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Research Article

Existence of Symmetric Positive Solutions for an m-Point Boundary Value Problem

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We study the second-order m-point boundary value problem $u''(t)+a(t)f(t,u(t))=0$, $0<t<1$, $u(0)=u(1)=\sum_{i=1}^{m-2}\alpha_i u(\eta_i)$, where $0<\eta_1<\eta_2<\cdots<\eta_{m-2}\leq 1/2$, $\alpha_i>0$ for $i=1,2,\ldots,m-2$ with $\sum_{i=1}^{m-2}\alpha_i<1, m\geq 3$. $a:(0,1)\to[0,\infty)$ is continuous, symmetric on the interval $(0,1)$, and maybe singular at $t=0$ and $t=1$, $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous, and $f(\cdot,x)$ is symmetric on the interval $[0,1]$ for all $x\in[0,\infty)$ and satisfies some appropriate growth conditions. By using Krasnoselskii's fixed point theorem in a cone, we get some existence results of symmetric positive solutions.

1. Introduction

The m-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. In the past few years, the existence of positive solutions for nonlinear second-order multipoint boundary value problems has been studied by many authors by using the Leray-Schauder continuation theorem, nonlinear alternative of Leray Schauder, coincidence degree theory, Krasnoselskii's fixed point theorem, Leggett-Wiliams fixed point theorem, or lower- and upper-solutions method (see [1–21] and references therein). On the other hand, there is much current attention focusing on questions of symmetric positive solutions for second-order two-point boundary value problems, for example, Avery and Henderson [22], Henderson and Thompson [23] imposed conditions on f to yield at least three symmetric positive solutions to the problem

$$\begin{gathered} y''+f(y)=0, \quad 0\leq t\leq 1, \\ u(0)=u(1)=0, \end{gathered} \tag{1.1}$$

where $f : \mathbb{R} \to [0,+\infty)$ is continuous. Both of the papers [22, 23] make an application of an extension of the Leggett-Williams fixed point theorem. Li and Zhang [24] considered the existence of multiple symmetric nonnegative solutions for the second-order boundary value problem

$$\begin{gathered} -x'' = f(x,x'), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.2}$$

where $f : \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ is continuous. The main tool is the Leggett-Williams fixed point theorem. Yao [25] gave the existence of n symmetric positive solutions and established a corresponding iterative scheme for the two-point boundary value problem

$$\begin{gathered} w''(t) + h(t)f(w(t)) = 0, \quad 0 < t < 1, \\ \alpha w(0) - \beta w(0) = 0, \qquad \alpha w(1) + \beta w(1) = 0, \end{gathered} \tag{1.3}$$

where $\alpha > 0$, $\beta \ge 0$, and the coefficient $h(t)$ may be singular at both end points $t = 0$ and $t = 1$. The main tool is the monotone iterative technique. Very recently, by using the Leggett-Wiliams fixed point theorem and a coincidence degree theorem of Mawhin, Kosmatov [26, 27] studied the existence of three positive solutions for a multipoint boundary value problem

$$\begin{gathered} -u''(t) = a(t)f(t,u(t),|u'(t)|), \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{n} \mu_i u(\xi_i), \qquad u(1-t) = u(t), \quad t \in [0,1], \end{gathered} \tag{1.4}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_n \le 1/2$, $\mu_i > 0$ for $i = 1,2,\ldots,n$, with $\sum_{i=1}^{n} \mu_i < 1$, $n \ge 2$.

In this paper, we are concerned with the existence of symmetric positive solutions for the following second-order m-point boundary value problem (BVP):

$$u''(t) + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, \tag{1.5}$$

$$u(0) = u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.6}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} \le 1/2$, $\alpha_i > 0$ for $i = 1,2,\ldots,m-2$, with $\sum_{i=1}^{m-2} \alpha_i < 1$, $m \ge 3$. $a : (0,1) \to [0,\infty)$ is continuous, symmetric on the interval $(0,1)$, and may be singular at both end points $t = 0$ and $t = 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and $f(1-t,x) = f(t,x)$ for all $(t,x) \in [0,1] \times [0,\infty)$. We use Krasnoselskii's fixed point theorem in cones and combine it with an available transformation to establish some simple criteria for the existence of at least one, at least two, or many symmetric positive solutions to BVP (1.5)-(1.6).

