = \sup_{\partial B_\rho(y) \cap H} u$. Define a solution u of

$$\Delta_\infty u(z) = 0, \quad z \in H, \qquad u|_{P_r(y)} = 1, \qquad u|_{Q_r(y)} = 0, \tag{1.3}$$

to be infinity-harmonic in Ω, in the sense of viscosity, $0 \leq u \leq 1$, continuous up to $P_r(y)$ and $Q_r(y)$, and $\limsup_{\rho\to\infty} M_y^u(\rho) = 0$. We will address the existence and uniqueness of such solutions in Lemma 3.4. We now state a result about the doubling property of solutions of (1.2) and (1.3). For $r > 0$, set $o_{3r} = 3re_n$.

THEOREM 1.2. (a) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For $y \in \partial\Omega$, assume that $P_r(y)$ lies on a connected component of $\partial\Omega$. Let u^r be a bounded solution of (1.2) in Ω and let r be small. Then there are positive constants c, C independent of r, such that $u^r(y_r) \geq c$ and*

$$u^{2r}(z) \leq Cu^r(z), \quad z \in \Omega \setminus B_{3r}(y). \tag{1.4}$$

(b) *Let H be the half-space in $\mathbb{R}^n$. Let u_o^r be the unique infinity-harmonic measure in H. Then there exist universal constants $C_1 > 0$ and $0 < C_2 < 1$ such that*

$$u_o^{2r}(z) \leq C_1 u_o^r(z), \quad z \in H \setminus B_{3r}(o), \qquad u_o^r(o_s) \leq C_2 u_o^{2r}(o_s), \quad s \geq 3r. \tag{1.5}$$

Estimates in Theorem 1.2 are well known for linear equations [3] and also for the linearized version for the p-Laplacian [4, 5]. While we are able to prove the doubling property for any C^2 domain (see Lemma 3.3), it is unclear how a halving property (i.e., $f(t) \leq cf(2t)$, f positive, increasing, and $c < 1$) may be proven if true. In particular, it would be interesting to know if this is true when Ω is the unit ball. We now introduce notations for Theorem 1.3. For $\alpha > 0$, let C_α stand for the interior of the half-infinite cone in H, with apex at o, the x_n-axis as the axis of symmetry, and aperture 2α. For $r > 0$, let $M^u(r) = \sup_{z \in \partial B_r(o) \cap C_\alpha} u(z)$. We extend the result in [7] to show optimality of the Aronsson singular examples [6].

THEOREM 1.3. *For $\alpha > 0$, let C_α be as described above. Let u, v be positive infinity-harmonic functions in C_α. Assume that* (i) *both u and v vanish continuously on $\partial C_\alpha \setminus \{o\}$,* (ii) $\sup_{r>0} M^u(r) = \infty$, $\sup_{r>0} M^v(r) = \infty$, *and* (iii) $\lim_{r\to\infty} M^u(r) = \lim_{r\to\infty} M^v(r) = 0$. *Then there exists a constant C, depending on α, u, and v such that*

$$\frac{1}{C} \leq \frac{u(z)}{v(z)} \leq C, \quad z \in C_\alpha. \tag{1.6}$$

Moreover, for every $m = 1, 2, 3, \ldots$, if $\alpha = \pi/2m$ and ω is a direction in $C_{\pi/2m}$, then for an appropriate $\widehat{C} = \widehat{C}(\omega)$,

$$\frac{1}{\widehat{C}|z|^{m^2/(2m+1)}} \leq u(z) \leq \frac{\widehat{C}}{|z|^{m^2/(2m+1)}}, \quad z \in C_{\pi/2m} \text{ with } z = |z|\omega. \tag{1.7}$$

The last conclusion in Theorem 1.3 will follow from the works [6, 7]. While Theorem 1.3 applies to special situations, the main purpose is to understand better the blowup rates of singular solutions, and in some situations decay rates.

We now state some well-known results that will be used in this work. Let $u > 0$ be infinity-harmonic in a domain Ω, suppose that $a, b \in \Omega$ such that the segment ab is at least $\eta > 0$ away from $\partial\Omega$, then the following Harnack inequality holds:

$$u(a)e^{|a-b|/\eta} \geq u(b). \tag{1.8}$$

Let $B_r(a) \subset \Omega$, if ω is a unit vector and $0 \leq t \leq s < r$, then

$$\frac{u(a+t\omega)}{r-t} \leq \frac{u(a+s\omega)}{r-s}, \qquad u(a+s\omega)(r-s) \leq u(a+t\omega)(r-t). \tag{1.9}$$

We will refer to (1.9) as the monotonicity property of u. For (1.8) and (1.9), see [8, 1, 11, 7, 12, 13]. Moreover, u is locally Lipschitz (C^1 if $n = 2$ [14]) and satisfies the comparison principle [15].

Finally, we mention that it is unclear if a boundary Holder continuity of the quotient of two infinity-harmonic functions holds for smooth domains. Such a result for general Lipschitz domains would undoubtedly be quite useful. For p-harmonic functions (finite p), we direct the reader to the recent work by John Lewis and Kaj Nystrom "Boundary Behaviour for p Harmonic Functions in Lipschitz and Starlike Lipschitz Ring Domains." We thank John Lewis for sending us this work.

2. Proof of Theorem 1.1

Our proof is an adaptation of the methods developed in [2, 1, 3]. Since Δ_∞ is translation and rotation invariant, we may assume that the origin $o \in \partial\Omega$. Set $\operatorname{osc}_A u = \sup_{z\in A} u(z) - \inf_{z\in A} u(z)$ to be the oscillation function of u on the set A. Recall that $\Omega_r(y) = \Omega \cap B_r(y), y \in \partial\Omega$.

Step 1 (oscillation estimate near the boundary). Let $u > 0$ be infinity-harmonic in Ω and vanishing on a neighborhood of o, in $\partial\Omega$. Let $M^u(r) = \sup_{z\in\Omega_r(o)} u(z)$. By the maximum principle, $M^u(r) > 0$ and $u(z) \leq M^u(r)$, $z \in \Omega_r(o)$. For $0 < \alpha \leq \beta$, consider the function $w(z) = M^u(\alpha) + [M^u(\beta) - M^u(\alpha)](|z| - \alpha)/(\beta - \alpha)$, $z \in \Omega_\beta \setminus \Omega_\alpha$. Clearly, $u \leq w$ on $\partial(\Omega_\beta \setminus \Omega_\alpha)$. Thus $u \leq w$ in $\Omega_\beta \setminus \Omega_\alpha$. Thus

$$M^u(\gamma) \leq M^u(\alpha) + \left[M^u(\beta) - M^u(\alpha)\right]\frac{\gamma-\alpha}{\beta-\alpha}, \quad \alpha \leq \gamma \leq \beta. \tag{2.1}$$

This implies that $\operatorname{osc}_{\Omega_r(o)} u = M^u(r)$ is convex in r. Since $u(o) = 0$, it follows that $2\operatorname{osc}_{\Omega_r(o)} u \leq \operatorname{osc}_{\Omega_{2r}(o)} u$.

Step 2 (Carleson inequality). We now use the interior ball condition. Since $\partial\Omega \in C^2$, $R_x \geq R_o/2$, $x \in P_{4\delta}(o)$, with $4\delta < \inf(\delta_y, R_o/2)$. For every $x \in \partial\Omega$, let ν_x denote the unit inner normal at x, and set $x_t = x + t\nu_x$, $0 \leq t \leq R_x$. We will prove that $u(z) \leq Cu(o_\delta)$, $z \in \Omega_\delta(o)$. We will adapt a device, based on the Harnack inequality, from [3]. For $z \in \Omega_\delta$, define $x_z \in \partial\Omega$ to be the point nearest to z. Also set $d(z) = |x_z - z|$. Then $z = x_z + d(z)\nu_{x_z} = (x_z)_{d(z)}$; set $z^s = x_z + 2^{s-1}d(z)\nu_{x_z}$, $s = 1, 2, 3, \ldots$. By the Harnack inequality (1.8), for $z \in \Omega_{3\delta}(o)$,

$$u(z) \leq \begin{cases} Mu(z^2) : 0 < d(z) < \dfrac{3\delta}{2}, \\[2ex] Mu(o_\delta) : \delta < d(z) < 3\delta. \end{cases} \tag{2.2}$$

We take $M = e^8$. We now make an observation which will be used repeatedly in what follows. If $d(z) \geq \delta/2^s$, then

$$u(z) \leq Mu(z^2) \leq \cdots \leq M^s u(z^s) \leq M^{s+1} u(o_\delta). \tag{2.3}$$