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. In Section 3, we discuss the existence of at least one symmetric positive solution for BVP

(1.5)-(1.6). Then we will prove the existence of two or many positive solutions in Section 4, where n is an arbitrary natural number.

2. Preliminaries and lemmas

In this section, we introduce some necessary definitions and preliminary results that will be used to prove our main results. A function w is said to be concave on $[0,1]$ if

$$w(rt_1+(1-r)t_2) \geq rw(t_1)+(1-r)w(t_2), \quad r, t_1, t_2 \in [0,1]. \tag{2.1}$$

A function w is said to be symmetric on $[0,1]$ if

$$w(t) = w(1-t), \quad t \in [0,1]. \tag{2.2}$$

A function u^* is called a symmetric positive solution of BVP (1.5)-(1.6) if $u^*(t) > 0$, $u^*(1-t) = u^*(t)$, $t \in [0,1]$, and (1.5) and (1.6) are satisfied.

We will consider the Banach space $C[0,1]$ equipped with norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$. Set

$$C^+[0,1] = \{w \in C[0,1] : w(t) \geq 0,\ t \in [0,1]\}. \tag{2.3}$$

We consider first the m-point BVP:

$$u'' + h(t) = 0, \quad 0 < t < 1, \tag{2.4}$$

$$u(0) = u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{2.5}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$.

Lemma 2.1. *Let* $\sum_{i=1}^{m-2}\alpha_i \neq 1$, $h \in C[0,1]$. *Then the m-point BVP (2.4)-(2.5) has a unique solution*

$$u(t) = \int_0^1 H(t,s)h(s)ds, \tag{2.6}$$

where

$$H(t,s) = G(t,s) + E(s), \tag{2.7}$$

$$G(x,y) = \begin{cases} x(1-y), & 0 \leq x \leq y \leq 1, \\ y(1-x), & 0 \leq y \leq x \leq 1, \end{cases} \qquad E(s) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i G(\eta_i, s). \tag{2.8}$$

Proof. From (2.4), we have

$$u(t) = -\int_0^t (t-s)h(s)ds + Bt + A. \tag{2.9}$$

In particular,

$$\begin{aligned} u(0) &= A, \\ u(1) &= -\int_0^1 (1-s)h(s)ds + B + A, \\ u(\eta_i) &= -\int_0^{\eta_i} (\eta_i - s)h(s)ds + B\eta_i + A. \end{aligned} \tag{2.10}$$

Combining with (2.5), we conclude that

$$\begin{aligned} B &= \int_0^1 (1-s)h(s)ds, \\ A &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)h(s)ds. \end{aligned} \tag{2.11}$$

Therefore, the m-point BVP (2.4)-(2.5) has a unique solution

$$\begin{aligned} u(t) &= -\int_0^t (t-s)h(s)ds + t\int_0^1 (1-s)h(s)ds + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds + \int_0^1 E(s)h(s)ds = \int_0^1 H(t,s)h(s)ds. \end{aligned} \tag{2.12}$$

This completes the proof. □

Lemma 2.2. *Suppose* $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} \le 1/2$, $\alpha_i > 0$ *for* $i = 1,2,\dots,m-2$, *with* $\sum_{i=1}^{m-2}\alpha_i < 1$. *Then*

(1) $H(t,s) \ge 0$, $t,s \in [0,1]$, $H(t,s) > 0$, $t,s \in (0,1)$;

(2) $G(1-t,1-s) = G(t,s)$, $t,s \in [0,1]$;

(3) $\gamma H(s,s) \le H(t,s) \le H(s,s)$, $t,s \in [0,1]$, *where*

$$\gamma = \frac{\sum_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum_{i=1}^{m-2}\alpha_i + \sum_{i=1}^{m-2}\alpha_i\eta_i}. \tag{2.13}$$