Suppose now that there is a $\xi_0 \in \Omega_\delta(o)$ such that $u(\xi_0) \geq M^{l+2} u(o_\delta)$, where $l = l(\delta)$ is large and will be determined later. Using the aforementioned observation, we obtain

$$\operatorname{dist}(\xi_0, \partial\Omega) \leq \frac{\delta}{2^l}. \tag{2.4}$$

Let $p_0 \in \partial\Omega$ be the nearest point to ξ_0. Clearly, $\xi_0 \in \Omega_{2^{-l}\delta}(p_0) \subset \Omega_{2\delta}(o)$. Thus, $\operatorname{osc}_{\Omega_{\delta 2^{-l}}(p_0)} u \geq u(\xi_0)$; thus by Step 1, for $m = 1, 2, 3 \ldots$,

$$\operatorname{osc}_{\Omega_{\delta 2^{-l+m}}(p_0)} u \geq 2^m \operatorname{osc}_{\Omega_{\delta 2^{-l}}(p_0)} u \geq 2^m u(\xi_0), \tag{2.5}$$

where $2^m \geq M^3 = e^{24}$. Select $m = 60$; thus $\operatorname{osc}_{\Omega_{\delta 2^{-l+m}}(p_0)} u \geq 2^m u(\xi_0) \geq M^{l+5} u(o_\delta)$. Thus there is a $\xi_1 \in \Omega_{\delta 2^{-l+m}}(p_0)$ such that $u(\xi_1) \geq M^{l+5} u(o_\delta)$. Arguing as done in (2.4), we see $\operatorname{dist}(\xi_1, \partial\Omega) \leq \delta 2^{-l-3}$. Letting $p_1 \in \partial\Omega$ to be closest to ξ_1, we see that $p_1 \in \Omega_{2\delta}(o)$. Repeating our previous argument,

$$\operatorname{osc}_{\Omega_{\delta 2^{-l-3+m}}(p_1)} u \geq 2^m \operatorname{osc}_{\Omega_{\delta 2^{-l-3}}(p_1)} u \geq 2^m u(\xi_1) \geq M^{l+8} u(o_\delta). \tag{2.6}$$

Thus we may find a $\xi_2 \in \Omega_{\delta 2^{-l-3+m}}(p_1)$ such that $u(\xi_2) \geq M^{l+8} u(o_\delta)$, and $\operatorname{dist}(\xi_2, \partial\Omega) \leq \delta 2^{-l-6}$. Thus we obtain a sequence of points $\xi_k \in \Omega$ and $p_k \in \partial\Omega$, $k = 1, 2, 3 \ldots$, such that

$$u(\xi_k) \geq M^{l+2+3k} u(o_\delta), \quad \operatorname{dist}(\xi_k, \partial\Omega) \leq \delta 2^{-l-3k}, \quad \xi_k \in \Omega_{\delta 2^{-l-3(k-1)+m}}(p_{k-1}). \tag{2.7}$$

Note that

$$|\xi_k - o| \leq \sum_{i=1}^{k-1} |\xi_{i+1} - \xi_i| + |\xi_0 - o| \leq \delta\left(1 + 2\sum_{i=0}^{k-1} 2^{-l-3i+m}\right). \tag{2.8}$$

Choose $l \geq 70$, then $|\xi_k - o| \leq 2\delta$. Noting that u vanishes continuously on $\partial\Omega$ and letting $k \to \infty$ in (2.7) result in a contradiction. Thus the Carleson inequality in Theorem 1.1 follows.

Step 3 (bounds near the boundary). We first derive a lower bound in terms of the distance to the boundary. For every $z \in \Omega_\delta(o)$, let x_z and $d(z)$ be as in Step 2. Note that $d(z) \leq |z - o| \leq \delta$. Thus $x_z \in \Omega_{2\delta}(o)$. Call $\zeta_z = x_z + \delta\nu_{x_z}$, observe that $\zeta_z \in \Omega_{3\delta}(o)$. By monotonicity (1.9) and the interior ball condition, we have

$$\frac{u(z)}{d(z)} \geq \frac{u(\zeta_z)}{\delta} \geq e^{-6} \frac{u(o_\delta)}{\delta}, \tag{2.9}$$

since $|\zeta_z - o_\delta| \leq |x_z + \delta\nu_{x_z} - \delta\nu_o| \leq 4\delta$.

Let $z \in \Omega_\delta(o)$. As noted previously, $x_z \in \Omega_{2\delta}(o)$ and $\Omega_\delta(x_z) \subset \Omega_{3\delta}(o)$. Note that $z \in \Omega_\delta(x_z)$. Set $\mu_z = \sup_{\Omega_\delta(x_z)} u$, then by comparison $u(\xi) \leq \mu_z |\xi - x_z|/\delta$, $\xi \in \Omega_\delta(x_z)$. Thus

$u(z) \le \mu_z d(z)/\delta$. By the Carleson inequality, $\mu_z \le Cu(\zeta_z)$. Note that $|x_\delta - o_\delta| = |x_z + \delta\nu_x - \delta\nu_o| \le 4\delta$. By the Harnack inequality, $u(\zeta_z) \le e^4 u(o_\delta)$. Thus there is universal $\widehat{C}$, such that

$$\frac{u(z)}{d(z)} \le \widehat{C}\frac{u(o_\delta)}{\delta}, \quad z \in \Omega_\delta(o). \tag{2.10}$$

If u, v are two positive infinity-harmonic functions in $\Omega_{4\delta}(o)$, then by (2.9) and (2.10), there exist universal constants C_1 and C_2 such that

$$C_1\frac{u(o_\delta)}{v(o_\delta)} \le \frac{u(z)}{v(z)} \le C_2\frac{u(o_\delta)}{v(o_\delta)}, \quad z \in \Omega_\delta(o). \tag{2.11}$$

This proves Theorem 1.1.

Remark 2.1. We comment that the distance function $d(z) = \operatorname{dist}(z, \partial\Omega)$, $z \in \Omega$, is C^2 and infinity-harmonic near $\partial\Omega$. Also the oscillation estimate in Step 1 continues to hold for Lipschitz boundaries. One could show a Carleson inequality by following the ideas in [2].

3. Proof of Theorem 1.2

In this section, we will assume that Ω is a bounded C^2 domain. For $y \in \partial\Omega$ and $r > 0$, recall the definitions of $P_r(y)$ and $Q_r(y)$. Note that both $P_r(y)$ and $Q_r(y)$ are relatively open in $\partial\Omega$. Let u be a solution of (1.2). As in Section 2, for $x \in \partial\Omega$, ν_x and $x_t = x + t\nu_x$, $t > 0$, are as defined in Section 2. We will assume that Ω is bounded but we can extend our arguments to the case of the half-space H. We will always take u to be bounded in this section. This will imply the maximum principle. At this time, it is not clear whether unbounded solutions to (1.2) exist. Let C_y be the connected component of $\partial\Omega$ that contains y. In Lemma 3.1, we assume that $B_r(y) \cap \partial\Omega = B_r(y) \cap C_y$.

Lemma 3.1. *Let $\Omega \in C^2$ be a bounded domain. Let $y \in \partial\Omega$ and $r > 0$. The following holds.*

(i) *There exists a solution u of the problem in (1.2) such that* $0 < u < 1$ *in* Ω.

(ii) *If v is any bounded solution of (1.2), then* $0 < v < 1$ *in* Ω.

(iii) *There are a maximal solution u_y^r and a minimal solution $\hat{u}_y^r$, in Ω such that if v is any bounded solution of (1.2), then* $\hat{u}_y^r \le v \le u_y^r$.

(iv) *If $t < r$, then* $u_y^t \le \lim_{\rho\uparrow r} u_y^\rho = \hat{u}_y^r \le u_y^r = \lim_{\hat{r}\downarrow r} u_y^{\hat{r}}$.

Moreover, u_y^r satisfies the following comparison principle: if ω, $w \in C(\widehat{\Omega})$ are infinity-harmonic, and $\omega \le u_y^r \le w$ on $\partial\Omega$, then $\omega \le u_y^r \le w$ in Ω.

Proof. Fix $y \in \partial\Omega$ and $r > 0$. We have broken up our proof into five steps. We first start with the existence of bounded solutions.

Step 1 (existence). We use the existence results proven in [8, 15], for Lipschitz boundary data. Let $\eta > 0$ be small. Set $I_r(y) = \partial B_r(y) \cap \partial\Omega$, and for $t > 0$, set $S_t = P_r(y) \cup (\bigcup_{x\in I_r(y)} B_t(x) \cap \partial\Omega)$. The set S_t is obtained by appending a t-band to $P_r(y)$. For $l = 1, 2, 3, \ldots$, let f_l be such that

(i) $f_l \in C(\partial\Omega)$,

(ii) $f_l(x) = 1$, $x \in P_r(y)$,

(iii) $f_l(x) = 0, x \in \partial\Omega \setminus S_{\eta/l}$,
(iv) $f_l(x) = (\eta/l) - \text{dist}(x, P_r(y))/(\eta/l), x \in S_{\eta/l}$.