Proof. The conclusions (1), (2), and the second inequality of (3) are evident. Now we prove that the first inequality of (3) holds. In fact, from $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} \le 1/2$, we know $1-\eta_i \ge \eta_i$, thus for $s \in [0,1]$, we have

$$G(\eta_i,s) = \begin{cases} (1-\eta_i)s, & 0 \le s \le \eta_i \\ \eta_i(1-s), & \eta_i \le s \le 1 \end{cases} \ge \eta_i s(1-s) = \eta_i G(s,s), \tag{2.14}$$

which means that

$$\alpha_i G(\eta_i,s) \ge \alpha_i\eta_i G(s,s), \quad i = 1,2,\dots,m-2, \tag{2.15}$$

and summing both sides from 1 to $m-2$, we get

$$\sum_{i=1}^{m-2}\alpha_i G(\eta_i,s) \geq \left(\sum_{i=1}^{m-2}\alpha_i\eta_i\right)G(s,s). \tag{2.16}$$

So

$$\sum_{i=1}^{m-2}\alpha_i G(\eta_i,s) + \sum_{i=1}^{m-2}\alpha_i\eta_i E(s) \geq \left(\sum_{i=1}^{m-2}\alpha_i\eta_i\right)[G(s,s)+E(s)]. \tag{2.17}$$

Thus

$$\left(1-\sum_{i=1}^{m-2}\alpha_i+\sum_{i=1}^{m-2}\alpha_i\eta_i\right)E(s) \geq \left(\sum_{i=1}^{m-2}\alpha_i\eta_i\right)[G(s,s)+E(s)] = \left(\sum_{i=1}^{m-2}\alpha_i\eta_i\right)H(s,s). \tag{2.18}$$

Subsequently,

$$E(s) \geq \frac{\sum_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum_{i=1}^{m-2}\alpha_i+\sum_{i=1}^{m-2}\alpha_i\eta_i}H(s,s) = \gamma H(s,s). \tag{2.19}$$

Therefore,

$$H(t,s) = G(t,s)+E(s) \geq E(s) \geq \gamma H(s,s), \quad t,s \in [0,1]. \tag{2.20}$$

This completes the proof. □

Lemma 2.3. *Let* $\sum_{i=1}^{m-2}\alpha_i \neq 1$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $h(t)$ *be symmetric on* $[0,1]$. *Then the unique solution* $u(t)$ *of BVP (2.4)-(2.5) is symmetric on* $[0,1]$.

Proof. For any $t,s \in [0,1]$, from (2.7) and Lemma 2.2, we have

$$\begin{aligned} u(1-t) &= \int_0^1 H(1-t,s)h(s)ds = \int_0^1 G(1-t,s)h(s)ds + \int_0^1 E(s)h(s)ds \\ &= \int_1^0 G(1-t,1-s)h(1-s)d(1-s) + \int_0^1 E(s)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds + \int_0^1 E(s)h(s)ds \\ &= \int_0^1 H(t,s)h(s)ds \\ &= u(t). \end{aligned} \tag{2.21}$$

Therefore,

$$u(1-t) = u(t), \quad t \in [0,1], \tag{2.22}$$

that is, $u(t)$ is symmetric on [0,1]. □

Without loss of generality, all constants η_i in the boundary value condition (1.6) are placed in the interval $(0,1/2]$ because of the symmetry of the solution.

Lemma 2.4. *Let* $\alpha_i > 0$ *for* $i = 1,2,\dots,m-2$ *with* $\sum_{i=1}^{m-2}\alpha_i < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $h \in C^+[0,1]$. *Then the unique solution* $u(t)$ *of BVP (2.4)-(2.5) is nonnegative on* $[0,1]$, *and if* $h(t) \not\equiv 0$, *then* $u(t)$ *is positive on* $[0,1]$.

Proof. Let $h \in C^+[0,1]$. From the fact that $u''(t) = -h(t) \le 0$, $t \in [0,1]$, we know that $u(t)$ is concave on $[0,1]$. From (2.5) and (2.6), we have

$$u(1) = u(0) = \int_0^1 H(0,s)h(s)ds = \int_0^1 E(s)h(s)ds \ge 0. \tag{2.23}$$

It follows that $u(t) \ge 0$, $t \in [0,1]$, and if $h(t) \not\equiv 0$, then $u(t) > 0$, $t \in [0,1]$. □

From the proof of Lemma 2.4, we know that if $\sum_{i=1}^{m-2}\alpha_i > 1$, $h \in C^+[0,1]$, then the BVP (2.4)-(2.5) has no positive solution. So in order to obtain positive solution of the BVP (2.4)-(2.5), in the rest of the paper we assume that $\sum_{i=1}^{m-2}\alpha_i \in (0,1)$.

Lemma 2.5. *Let* $\sum_{i=1}^{m-2}\alpha_i \in (0,1)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} \le 1/2$, $h \in C^+[0,1]$. *Then the unique solution* $u(t)$ *of BVP (2.4)-(2.5) satisfies*

$$\min_{t\in[0,1]} u(t) \ge \gamma\|u\|, \tag{2.24}$$

where γ *is as in Lemma 2.2.*

Proof. Applying (2.6) and Lemma 2.2, we find that for $t \in [0,1]$,

$$u(t) = \int_0^1 H(t,s)h(s)ds \le \int_0^1 H(s,s)h(s)ds. \tag{2.25}$$

Therefore,

$$\|u\| \le \int_0^1 H(s,s)h(s)ds. \tag{2.26}$$

On the other hand, for any $t \in [0,1]$, by (2.7) and Lemma 2.2, we have

$$u(t) = \int_0^1 H(t,s)h(s)ds \ge \int_0^1 \gamma H(s,s)h(s)ds = \gamma\int_0^1 H(s,s)h(s)ds. \tag{2.27}$$

From (2.26) and (2.27) we know that (2.24) holds. □

We will use the following assumptions.

(A_1) $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} \le 1/2$, $\alpha_i > 0$ for $i = 1,2,\dots,m-2$, with $\sum_{i=1}^{m-2}\alpha_i < 1$;

(A_2) $a : (0,1) \to [0,\infty)$ is continuous, symmetric on $(0,1)$, and

$$0 < \int_0^1 H(s,s)a(s)ds < +\infty; \tag{2.28}$$

(A_3) $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous and $f(\cdot,x)$ is symmetric on $[0,1]$ for all $x \ge 0$.

Define

$$K = \{w \in C^+[0,1] : w(t) \text{ is symmetric, concave on } [0,1],\ \min_{0\le t\le 1} w(t) \ge \gamma \|w\|\}. \tag{2.29}$$

It is easy to see that K is a cone of nonnegative functions in $C[0,1]$. Define an integral operator $T : E \to E$ by

$$Tu(t) = \int_0^1 H(t,s)a(s)f(s,u(s))\,ds, \quad t \in [0,1]. \tag{2.30}$$

It is easy to see that BVP (1.5)-(1.6) has a solution $u = u(t)$ if and only if u is a fixed point of the operator T defined by (2.30).

Lemma 2.6. *Suppose that* (A_1), (A_2), *and* (A_3) *hold, then* T *is completely continuous and* $T(K) \subset K$.

Proof. $(Tu)''(t) = -a(t)f(t,u(t)) \le 0$ implies that Tu is concave, thus from Lemmas 2.3, 2.4, and 2.5, we know that $T(K) \subset K$. Now we will prove that the operator T is completely continuous. For $n \ge 2$, define a_n by

$$a_n(t) = \begin{cases} \inf_{0<s\le 1/n} a(s), & 0 < t \le \dfrac{1}{n}, \\ a(t), & \dfrac{1}{n} < t < 1 - d\dfrac{1}{n}, \\ \inf_{1-1/n\le s<1} a(s), & 1 - \dfrac{1}{n} \le t < 1, \end{cases} \tag{2.31}$$

and define $T_n : K \to K$ by

$$T_n u(t) = \int_0^1 H(t,s)a_n(s)f(s,u(s))\,ds. \tag{2.32}$$

Obviously, T_n is compact on K for any $n \ge 2$ by an application of Ascoli-Arzela theorem [28]. Denote $B_R = \{u \in K : \|u\| \le R\}$. We claim that T_n converges on B_R uniformly to T as $n \to \infty$. In fact, let $M_R = \max\{f(s,x) : (s,x) \in [0,1] \times [0,R]\}$, then $M_R < \infty$. Since $0 < \int_0^1 H(s,s)a(s)\,ds < +\infty$, by the absolute continuity of integral, we have

$$\lim_{n\to\infty} \int_{e(1/n)} H(s,s)a(s)\,ds = 0, \tag{2.33}$$

where $e(1/n) = [0,1/n] \cup [1-1/n,1]$. So, for any $t \in [0,1]$, fixed $R > 0$, and $u \in B_R$,