Now let $u_l \in C(\widehat{\Omega})$ be the unique viscosity solution of the problem

$$\Delta_\infty u_l(z) = 0, \quad z \in \Omega, \quad u_l|_{\partial\Omega} = f_l. \tag{3.1}$$

Clearly, $0 < u_l < 1$ in Ω. Since $f_l \geq f_{l+1}$, by comparison, there is a function u_η such that $u_l \downarrow u_\eta$ in Ω. We first show that if $x \in P_r(y)$ and $z \to x \in P_r(y)$, $z \in \Omega$, then $u_\eta(z) \to 1$. Consider the set $\Omega_\delta(x)$, where $\delta = \inf_{\xi \in Q_r(y)} |x - \xi|/2$. For $z \in \Omega_\delta(x)$, set $w(z) = 1 - |z - x|/\delta$. By comparison, for every l, $w \leq u_l \leq 1$ in $\Omega_\delta(x)$. Thus $1 - |z - x|/\delta \leq u_\eta(z) \leq 1$, $z \in \Omega_\delta(x)$. We see that $\lim_{z \to x} u_\eta(z) = 1$. For $x \in Q_r(y)$ and $\delta = \inf_{\xi \in P_r(y)} |\xi - x|/2$, it is clear that for l large, $u_l(z) \leq |z - x|/\delta$, $z \in B_\delta(x)$. Thus $u_\eta(z) \to 0$ as $z \to x$. Moreover, the limit function u_η does not depend on the width η of the appended band S_η. An argument based on comparison shows easily that for any η_1, $\eta_2 > 0$, $u_{\eta_1} = u_{\eta_2}$. Set $u = u_\eta$. Our next step is to show that u is a viscosity solution in Ω. We first show that u is locally Lipschitz in Ω. To see this, take $x_1 \in \Omega$ and $t > 0$ such that $B_{4t}(x_1) \subset \Omega$. Select $x_2 \in B_t(x_1)$; set $\mu_l = sup_{B_{4t}(x_1)} u_l$. Applying monontonicity (1.9) in $B_t(x_1)$, we have for every l, $(\mu_l - u_l(x_1))/t \leq (\mu_l - u_l(x_2))/(t - |x_1 - x_2|)$. Rearranging terms (see [1, Lemma 3.6], also see [12]), noting that $u_l(x_1)$, $u_l(x_2) \geq 0$ and $\mu_l \leq \mu_1 \leq 1$, we obtain $|u_l(x_2) - u_l(x_1)|/|x_1 - x_2| \leq 1/t$. Fixing x_1, x_2 and letting $l \to \infty$, we obtain that u is locally Lipschitz. Fix $\xi \in \Omega$ and for $0 \leq t < \text{dist}(\xi, p\Omega)$, set $M_l(t) = \sup_{B_t(\xi)} u_l$, $m_l = \inf_{B_t(\xi)} u_l$, $M(t) = \sup_{B_t(\xi)} u$, and $m(t) = \inf_{B_t(\xi)} u$. Using that (i) $u_k \leq u_j \leq u_l$ when $l < j < k$, (ii) M_l is convex and m_l is concave in t, it follows that for $a < c < b$ and $z \in \partial B_c(\xi)$,

$$\frac{b-c}{b-a} m_k(a) + \frac{c-a}{b-a} m_k(b) \leq m_k(c) \leq u_j(z) \leq M_l(c) \leq \frac{b-c}{b-a} M_l(a) + \frac{c-a}{b-a} M_l(b). \tag{3.2}$$

Now in (3.2) first letting $k \to \infty$, replacing $u_j(z)$ by $u(z)$, and then letting $l \to \infty$, we obtain that $M(t)$ is convex and $m(t)$ is concave. This implies that u is a viscosity solution [8]. Part (i) now follows. A proof also could be worked by showing cone comparison.

Throughout the rest of the proof, u will stand for the solution constructed in Step 1.

Step 2 (comparison). We now prove an easy comparison result for u. Let $f \in C(\partial\Omega)$ and let $u_f \in C(\widehat{\Omega})$ be the unique infinity-harmonic function with boundary values f. Let $f \leq \chi_{P_r(y)}$. Using comparison, we see that for every l, $u_f \leq u_l$ in Ω. Thus $u_f \leq u$ in Ω. Now let $f \geq \chi_{P_r(y)}$, set $\varepsilon > 0$. Since $f \geq 1$ in $P_r(y)$, there exists a $\delta > 0$ such that $f + \varepsilon \geq 1$ in $B_\delta(x) \cap \partial\Omega$, for every $x \in \partial\Omega \cap \partial B_r(y)$. Take l large so that $\eta/l \leq \delta/2$. By comparison, $u_l \leq u_f + \varepsilon$ in Ω. Thus we have $u \leq u_f$ in Ω.

Step 3 (maximum principle). We now prove part (ii). Let v be any bounded solution of (1.2). We will adapt an argument used in [11]. We observe that there is an $R_0 > 0$ such that for $x \in P_{2r}(y)$, $R_x \geq R_0$, and consequently, $\bigcup_{x \in P_{2r}(y)} B_{R_0/4}(x_{R_0/4}) \subset \Omega$. In what follows we take the quantities σ, $\eta < R_0/10$. We exploit the special geometry of $P_r(y)$ to achieve our proof.

Set $I_r(y) = \partial\Omega \cap \partial B_r(y)$; for every $x \in I_r(y)$ and $\sigma > 0$, define $m_x(\sigma) = \inf_{\partial B_\sigma(x) \cap \Omega} v$ and $M_x(\sigma) = \sup_{\partial B_\sigma(x) \cap \Omega} v$. Clearly, $m_x(\sigma) \leq 0$ and $M_x(\sigma) \geq 1$. We claim that M_x is convex and m_x is concave in σ. To see this, take $z \in \Omega$ with $0 < a \le |z-x| \le b$. Set $w(z) = m_x(a) + [m_x(b) - m_x(a)](|z-x| - a)/(b-a)$. Clearly, $w \le 0$. By comparison, $w \le v$ in $(B_b(x) \setminus B_a(x)) \cap \Omega$. Thus $m_x(\sigma)$ is concave in σ, and one can show analogously that $M_x(\sigma)$ is convex. Define $m^y(\sigma) = \inf_{x \in I_r(y)} m_x(\sigma)$ and $M^y(\sigma) = \sup_{x \in I_r(y)} M_x(\sigma)$, then for $\sigma > 0$,

$$\begin{aligned} &\text{(i) } M^y(\sigma) \ge 1 \text{ is convex}, \qquad m^y(\sigma) \le 0 \text{ is concave in } \sigma, \\ &\text{(ii) } m^y(\sigma) \le v(z) \le M^y(\sigma), \quad z \in \Omega \setminus \cup_{x \in I_r(y)} B_\sigma(x), \\ &\text{(iii) } M^y(\sigma) \uparrow, \quad m^y(\sigma) \downarrow \quad \text{as } \sigma \downarrow 0. \end{aligned} \tag{3.3}$$

Note that $v = 0$ or 1 on $\partial\Omega \setminus \bigcup_{x \in I_r(y)} B_\sigma(x)$. Thus (3.3)(i) follows easily. Now using (3.3)(i) and comparison in the set $\Omega \setminus \bigcup_{x \in I_r(y)} B_\sigma(x)$ yields (3.3)(ii). Clearly, $M^y(\sigma)(m^y(\sigma))$ is the supremum (infimum) of v on $\Omega \setminus \bigcup_{x \in I_r(y)} B_\sigma(x)$. The conclusion in (3.3)(iii) follows by observing that $\bigcup_{x \in I_r(y)} B_{\sigma_1}(x) \subset \bigcup_{x \in I_r(y)} B_{\sigma_2}(x)$, when $\sigma_1 > \sigma_2$. By (3.3), the quantities $M(0) = \lim_{\sigma \to 0} M^y(\sigma)$ and $m(0) = \lim_{\sigma \to 0} m^y(\sigma)$ exist. By our assumptions, $-\infty < m(0) \le v \le M(0) < \infty$. We show that $m(0) = 0$. Assume instead that $m(0) < 0$. Recall that v is continuous up to $Q_r(y)$ and $P_r(y)$. For $x \in \partial\Omega$, let $\rho(x) = \operatorname{dist}(x, P_r(y))$ and $\hat{\rho}(x) = \operatorname{dist}(x, Q_r(y))$. For $x \in Q_r(y)$, define $w_x(z) = m(0)|z-x|/\rho(x)$ in the set $\Omega_{\rho(x)}(x)$. By comparison $w_x \le v$ in $\Omega_{\rho(x)}(x)$, and $v \ge m(0)/2$, in $\Omega_{\rho(x)/2}(x)$. For $x \in P_r(y)$, define $\omega_x(z) = 1 + (m(0)-1)|z-x|/\hat{\rho}(x)$ in $\Omega_{\hat{\rho}(x)}(x)$. Then $v \ge \omega_x$ in $\Omega_{\hat{\rho}(x)}(x)$ and $v \ge m(0)/2$, in $\Omega_{\hat{\rho}(x)/2}(x)$. Let $\eta > 0$ be small. Set $A_\eta = \{x \in \partial\Omega : \rho(x) \ge \eta\}$ and $B_\eta = \{x \in P_r(y) : \hat{\rho}(x) \ge \eta\}$. We now apply the above observations to obtain

$$v(z) \ge \begin{cases} \dfrac{m(0)}{2} : z \in \Omega_{\rho(x)/2}(x), \quad x \in A_\eta, \\[2ex] \dfrac{m(0)}{2} : z \in \Omega_{\hat{\rho}(x)/2}(x), \quad x \in B_\eta. \end{cases} \tag{3.4}$$

Set $S = \bigcup_{\eta > 0} \bigcup_{x \in A_\eta} \Omega_{\rho(x)/2}(x)$ and $T = \bigcup_{\eta > 0} \bigcup_{x \in B_\eta} \Omega_{\hat{\rho}(x)/2}(x)$, and call $G_y = \Omega \setminus (S \cup T)$. For $l = 1, 2, 3 \ldots$, let $z_l \in \Omega$ be such that $v(z_l) \le 7m(0)/8$ and $v(z_l) \to m(0)$, as $l \to \infty$. By (3.4), $z_l \in \Omega \setminus G_y$, and by the maximum principle, $\operatorname{dist}(z_l, I_r(y)) \to 0$.

In the discussion that follows, we will assume that $n > 2$. Recalling that $I_r(y) = \partial B_r(y) \cap \partial\Omega$, it follows that $I_r(y)$ is smooth. For every l, let $x_l \in I_r(y)$ be the closest point to z_l and $d_l = |x_l - z_l|$. Note that the segment $x_l z_l$ is normal to $I_r(y)$. Since $x_l \in \partial B_r(y)$, $yx_l \perp \partial B_r(y)$, and so $yx_l \perp I_r(y)$. Let T_l be the hyperplane tangential to $\partial\Omega$ at x_l, and let Π_l be the 2-dimensional plane containing the segments yx_l and yz_l. Thus $\Pi_l \perp I_r(y)$ at x_l and ν_{x_l} lies in Π_l. Note that $\Pi_l \perp T_l$ and $I_r(y)$ is tangential to T_l at x_l. Call $J_l = \partial\Omega \cap \Pi_l$, observe that the curve $J_l \perp I_r(y)$ at x_l. It is easy to see that if $x \in J_l$ is close to x_l, then (i) $\rho(x) = |x - x_l|$ if $x \in P_r(y)$, and (ii) $\hat{\rho}(x) = |x - x_l|$ if $x \in Q_r(y)$. Now consider the set $C_l = \Pi_l \cap \partial B_{d_l}(x_l) \setminus G_y$. As noted above $z_l \in C_l$, moreover one can find $\alpha_l \in C_l$ such that $v(\alpha_l) = 3m(0)/4$. We will apply the Harnack inequality in C_l to obtain a contradiction. In (3.4), take $\eta = d_l$ and we observe the following. Since $\partial\Omega \in C^2$ and x_l's lie in a compact set, it follows that for $q \in C_l$, $\operatorname{dist}(q, \partial\Omega) \approx \operatorname{dist}(q, T_l) = O(d_l)$, as $d_l \to 0$. In other words, $\operatorname{dist}(q, \partial\Omega)$ has a lower bound of the order of d_l. We show this as follows. First note that since $\partial\Omega \in C^2$, it permits a local parametrization near x_l, where $x_n = \nu_{x_l}$, $x_n = 0$ is T_l, and $x_n = \phi(x_1, \ldots, x_{n-1})$ describes $\partial\Omega$. Clearly, $\operatorname{dist}(q, \partial\Omega) \le |q - x_l| = d_l$. We will show that (a) $\operatorname{dist}(q,\partial\Omega) \geq \operatorname{dist}(q,T_l) + O(d_l^2)$ and (b) $\operatorname{dist}(q,T_l) \approx O(d_l)$, uniformly in l. (a) Let (i) $q_{\partial\Omega}$ be the point on $\partial\Omega$ closest to q, (ii) let q_{T_l} be the point on T_l closest to q, (iii) q_{int} the point of intersection of the line, containing the segment $qq_{\partial\Omega}$, and T_l, and (iv) let $q_{\partial\Omega}^{T_l}$ be the point on T_L closest to $q_{\partial\Omega}$. Clearly, $|q - q_{T_l}| \leq |q - x_l| = d_l$ and $q_{\partial\Omega} \in B_{2d_l}(x_l)$. Since $\partial\Omega \in C^2$, $|q_{\partial\Omega} - q_{\partial\Omega}^{T_l}| = O(d_l^2)$. If $|q - q_{\partial\Omega}| \geq |q - q_{\text{int}}|$, then $|q - q_{\partial\Omega}| \geq |q - q_{\text{int}}| + \operatorname{dist}(q_{\partial\Omega}, T_l) = |q - q_{\text{int}}| + O(d_l^2) \geq |q - q_{T_l}| + O(d_l^2)$. Let $|q - q_{\partial\Omega}| < |q - q_{\text{int}}|$. If $|q - q_{\partial\Omega}| \geq |q - q_{T_l}|$, then we are done. Otherwise, $|q - q_{\partial\Omega}| + |q_{\partial\Omega} - q_{\partial\Omega}^{T_l}| \geq |q - q_{\partial\Omega}^{T_l}| \geq |q - q_{T_l}|$. Thus $|q - q_{\partial\Omega}| \geq |q - q_{T_l}| + O(d_l^2)$.

(b) We now estimate $|q - q_{T_l}|$. Let $p_l = J_l \cap \partial B_{d_l}(x_l)$, then $|q - p_l| \geq d_l/2$. See the paragraph preceding proof of (a). Note that $\operatorname{dist}(p_l, T_l) = O(d_l^2)$, since $\partial\Omega \in C^2$. If $\langle p_l - x_l, \nu_{x_l} \rangle \geq 0$, then $\operatorname{dist}(q, T_l) \geq d_l/3$. If $\langle p_l - x_l, \nu_{x_l} \rangle < 0$, it again follows that $\operatorname{dist}(q, T_l) \geq d_l/3$.

We now apply the Harnack inequality, employing the above estimate for $(q,\partial\Omega)$, to see that for some $c > 0$ independent of d_l,

$$(v(z_l) - m(0)) \geq e^{-c}(v(\alpha_l) - m(0)) \geq \frac{e^{-c}|m(0)|}{4}. \tag{3.5}$$

Letting $l \to \infty$, we get $0 \geq |m(0)|/4$. Thus $m(0) = 0$. To show that $M(0) = 1$, we work with function $1 - u$ and in place of $m(0)$, we take $1 - M(0)$. Arguing analogously, one may now show that $M(0) = 1$. When $n = 2$, $I_r(y)$ reduces to two points and one may again adapt the above argument to obtain part (ii).

Step 4 (maximal solution u_y^r). Our next goal is to show that $u \geq v$, where v is any bounded solution of (1.2). Recall that for $x \in \partial\Omega$, ν_x is the unit inner normal to $\partial\Omega$ at x and $x_s = x + s\nu_x$. Since $\partial\Omega \in C^2$ and is bounded, there exists a $\delta > 0$ such that for every $x \in \partial\Omega$, $R_x \geq \delta$. Let $\varepsilon > 0$, small, with $\varepsilon \leq \min(1/10^4, \delta^2/10^4, r^2/10^4)$. For every $x \in \partial\Omega$, set $\Omega^\varepsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq \varepsilon\}$. Then $\partial\Omega^\varepsilon = \{x_\varepsilon : x \in \partial\Omega\}$. We will estimate u and v on $\partial\Omega^\varepsilon$. To this end, set $P_\varepsilon = \{x_\varepsilon : x \in P_r(y)\}$ and $Q_\varepsilon = \{x_\varepsilon : x \in Q_r(y)\}$. Note that

$$Q_\varepsilon = \partial\Omega^\varepsilon \setminus \widehat{P}_\varepsilon, \qquad \operatorname{dist}(\partial\Omega, \partial\Omega^\varepsilon) = \varepsilon, \qquad \Omega^\varepsilon \uparrow \Omega, \quad \text{as } \varepsilon \downarrow 0. \tag{3.6}$$