$$\begin{aligned} |T_n u(t) - Tu(t)| &= \left| \int_0^1 [a(s) - a_n(s)]H(t,s)f(s,u(s))\,ds \right| \\ &\le M_R \int_0^1 |a(s) - a_n(s)| H(t,s)\,ds \\ &\le M_R \int_{e(1/n)} a(s)H(s,s)\,ds \longrightarrow 0 (n \longrightarrow \infty), \end{aligned} \tag{2.34}$$

where we have used assumptions (A_1), (A_2), and (A_3) and the fact that $H(t,s) \le H(s,s)$ for $t,s \in [0,1]$. Hence the completely continuous operator T_n converges uniformly to T as $n \to \infty$ on any bounded subset of K, and therefore T is completely continuous. □

We will use the following notations:

$$f_0 = \liminf_{x\to+0} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_\infty = \liminf_{x\to+\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f^0 = \limsup_{x\to+0} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f^\infty = \limsup_{x\to+\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \tag{2.35}$$
$$\Lambda = \left(\int_0^1 H(s,s)a(s)ds\right)^{-1}.$$

Now we formulate a fixed point theorem which will be used in the sequel (cf. [29, 30]).

Theorem 2.7. *Let E be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

(A) $\|Tu\| \le \|u\|$, *for all* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, *for all* $u \in K \cap \partial\Omega_2$; *or*

(B) $\|Tu\| \ge \|u\|$, *for all* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, *for all* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. The existence of single positive solution

In this section, we will impose growth conditions on f which allow us to apply Theorem 2.7 with regard to obtaining the existence of at least one symmetric positive solution for BVP (1.5)-(1.6). We obtain the following existence results.

Theorem 3.1. *Assume that (A_1), (A_2), and (A_3) hold. If there exist two constants R_1, R_2 with $0 < R_1 \le \gamma R_2$ such that*

(D_1) $f(t,x) \le \Lambda R_1$, *for all* $(t,x) \in [0,1] \times [\gamma R_1, R_1]$, *and* $f(t,x) \ge (1/\gamma)\Lambda R_2$, *for all* $(t,x) \in [0,1] \times [\gamma R_2, R_2]$; *or*

(D_2) $f(t,x) \ge (1/\gamma)\Lambda R_1$, *for all* $(t,x) \in [0,1] \times [\gamma R_1, R_1]$, *and* $f(t,x) \le \Lambda R_2$, *for all* $(t,x) \in [0,1] \times [\gamma R_2, R_2]$,

then BVP (1.5)-(1.6) has at least one symmetric positive solution u^* satisfying

$$R_1 \le \|u^*\| \le R_2. \tag{3.1}$$

Proof. We only prove the case (D_1). Let

$$\Omega_1 = \{u : u \in E,\ \|u\| < R_1\}, \qquad \Omega_2 = \{u : u \in E,\ \|u\| < R_2\}. \tag{3.2}$$

For $u \in K$, from Lemma 2.5 we know that $\min_{0\le s\le 1} u(s) \ge \gamma\|u\|$. Therefore, for $u \in K \cap \partial\Omega_1$, we have $u(s) \in [\gamma R_1, R_1]$, $s \in [0,1]$, which imply that $f(s,u(s)) \le \Lambda R_1$. Thus for

$t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)a(s)f(s,u(s))\,ds \le \int_0^1 H(s,s)a(s)f(s,u(s))\,ds \\ &\le \Lambda R_1 \int_0^1 H(s,s)a(s)ds = R_1 = \|u\|. \end{aligned} \tag{3.3}$$

Therefore,

$$\|Tu\| \le \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.4}$$

On the other hand, for $u \in K \cap \partial\Omega_2$, we have $u(s) \in [\gamma R_2, R_2]$, $s \in [0,1]$, which imply that $f(s,u(s)) \ge (1/\gamma)\Lambda R_2$. Thus for $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)a(s)f(s,u(s))\,ds \ge \frac{1}{\gamma}\Lambda R_2 \int_0^1 H(t,s)a(s)ds \\ &\ge \frac{1}{\gamma}\Lambda R_2 \int_0^1 \gamma H(s,s)a(s)ds = R_2 = \|u\|, \end{aligned} \tag{3.5}$$

which implies that

$$\|Tu\| \ge \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{3.6}$$

Therefore, from (3.4), (3.6), and Theorem 2.7, it follows that T has a fixed point $u^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. So, u^* is a symmetric positive solution of BVP (1.5)-(1.6) with $R_1 \le \|u^*\| \le R_2$. □