For $z \in \partial\Omega^\varepsilon$, let z^ε be the nearest point on $\partial\Omega$. Clearly, $z = (z^\varepsilon)_\varepsilon$. If $z \in Q_\varepsilon$, then $u(z^\varepsilon) = 0$, and if $z \in P_\varepsilon$, then $u(z^\varepsilon) = 1$. Set

$$N_\varepsilon = \{z \in Q_\varepsilon : \operatorname{dist}(z^\varepsilon, P_r(y)) \geq \sqrt{\varepsilon}\}, \qquad O_\varepsilon = \{z \in P_\varepsilon : \operatorname{dist}(z^\varepsilon, Q_r(y)) \geq \sqrt{\varepsilon}\}. \tag{3.7}$$

For v, we use comparison as follows. For $x \in Q_r(y)$ with $\operatorname{dist}(x, P_r(y)) \geq \sqrt{\varepsilon}$,

$$v(\xi) \leq \frac{|\xi - x|}{\sqrt{\varepsilon}}, \quad \xi \in B_{\sqrt{\varepsilon}}(x) \cap \Omega, \text{ implying that } 0 < v(x_\varepsilon) \leq \sqrt{\varepsilon}. \tag{3.8}$$

Similarly, for $x \in P_r(y)$ with $\operatorname{dist}(x, Q_r(y)) \geq \sqrt{\varepsilon}$,

$$1 - v(\xi) \leq \frac{|\xi - x|}{\sqrt{\varepsilon}}, \quad \xi \in B_{\sqrt{\varepsilon}}(x) \cap \Omega, \text{ implying that } 1 - \sqrt{\varepsilon} \leq v(x_\varepsilon) \leq 1. \tag{3.9}$$

Thus from (3.8) and (3.9), we obtain that

$$0 < v \le \sqrt{\varepsilon} \quad \text{on } N_\varepsilon, \qquad 1-\sqrt{\varepsilon} \le v < 1 \quad \text{on } O_\varepsilon. \tag{3.10}$$

Note that (3.10) is satisfied by any solution of (1.2), and in particular holds also for u. However, we will work with u_l instead. Fix $\eta > 0$, and recall from Step 1 that for $l = 1,2,3,\ldots$,

$$f_l(x) = \frac{(\eta/l) - \operatorname{dist}(x, P_r(y))}{(\eta/l)}, \quad x \in S_{\eta/l},\ f_l(x) = 0,\ x \in \partial\Omega \setminus S_{\eta/l}. \tag{3.11}$$

For ease of presentation, set $j = 4l/\eta$. We will work with l's such that $j\sqrt{\varepsilon} < 1$. For $x \in \partial\Omega$ with $\operatorname{dist}(x, P_r(y)) \le 3\sqrt{\varepsilon}$, we see that

$$u_l(x) = 1, \quad x \in P_r(y), \qquad u_l(x) \ge \frac{(\eta/l) - 3\sqrt{\varepsilon}}{(\eta/l)} \ge 1 - j\sqrt{\varepsilon}, \quad x \notin P_r(y). \tag{3.12}$$

We now use comparison in $B_{\sqrt{\varepsilon}}(x) \cap \Omega$, with $\operatorname{dist}(x, P_r(y)) \le 2\sqrt{\varepsilon}$. Set $w_x(z) = j\sqrt{\varepsilon} + (1 - j\sqrt{\varepsilon})|z-x|/\sqrt{\varepsilon}$. Clearly, $w_x \ge 1 - u_l$ in $B_{\sqrt{\varepsilon}}(x) \cap \Omega$. Using (3.9) and noting that $u_l \ge u$, we have for $x \in \partial\Omega$,

(i) $u_l(x_\varepsilon) \ge 1 - \sqrt{\varepsilon}$, $x \in P_r(y)$, with $\operatorname{dist}(x, Q_r(y)) \ge \sqrt{\varepsilon}$,

(ii) $u_l(x_\varepsilon) \ge (1 - j\sqrt{\varepsilon})(1-\sqrt{\varepsilon})$, with $\operatorname{dist}(x, P_r(y)) \le 2\sqrt{\varepsilon}$.

Call $J_\varepsilon = \{x_\varepsilon : x \in \partial\Omega \text{ and } \operatorname{dist}(x, P_r(y)) \le 2\sqrt{\varepsilon}\}$. From (3.7), $J_\varepsilon \supset O_\varepsilon$, $J_\varepsilon \cap N_\varepsilon \ne \varnothing$, and $u_l(x_\varepsilon) \ge (1 - j\sqrt{\varepsilon})(1-\sqrt{\varepsilon})$, $x \in J_\varepsilon$. Using (3.8) and (3.9), we see that $u_l + 2j\sqrt{\varepsilon} \ge v$ on $\partial\Omega^\varepsilon$. By comparison, $u_l + 2j\sqrt{\varepsilon} \ge v$ in Ω^ε. Letting $\varepsilon \to 0$, we obtain $u_l \ge v$ in Ω. Now letting $l \to \infty$, we see that $u \ge v$ in Ω.

From here on, we call $u_y^r = u$ and refer to it as the maximal solution of (1.2); clearly, $v \le u_y^r$.

Step 5 (minimal solution $\hat{u}_y^r$). It is clear from Step 1 that for $r_1 < r_2$, $u_y^{r_1} \le u_y^{r_2}$ (working with the corresponding u_l's). Note that u_y^r is locally Lipschitz but uniformly so in r. Set $\hat{u}_y^r = \sup_{t<r} u_y^t$. Using Step 1, $\hat{u}_y^r$ is a solution of (1.2) and $\hat{u}_y^r \le u_y^r$. The comparison principle in Step 2 also holds. We now show that $\hat{u}_y^r \le v$, where v is any solution of (1.2). We do this by showing that $u_y^t \le v$ whenever $t < r$. Fix $t < r$; we proceed as in Step 4. Let $\delta > 0$ be as in Step 4. Let $d > 0$, small, such that $0 < d \le \min(\delta^2/10^4, r^2/10^4, (r-t)^2/100)$; set $\Omega^d = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \ge d\}$. As in Step 4, define $P_{d,s} = \{x_d : x \in P_s(y)\}$, where s is either r or t. Now define $Q_{d,s}$ analogously. Then (3.6) holds. For $z \in \partial\Omega^d$, recall that $z^d \in \partial\Omega$ is such that $|z - z^d| = d$. Now for each $s = r, t$, define the sets $N_{d,s}$ and $O_{d,s}$, both subsets of Ω^d, along the lines of (3.7). Using (3.8), (3.9), and (3.10), we obtain

(i) $0 < u_y^t(\xi) \le \sqrt{d}$, $\xi \in N_{d,t}$,

(ii) $1 - \sqrt{d} \le u_y^t(\xi) \le 1$, $\xi \in O_{d,t}$,

(iii) $0 < v(\xi) \le \sqrt{d}$, $\xi \in N_{d,r}$,

(iv) $1 - \sqrt{d} \le v(x_d) \le 1$, $x \in O_{d,r}$.

Clearly, $O_{d,r} \supset O_{d,t}$, $N_{d,t} \supset N_{d,r}$, and for small d, $N_{d,t} \cap O_{d,r} \ne \varnothing$. Thus $v + 2\sqrt{d} \ge u_y^t$ on $\partial\Omega^d$. Using comparison in Ω^d and taking $d \to 0$, we obtain $v \ge u_y^t$. The claim now follows. Call $\hat{u}_y^r$ the minimal solution. By Step 4 and arguments presented here, we have the first part of part (iii), namely,

$$u_y^t \leq \hat{u}_y^r \leq v \leq u_y^r, \quad \text{whenever } t < r. \tag{3.13}$$

Since $u_y^{\tilde{r}} \geq u_y^r$, $\tilde{r} > r$. Setting $\tilde{u} = \lim_{\tilde{r}\downarrow r} u_y^{\tilde{r}} \geq u_y^r$, imitating Step 1, one can show that $\tilde{u}$ solves (1.2). Thus by Step 4, $\lim_{\tilde{r}\downarrow r} u_y^{\tilde{r}} = u_y^r$. Part (iv) is also proven. $\square$

Remark 3.2. If one could prove that $\lim_{t\uparrow r} u_y^t = u_y^r$, then uniqueness would follow, however, equality here would be a stronger result. A proof of this is shown for the half-space in Lemma 3.4.