Theorem 3.2. *Assume that* (A_1), (A_2), *and* (A_3) *hold. If one of the following conditions is satisfied:*

(D_3) $f_0 > (1/\gamma^2)\Lambda$ *and* $f^\infty < \Lambda$ *(particularly,* $f_0 = \infty$ *and* $f^\infty = 0$*),*

(D_4) $f^0 < \Lambda$ *and* $f_\infty > (1/\gamma^2)\Lambda$ *(particularly,* $f^0 = 0$ *and* $f_\infty = \infty$*),*

then BVP (1.5)-(1.6) has at least one symmetric positive solution.

Proof. We only prove the case (D_3). From $f_0 > (1/\gamma^2)\Lambda$, we know that there exists $R_1 > 0$ such that $f(s,x) \ge (1/\gamma^2)\Lambda x$ for $(s,x) \in [0,1] \times [0,R_1]$. Let $\Omega_1 = \{u : u \in E,\ \|u\| < R_1\}$, then for $u \in K \cap \partial\Omega_1$ and $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)a(s)f(s,u(s))\,ds \ge \frac{1}{\gamma^2}\Lambda \int_0^1 H(t,s)a(s)u(s)ds \\ &\ge \frac{1}{\gamma^2}\Lambda \int_0^1 \gamma G(s,s)a(s)\gamma\|u\|ds = \|u\|. \end{aligned} \tag{3.7}$$

Therefore,

$$\|Tu\| \ge \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.8}$$

On the other hand, from $f^\infty < \Lambda$ we know that there exists $\overline{R} > 0$ such that $f(s,x) \le \Lambda x$ for $(t,x) \in [0,1] \times (\overline{R}, \infty)$. Let $R_2 > \max\{R_1, (1/\gamma)\overline{R}\}$, and $\Omega_2 = \{u : u \in E,$

$|u\| < R_2\}$. Then, for $u \in K \cap \partial\Omega_2$, we have $u(s) \geq \gamma\|u\| = \gamma R_2 > \overline{R}$, which implies that $f(u(s)) \leq \Lambda u(s)$ for $s \in [0,1]$. Thus,

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)a(s)f(s,u(s))\,ds \leq \int_0^1 H(t,s)a(s)\Lambda u(s)\,ds \\ &\leq \Lambda \int_0^1 H(s,s)a(s)\|u\|\,ds = \|u\|. \end{aligned} \tag{3.9}$$

Hence we have

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{3.10}$$

Therefore, from (3.8), (3.10), and Theorem 2.7, it follows that T has a fixed point $u^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and thus u^* is a symmetric positive solution of BVP (1.5)-(1.6). ☐

THEOREM 3.3. *Assume that* (A_1), (A_2), *and* (A_3) *hold. If there exists two constants* R_1, R_2 *with* $0 < R_1 \leq R_2$ *such that*

(D_5) $f(t,\cdot)$ *is nondecreasing on* $[0,R_2]$ *for all* $t \in [0,1]$,

(D_6) $f(s,\gamma R_1) \geq (1/\gamma)\Lambda R_1$, *and* $f(t,R_2) \leq \Lambda R_2$ *for all* $t \in [0,1]$,

then BVP (1.5)-(1.6) has at least one symmetric positive solution u^* *satisfying*

$$R_1 \leq \|u^*\| \leq R_2. \tag{3.11}$$

Proof. Let

$$\Omega_1 = \{u : u \in E,\ \|u\| < R_1\}, \qquad \Omega_2 = \{u : u \in E,\ \|u\| < R_2\}. \tag{3.12}$$

For $u \in K$, from Lemma 2.5, we know that $\min_{0\leq t\leq 1} u(t) \geq \gamma\|u\|$. Therefore, for $u \in K \cap \partial\Omega_1$, we have $u(s) \geq \gamma\|u\| = \gamma R_1$ for $s \in [0,1]$, thus by (D_5) and (D_6), we have