Let $v_y^r > 0$ satisfy the following:

$$\Delta_\infty v_y^r(z) = 0, \quad z \in \Omega, \qquad v_y^r|_{Q_r(y)} = 0, \qquad v \in C(\hat{\Omega} \setminus \Omega_r(y)). \tag{3.14}$$

LEMMA 3.3. *For $y \in \partial\Omega$, $r > 0$, let u^r be a bounded solution of (1.2) and let v_y^r be as in (3.14). Assume that r is small. Then $u^r(y_r) \geq c$, for some universal constant $c > 0$. Moreover, there exist universal constants $C > 0$ and $\overline{C} > 0$ such that*

(i) $u^{2r}(z) \leq Cu^r(z)$, $z \in \Omega \setminus \Omega_{3r}(y)$,

(ii) $u^r(z)/\overline{C} \leq v_y^r(z)/v_y^r(y_r) \leq \overline{C}u^r(z)$, $z \in \Omega \setminus \Omega_{2r}(y)$.

Proof. By the maximum principle, $0 < u^r < 1$ in Ω. Let $w(z) = (r - |z-y|)/r$, $z \in \Omega_r(y)$. Then $u^r \geq w$ on $\partial B_r(y) \cap \Omega$, and $u^r \geq w$ on $P_r(y)$. By comparison, $u^r \geq w$ in $\Omega_r(y)$. Thus $u^r(y_{r/2}) \geq w(y_{r/2}) = 1/2$. We may now use the Harnack inequality to conclude that

$$u^r(y_r) \geq e^{-2}u^r(y_{r/2}) \geq \frac{1}{2e^2}. \tag{3.15}$$

We now prove part (i), the "doubling" property of u^r in $\Omega \setminus \Omega_{3r}(y)$. We will use the boundary Harnack inequality and comparison. Note that $u^{2r} = u^r = 0$, $x \in \partial\Omega \setminus P_{5r/2}(y)$. We consider $\Omega_{r/4}(x)$, $x \in \partial B_{5r/2}(y) \cap \partial\Omega$. By Theorem 1.1(ii),

$$\frac{u^r(z)}{u^{2r}(z)} \geq C_1 \frac{u^r(x_{r/4})}{u^{2r}(x_{r/4})}, \quad z \in \partial B_{5r/2}(y) \cap \Omega_{r/4}(x). \tag{3.16}$$

We now use the Harnack inequality and (3.15) to conclude that there are universal constants C_2, C_3, and C_4 such that

$$\frac{u^r(z)}{u^{2r}(z)} \geq C_2 \frac{u^r(y_{5r/2})}{u^{2r}(y_{5r/2})} \geq C_3 \frac{u^r(y_r)}{u^{2r}(y_r)} \geq C_4, \quad z \in \partial B_{5r/2}(y) \cap \Omega. \tag{3.17}$$

We now use comparison in $\Omega \setminus \Omega_{5r/2}(y)$ to conclude part (i). We now prove part (ii). For every $x \in \partial\Omega \cap \partial B_{2r}(y)$, we have by Theorem 1.1(ii) that

$$C_1 \frac{u^r(x_{r/2})}{v_y^r(x_{r/2})} \leq \frac{u^r(z)}{v_y^r(z)} \leq C_2 \frac{u^r(x_{r/2})}{v_y^r(x_{r/2})}, \quad z \in \partial B_{2r}(y) \cap \Omega_{r/2}(x). \tag{3.18}$$

As done before, we may use the Harnack inequality to conclude that

$$C_3 \frac{u^r(y_r)}{v_y^r(y_r)} \leq \frac{u^r(z)}{v_y^r(z)} \leq C_4 \frac{u^r(y_r)}{v^r(y_r)}, \quad z \in \partial B_{2r}(y) \cap \Omega. \tag{3.19}$$

Thus using (3.15), we obtain

$$\frac{v_y^r(y_r)u^r(z)}{C} \leq v_y^r(z) \leq C v_y^r(y_r)u^r(z), \quad z \in \partial B_{2r}(y) \cap \Omega. \tag{3.20}$$

The claim follows by comparison. □

We now look at the case of the half-space $H = \{x \in \mathbb{R}^n : x_n > 0\}$. Set $T = \{x \in \mathbb{R}^n : x_n = 0\}$; for $y \in T$, define $P_r(y) = H \cap B_r(y)$, $Q_r(y) = T \setminus \widehat{P}_r(y)$, and $M_y^u(\rho) = \sup_{\partial B_\rho(y) \cap H} u$. Define a solution u of

$$\Delta_\infty u(z) = 0, \quad z \in H, \ u|_{P_r(y)} = 1, \ u|_{Q_r(y)} = 0, \tag{3.21}$$

to satisfy the equation in the sense of viscosity, $0 \leq u \leq 1$, u is continuous up to $P_r(y)$ and $Q_r(y)$, and $\limsup_{\rho \to \infty} M_y^u(\rho) = 0$. Set $L_y = \{y + s e_n : s \in \mathbb{R}\}$. In the proof of Lemma 3.4, we make use of an example of a positive singular infinity-harmonic function in the half-space [6, 11]. We utilize the definition in Step 2 of Theorem 1.3 as appears in Section 4. For Lemma 3.4, we define $\phi(x) = f(\theta)/|x|^{1/3}$, where θ is the conical angle at y and $f(\theta)$ is the function $f_m(\theta)$ when $m = 1$. Then $\phi(x)$ blows up at y, vanishes elsewhere on T, and decays to zero at inifinity. In what follows, we make frequent use of the results in [7].

Lemma 3.4 (Half-Space). *Let $H = \{x \in \mathbb{R}^n : x_n > 0\}$. Then there exists a unique solution u_y^r of the problem in (3.21) such that $0 < u_y^r < 1$. Moreover, if $\sigma^2 = \sum_{i=1}^{n-1}(z_i - y_i)^2$, then $u_y^r(z)$ is symmetric about the line L_y, and $u_y^r(z) = u_y^r(\sigma, z_n)$ is decreasing in σ.*

Proof. Let $0 < r < \rho$ and set $H_\rho = H \cap B_\rho(y)$. For $l = 1, 2, 3, \ldots$, and $\eta > 0$, define $u_l^{r,\rho}$ to be the unique solution of the problem

$$\Delta_\infty u_l^{r,\rho}(z) = 0, \quad z \in H \cap B_\rho(y), \qquad u_l^{\rho,r}|_T = f_l, \tag{3.22}$$

where f_l is the function as defined in Step 1 of Lemma 3.1. Clearly, $0 < u_l^{r,\rho} < 1$ and by comparison, $u_l^{r,\rho} \uparrow u_l^r$ as $\rho \uparrow \infty$. Arguing as in Step 1 of Lemma 3.1, we may show that u_l^r is infinity-harmonic in H and $u_l^r|_T = f_l$. Set $M_l(s) = \sup_{\partial B_s(y) \cap H} u_l^r$; we now show that $M_l(s) \to 0$ as $s \to \infty$. The following are true: (i) $0 < M_l(s) \leq 1$, $s > 0$, (ii) $M_l(s) = 1$, $0 < s \leq r$, and (iii) $M_l(s) < 1$ and is convex in s, whenever $s > r + (\eta/l)$. Thus $M_l(s)$ is non-increasing. Let $\phi > 0$ be the Aronsson singular solution on H as described above. Adapting the argument in Step 2 of Theorem 1.2 (this follows below), one may show that for some $C > 0$ depending only on $\phi(y_{2r})$, $u_l^{r,\rho}(z) \leq C\phi(z)$, $z \in \partial B_{2r}(y) \cap H$, and $0 = u_l^{r,\rho} < \phi$ on $\partial B_\rho(y) \cap H$. Thus $u_l^{r,\rho}(z) \leq C\phi(z)$, $z \in H_\rho \setminus H_{2r}$. Clearly, $u_l^r \leq C\phi$, in H, and $M_l(s) \to 0$ as $s \to \infty$. Hence u_l^r solves (3.21) with modified boundary data f_l. Since u_l^r decreases in l, it follows that $u_l^r \to u_y^r$, where now u_y^r solves (3.21). Let v be any solution of (3.21). We show that $u_y^r \geq v$ in H. Let $\varepsilon > 0$ be small. We adapt Step 4 in Lemma 3.1 and use the comparison lemma [7, Lemma 2.2]. We work with u_l^r and estimate both u_l^r and v on the set $\{x \in H : x_n = \varepsilon\}$. Let $r_0 > 100\sqrt{\varepsilon}$ be such that $0 \le \sup(M_y^v(\rho), M_l^r) \le \varepsilon$, $\rho \ge r_0$. We work in $S_\varepsilon^{r_0} = \{x \in H : x_n \ge \varepsilon, \text{ and } |x| \le r_0\}$. By letting $r_0 \to \infty$ and, as done in Step 4 of Lemma 3.1, then letting $\varepsilon \to 0$, one can show that $u_l^r \ge v$ in H for any l. Thus $u_y^r \ge v$. Also we may show by adapting Step 5 that $u_y^t \le v \le u_y^r$, $t < r$. Call u_y^r the maximal solution. Note that u_y^r is symmetric about L_y, since by reflection, $u_l^{r,\rho}$ is symmetric about L_y. Writing $u_l^{r,\rho}(z) = u_l^{r,\rho}(\sigma, z_n)$ and using reflection and comparison (see [7, Lemma 2.6]), we see that $u_l^{r,\rho}(\sigma, z_n)$ decreases in σ. Thus the same holds for u_y^r.