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)a(s)f(s,u(s))\,ds \geq \int_0^1 H(t,s)a(s)f(s,\gamma R_1)\,ds \\ &\geq \int_0^1 \gamma H(s,s)a(s)\frac{1}{\gamma}\Lambda R_1\,ds = R_1 = \|u\|. \end{aligned} \tag{3.13}$$

Therefore,

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.14}$$

On the other hand, for $u \in K \cap \partial\Omega_2$, we have $u(s) \leq R_2$ for $s \in [0,1]$, thus by (D_5) and (D_6), we have

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)a(s)f(s,u(s))\,ds \leq \int_0^1 H(t,s)a(s)f(s,R_2)\,ds \\ &\leq \int_0^1 H(s,s)a(s)\Lambda R_2\,ds = R_2 = \|u\|. \end{aligned} \tag{3.15}$$

Hence we have

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{3.16}$$

Therefore, from (3.14), (3.16), and Theorem 2.7, it follows that T has a fixed point $u^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ satisfying $R_1 \leq \|u^*\| \leq R_2$, u^* is a symmetric positive solution of BVP (1.5)-(1.6). □

4. The existence of many positive solutions

Now we discuss the multiplicity of positive solutions for BVP (1.5)-(1.6). We obtain the following existence results.

Theorem 4.1. *Assume that* (A_1), (A_2) *and* (A_3) *hold. In addition, suppose that*

(D_7) $f_0 > (1/\gamma^2)\Lambda$ *and* $f_\infty > (1/\gamma^2)\Lambda$ *(particularly,* $f_0 = f_\infty = \infty$*);*

(D_8) *there exists a constant* ρ_1 *such that*

$$f(s,x) \leq \Lambda\rho_1, \quad (s,x) \in [0,1] \times [\gamma\rho_1, \rho_1]. \tag{4.1}$$

Then BVP (1.5)-(1.6) has at least two symmetric positive solutions u_1 *and* u_2 *satisfying* $0 < \|u_1\| \leq \rho_1 \leq \|u_2\|$.

Proof. At first, in view of $f_0 > (1/\gamma^2)\Lambda$, there exists $r \in (0, \rho_1)$ such that

$$f(s,x) \geq \frac{1}{\gamma^2}\Lambda x, \quad (s,x) \in [0,1] \times [0,r]. \tag{4.2}$$

Set $\Omega_r = \{u : u \in E,\ \|u\| < r\}$. Then for $u \in K \cap \partial\Omega_r$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds \geq \int_0^1 G(t,s)a(s)\frac{1}{\gamma^2}\Lambda u(s)\,ds \\ &\geq \frac{1}{\gamma^2}\Lambda \int_0^1 \gamma G(s,s)a(s)\gamma\|u\|\,ds = \|u\|, \end{aligned} \tag{4.3}$$

which implies that

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_r. \tag{4.4}$$

Next, since $f_\infty > (1/\gamma^2)\Lambda$, there exists $R \in (\rho_1, \infty)$ such that

$$f(s,x) \geq \frac{1}{\gamma^2}\Lambda x, \quad (s,x) \in [0,1] \times [0,R]. \tag{4.5}$$

Set $\Omega_R = \{u : u \in E,\ \|u\| < R\}$. For $u \in K$, from Lemma 2.5, we know that $u(s) \geq \gamma\|u\|$, for $s \in [0,1]$. Therefore, for $u \in K \cap \partial\Omega_R$, we have $u(s) \in [\gamma R, R]$, $s \in [0,1]$, which imply that $f(s,u(s)) \geq (1/\gamma^2)\Lambda u(s) \geq (1/\gamma)\Lambda\|u\|$. Thus,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds \geq \int_0^1 G(t,s)a(s)\frac{1}{\gamma}\Lambda\|u\|\,ds \\ &\geq \frac{1}{\gamma}\Lambda\int_0^1 \gamma G(s,s)a(s)\|u\|\,ds = \|u\|, \end{aligned} \tag{4.6}$$

which implies that

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_R. \tag{4.7}$$

Finally, set $\Omega_{\rho_1} = \{u : u \in E,\ \|u\| < \rho_1\}$. For any $u \in K \cap \partial\Omega_{\rho_1}$, we have $u(s) \in [\gamma\rho_1, \rho_1]$, $s \in [0,1]$. Thus, from (2.30) and (D_8), we obtain