We now scale as follows. For $\theta > 0$, set $z^\theta = y + \theta(z - y)$, $z \in H$, then $\mu^\theta(z) = u_y^r(z^\theta)$ solves (3.21) with $P_r(y)$ replaced by $P_{r/\theta}(y)$. Clearly, by the maximality of $u_y^{r/\theta}$,

$$\mu^\theta(z) \le u_y^{r/\theta}(z), \quad \text{implying that } u_y^r(z^\theta) \le u_y^{r/\theta}(z),\ z \in H. \tag{3.23}$$

Using $\theta > 1$ and that u_y^r is maximal, we see that if v is any solution of (3.21), then (3.23) implies that $u_y^r(z^\theta) \le u_y^{r/\theta}(z) \le v(z) \le u_y^r(z)$, $z \in H$. Letting $\theta \uparrow 1$ and using continuity, we obtain uniqueness of u_y^r. □

Proof of Theorem 1.2. Set $T_s = \{x_n = s\}$, $s > 0$ and $M(s) = \sup_{T_s} u$. Let $u^r = u_o^r$ solve (3.21). We will assume that $\lim_{\rho\to\infty} \sup_{\partial B_\rho(o)\cap H} u^r = 0$. Our proof will use and adapt results from [7].

Step 1. By Lemma 3.4, u^r is unique. To show the doubling property of u^r, we use Lemma 3.3(ii), the comparison result in [7, Lemma 2.2], and (3.15), that is, $u^r(o_r) \ge c > 0$. We now focus on the halving property. By Lemma 3.4, u^r is unique and $u^{2r}(x) = u^r(x/2)$, $x \in H$. Thus our goal is to show that

$$u^r(o_{4r}) \le \alpha u^{2r}(o_{4r}) = \alpha u^r(o_{2r}), \tag{3.24}$$

for some universal $0 < \alpha < 1$. We now make an observation. By Lemma 3.4, for $t > 1$, $u^r(x) \le u^{rt}(x) = u^r(x/t)$. If v is a unit vector with $\langle v, e_n\rangle \ge 0$, then $u^r(sv)$, $s > 0$, is a decreasing function of s. In particular, writing a point on the x_n-axis as $(0, x_n)$, $u^r(0, x_n)$ decreases in x_n. By Lemma 3.4, $u(0,s) = M(s)$, $s > 0$, and $M(s)$ is decreasing. To see that $M(s)$ is convex in s, for $0 < s < t$, consider the set $H_{s,t} = \{x \in H : s < x_n < t\}$. The function $w(x) = M(s) + [M(t) - M(s)][x_n - s]/(t - s)$ is infinity-harmonic in $H_{s,t}$, and by comparison, $u^r \le w$ in $H_{s,t}$. The claim follows. Since $u^r(o_{2r}) = M(2r)$, $u(o_{4r}) = M(4r)$, and $\lim_{s\to\infty} M(s) = 0$, by convexity it is clear that $u(o_{4r})/u(o_{2r}) = \alpha < 1$. Our goal is to show that α is independent of r.

Step 2 (decay estimate). We show that $u^r(x)$ decays like $|x|^{-1/3}$. We use the work [7]. Let $v(x) = f(\theta)|x|^{-1/3}$, where $\theta = \theta(x) = \cos^{-1} x_n/|x|$, be the Aronsson example of a singular solution in the half-space H (see Section 4). Consider the set $A_t = H \cap \partial B_t(o)$, $t > 0$. Employing Theorem 1.1(ii), the Harnack inequality, and following the proof of Theorem 1.1 in [7], we see that there are universal constants C_1, C_2 such that

$$C_1 \frac{u^r(x)}{v(x)} \le \frac{u^r(o_t)}{v(o_t)} \le C_2 \frac{u^r(x)}{v(x)}, \quad x \in A_t,\ t \ge 2r. \tag{3.25}$$

Set $\Gamma(t) = \sup_{A_t} u^r/v$, $\gamma(t) = \inf_{A_t} u^r/v$, $t \geq 2r$. We now proceed as in [7, Corollary 2.3] to see that there is a universal constant C_3 such that for $t \geq 2r$ and $2r \leq t_1 \leq t_2$,

$$\gamma(t) \leq \Gamma(t), \qquad \Gamma(t) \leq C_3\gamma(t), \qquad \Gamma(t_2) \leq \Gamma(t_1), \qquad \gamma(t_2) \geq \gamma(t_1). \tag{3.26}$$

For generality, let $\lambda \geq 2$ and $v(x) = v_{\lambda r}(x) = f(\theta)(\lambda r/|x|)^{1/3}u^r(o_{\lambda r})/f(0)$. Then $v(o_{\lambda r}) = u^r(o_{\lambda r})$. Note that $f(0) = (16)^{-1/3}$, and by (3.15) and the Harnack inequality, $u^r(o_{\lambda r}) \geq e^{1-2\lambda}u^r(o_{r/2}) \geq e^{1-2\lambda}/2$. Using (3.25) and (3.26), we see that there are universal constants C_4 and C_5 such that

$$C_5 \leq \gamma(\lambda r) \leq \gamma(t) \leq \frac{u^r(x)}{v_{\lambda r}(x)} \leq \Gamma(t) \leq \Gamma(\lambda r) \leq C_4, \quad x \in A_t,\ t \geq \lambda r. \tag{3.27}$$

Thus $C_5 v_{\lambda r}(x) \leq u^r(x) \leq C_4 v_{\lambda r}(x)$, $x \in H \setminus B_t(o)$. Thus

$$C_5\frac{f(\theta)}{f(0)}\left(\frac{\lambda r}{|x|}\right)^{1/3} \leq \frac{u^r(x)}{u^r(o_{\lambda r})} \leq C_4\frac{f(\theta)}{f(0)}\left(\frac{\lambda r}{|x|}\right)^{1/3}, \quad |x| \geq \lambda r. \tag{3.28}$$

Step 3. Using (3.28) and Step 1, it follows that for $\kappa > 1$ and $|x| = \kappa\lambda r$, there are universal constants C_6 and C_7 such that

$$\frac{C_6}{\kappa^{1/3}} \leq \frac{u(o_{\kappa\lambda r})}{u(o_{2r})} \leq \frac{C_7}{\kappa^{1/3}} \Longrightarrow \frac{C_6}{\kappa^{1/3}} \leq \frac{M(\kappa\lambda r)}{M(\lambda r)} \leq \frac{C_7}{\kappa^{1/3}}. \tag{3.29}$$

Choose $l = \sup(3, 3/C_7)$ and set $\kappa = (lC_7)^3 > 3$.

$$M(2\lambda r) \leq \frac{\kappa-2}{\kappa-1}M(\lambda r) + \frac{1}{\kappa-1}M(\kappa\lambda r) \leq \frac{1}{\kappa-1}\left(\kappa - 2 + \frac{1}{l}\right)M(\lambda r) \leq \frac{\kappa-5/3}{\kappa-1}M(\lambda r). \tag{3.30}$$

Clearly, $\alpha < 1$ in (3.24) and is universal. □

4. Proof of Theorem 1.3

In this section, we will present another application of Theorem 1.1. We show that any two positive singular infinity-harmonic singular functions, defined in a cone, are comparable. As a consequence, we will show the optimality of the blowup rates of the Aronsson examples [6]. This will extend the results in [7]. First we prove a version of monotonicity that holds in a cone.

Lemma 4.1 (Monotonicity). *For $0 < \alpha \leq \pi/2$, let C_α denote the interior of the cone $\{x : x_n > 0,\ x_n = \|x\| \cos\alpha\}$. Let $u > 0$ be ∞-harmonic in C_α. Suppose that v is a unit vector that lies in C_α, that is, $\langle v, e_n\rangle \geq \cos\alpha$ and let $\theta = \alpha - \cos^{-1}\langle e_n, v\rangle$, then for $0 < t < s$, $u(tv)/t^{\sin\theta} \geq u(sv)/s^{\sin\theta}$, and $u(tv)t^{\sin\theta} \leq u(sv)s^{\sin\theta}$.*

Proof. We use the version of the Harnack inequality proved in [7, Lemma 2.1]. Then $\sigma(\tau) = (t+\tau(s-t))\nu$, $0 \le \tau \le 1$, while $d(\tau) = \sigma(\tau)\sin\theta$. Thus

$$\begin{aligned} u(t\nu) &\geq u(s\nu)\exp\left(-\int_0^1 \frac{s-t}{\sin\theta(t+\tau(s-t))}\,d\tau\right) \\ &= u(s\nu)\exp\left(-\frac{\log(s/t)}{\sin\theta}\right) = u(s\nu)\left(\frac{t}{s}\right)^{1/\sin\theta}. \end{aligned} \tag{4.1}$$

Thus $u(t\nu)/t^{1/\sin\theta} \geq u(s\nu)/s^{1/\sin\theta}$. Switching $t\nu$ by $s\nu$ yields the second inequality. □

Proof of Theorem 1.3. Our proof will be an adaptation of the proof of Theorem 1.1 in [7]. First note that $M^u(r)$ is convex. By using comparison, we see that $u(x) \le M^u(t) + [M^u(s) - M^u(t)](|x| - t)/(s-t)$ in the annulus $C_\alpha \cap (B_s(o) \setminus B_t(o))$, $0 < t < s$. Thus $M^u(r)$ is decreasing, $\lim_{r\to 0} M^u(r) = \infty$, and $\lim_{r\to\infty} M^u(r) = 0$.

Step 1. We will prove that any two positive solutions u and v are comparable in C_α. Now consider the set $C_{\alpha,r} = C_\alpha \cap B_r(o)$. Then (i) for $x \in \partial C_\alpha \cap \partial B_r(o)$, $R_x = r\tan\alpha$, and (ii) for $y \in \partial C_\alpha \cap B_{r/4}(x)$, $R_y \geq (3r/4)\tan\alpha$. In Theorem 1.1, we may take $\delta = (r/8)\tan\alpha$. Thus there are universal constants C_1 and C_2 such that

$$C_1\frac{u(z)}{v(z)} \le \frac{u(x_\delta)}{v(x_\delta)} \le C_2\frac{u(z)}{v(z)}, \quad z \in C_\alpha \cap B_{(r\tan\alpha)/4}(x). \tag{4.2}$$

Set $p_r = re_n$; let $S_{r,x}$ be the great circle centered at o, has radius r, and passing through p_r and x. Using the Harnack inequality, we may conclude that for $\xi \in (C_\alpha \cap S_{r,x}) \setminus B_{(r\tan\alpha)/4}(x)$, there are constants $A_1 = A_1(\alpha)$ and $A_2 = A_2(\alpha)$ such that $A_1u(p_r) \le u(\xi) \le A_2u(p_r)$. This holds for every $x \in \partial C_\alpha \cap \partial B_r(o)$. Using (4.2), we obtain that there are constants $C_3 = C_3(\alpha)$ and $C_4(\alpha)$ such that

$$C_3\frac{u(z)}{v(z)} \le \frac{u(p_r)}{v(p_r)} \le C_4\frac{u(z)}{v(z)}, \quad z \in C_{\alpha,r}. \tag{4.3}$$

It is clear that (4.3) holds for every $r > 0$. Define $\Gamma(r) = \sup_{z\in C_{\alpha,r}} u(z)/v(z)$ and $\gamma(r) = \inf_{z\in C_{\alpha,r}} u(z)/v(z)$. Thus from (4.3), there is a constant $C_5 = C_5(\alpha)$ such that $\gamma(r) \le \Gamma(r) \le C_5\gamma(r)$, $r > 0$. By comparison, $\Gamma(r)$ is decreasing and $\gamma(r)$ is increasing in r, see [7, Lemma 2.2 and Corollary 2.3]. Thus using the above inequality, we see that $0 < \gamma(0) \le \gamma(r) \le \Gamma(r) \le \Gamma(0) \le C_5\gamma(0) < \infty$, $r > 0$. Thus there is a constant C such that $v(x)/C \le u(x) \le Cv(x)$, $x \in C_\alpha$.

Step 2. We now show the optimality of the Aronsson examples. We first observe that by arguing as in [7, Lemmas 2.5 and 2.6], any solution u is axially symmetric in C_α. If we set $\rho^2 = \sum_{i=1}^n x_i^2$ and $\theta = \cos^{-1}\langle x, e_n\rangle/|x|$, then $u(x) = u(\rho,\theta)$, $x \in C_\alpha$, and

$$\Delta_\infty u = u_\rho^2 u_{\rho\rho} + \frac{2u_\rho u_\theta u_{\rho\theta}}{\rho^2} + \frac{u_\theta^2 u_{\theta\theta}}{\rho^4} - \frac{u_\rho u_{\theta\theta}^2}{\rho^3} = 0. \tag{4.4}$$

Note that there is no explicit dependence on the dimension n. For each $m = 1,2,3,\dots$, set

$\alpha = \pi/2m$. The Aronsson example in the planar cone $C_{\pi/2m}$ is given by $w_m(x) = w_m(|x|, \theta) = f_m(\theta)/|x|^{m^2/(2m+1)}$, where

$$f_m(\theta) = \left|1 - \frac{\cos^2 t}{k}\right|^{(k-1)/2} \cos t, \quad \theta = \int_0^t \frac{\sin^2 s}{k - \cos^2 s} ds, \; k = -\frac{m^2}{2m+1}. \tag{4.5}$$

Note that $\theta = t - (1 + 1/m)\arctan(m \tan t/(m + 1))$. From above w_m is symmetric in θ and reinterpreting the polar angle θ to be the conical angle, we obtain an example in higher dimensions. This continues to be a viscosity solution in $C_{\pi/2m}$, see the appendix in [11, 7]. Note that $w_m(|x|, \theta) > 0$, $-\pi/2m \le \theta \le \pi/2m$, and $w_m(\pm\pi/2m) = 0$. We now have the desired conclusion by using Step 1. □

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References

[1] T. Bhattacharya, "On the properties of ∞-harmonic functions and an application to capacitary convex rings," *Electronic Journal of Differential Equations*, vol. 2002, no. 101, pp. 1–22, 2002.

[2] P. Bauman, "Positive solutions of elliptic equations in nondivergence form and their adjoints," *Arkiv för Matematik*, vol. 22, no. 2, pp. 153–173, 1984.

[3] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, "Boundary behavior of nonnegative solutions of elliptic operators in divergence form," *Indiana University Mathematics Journal*, vol. 30, no. 4, pp. 621–640, 1981.

[4] E. Fabes, N. Garofalo, S. Marín-Malave, and S. Salsa, "Fatou theorems for some nonlinear elliptic equations," *Revista Matemática Iberoamericana*, vol. 4, no. 2, pp. 227–251, 1988.

[5] J. J. Manfredi and A. Weitsman, "On the Fatou theorem for p-harmonic functions," *Communications in Partial Differential Equations*, vol. 13, no. 6, pp. 651–668, 1988.

[6] G. Aronsson, "Construction of singular solutions to the p-harmonic equation and its limit equation for $p = \infty$," *Manuscripta Mathematica*, vol. 56, no. 2, pp. 135–158, 1986.

[7] T. Bhattacharya, "A note on non-negative singular infinity-harmonic functions in the half-space," *Revista Matemática Complutense*, vol. 18, no. 2, pp. 377–385, 2005.

[8] G. Aronsson, M. G. Crandall, and P. Juutinen, "A tour of the theory of absolutely minimizing functions," *Bulletin of the American Mathematical Society. New Series*, vol. 41, no. 4, pp. 439–505, 2004.

[9] M. G. Crandall, L. C. Evans, and R. F. Gariepy, "Optimal Lipschitz extensions and the infinity Laplacian," *Calculus of Variations and Partial Differential Equations*, vol. 13, no. 2, pp. 123–139, 2001.

[10] T. Bhattacharya, E. DiBenedetto, and J. J. Manfredi, "Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems. Some topics in nonlinear PDEs (Turin, 1989)," *Rendiconti del Seminario Matematico. Università e Politecnico di Torino*, pp. 15–68 (1991), 1989, special issue.

[11] T. Bhattacharya, "On the behaviour of ∞-harmonic functions near isolated points," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 58, no. 3-4, pp. 333–349, 2004.

[12] T. Bhattacharya, "On the behaviour of ∞-harmonic functions on some special unbounded domains," *Pacific Journal of Mathematics*, vol. 219, no. 2, pp. 237–253, 2005.

[13] P. Lindqvist and J. J. Manfredi, "The Harnack inequality for ∞-harmonic functions," *Electronic Journal of Differential Equations*, vol. 1995, no. 4, pp. 1–5, 1995.

[14] O. Savin, "C^1 regularity for infinity harmonic functions in two dimensions," *Archive for Rational Mechanics and Analysis*, vol. 176, no. 3, pp. 351–361, 2005.

[15] G. Barles and J. Busca, "Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term," *Communications in Partial Differential Equations*, vol. 26, no. 11-12, pp. 2323–2337, 2001.

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