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds \leq \int_0^1 G(t,s)a(s)\Lambda u(s)\,ds \\ &\leq \Lambda\int_0^1 G(s,s)a(s)\|u\|\,ds = \|u\|, \end{aligned} \tag{4.8}$$

which yields

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_{\rho_1}. \tag{4.9}$$

Hence, since $r < \rho_1 < R$, from (4.4), (4.7), and (4.9), it follows from Theorem 2.7 that T has a fixed point $u_1 \in K \cap (\overline{\Omega}_{\rho_1} \setminus \Omega_r)$, and a fixed point $u_2 \in K \cap (\overline{\Omega}_R \setminus \Omega_{\rho_1})$. Both are symmetric positive solutions of BVP (1.5)-(1.6). □

Remark 4.2. From the proof, we know that if (D_8) holds and $f_0 > (1/\gamma^2)\Lambda$ (or $f_\infty > (1/\gamma^2)\Lambda$), then BVP (1.5)- (1.6) has a symmetric positive solution u satisfying $0 < \|u\| \leq \rho_1$ (or $\|u\| \geq \rho_1$).

In a similar way, we can get the following results.

THEOREM 4.3. *Assume that (A_1), (A_2), and (A_3) hold. If the following conditions are satisfied.*

(D_9) $f^0 < \Lambda$ *and* $f^\infty < \Lambda$ *(particularly,* $f^0 = f^\infty = 0$*).*

(D_{10}) *There exists a constant* ρ_2 *such that*

$$f(s,x) \geq \frac{1}{\gamma^2}\Lambda\rho_2, \quad (s,x) \in [0,1] \times [\gamma\rho_2, \rho_2]. \tag{4.10}$$

Then BVP (1.5)-(1.6) has at least two symmetric positive solutions u_1 *and* u_2 *satisfying* $0 < \|u_1\| < \rho_2 < \|u_2\|$.

Remark 4.4. If (D_{10}) holds and $f^0 < \Lambda$ (or $f^\infty < \Lambda$), then BVP (1.5)-(1.6) has a symmetric positive solution u satisfying $0 < \|u\| \leq \rho_2$ (or $\|u\| \geq \rho_2$).

THEOREM 4.5. *Assume that (A_1), (A_2), and (A_3) hold. If there exist* $2n$ *positive numbers* r_k, R_k, $k = 1,2,\ldots,n$, *with* $r_1 < \gamma R_1 < r_2 < \gamma R_2 < \cdots < r_n < \gamma R_n$ *such that*

(D_{11}) *$f(s,x) \leq \Lambda r_k$ for $(s,x) \in [0,1] \times [\gamma r_k, r_k]$, and $f(s,x) \geq (1/\gamma)\Lambda R_k$ for $(s,x) \in [0,1] \times [\gamma R_k, R_k]$, $k = 1,2,\dots,n$; or*

(D_{12}) *$f(s,x) \geq (1/\gamma)\Lambda r_k$ for $(s,x) \in [0,1] \times [\gamma r_k, r_k]$, and $f(s,x) \leq \Lambda R_k$ for $(s,x) \in [0,1] \times [\gamma R_k, R_k]$, $k = 1,2,\dots,n$,*

then BVP (1.5)-(1.6) has n symmetric positive solutions u_k satisfying $r_k \leq \|u_k\| \leq R_k$ for $k = 1,2,\dots,n$.

Theorem 4.6. *Assume that (A_1), (A_2), and (A_3) hold. If there exist $2n$ positive numbers $r_1 < R_1 < r_2 < R_2 < \cdots < r_n < R_n$ such that*

(D_{13}) *$f(t,\cdot)$ is nondecreasing on $[0,R_n]$ for all $t \in [0,1]$;*

(D_{14}) *$f(s,\gamma r_k) \geq (1/\gamma)\Lambda r_k$, and $f(s,R_k) \leq \Lambda R_k$, $k = 1,2,\dots,n$ for all $s \in [0,1]$,*

then BVP (1.5)-(1.6) has n symmetric positive solutions u_k satisfying $r_k \leq \|u_k\| \leq R_k$, $k = 1,2,\dots,n$.

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