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Research Article

Existence and Nonexistence Results for a Class of Quasilinear Elliptic Systems

Said El Manouni and Kanishka Perera

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Using variational methods, we prove the existence and nonexistence of positive solutions for a class of (p,q)-Laplacian systems with a parameter.

1. Introduction

In a recent paper, Perera [1] studied the existence, multiplicity, and nonexistence of positive classical solutions of the p-Laplacian problem

$$\begin{aligned} -\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $1 < p < \infty$, $\lambda > 0$ is a parameter, and f is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying

$$|f(x,t)| \leq Ct^{p-1} \quad \forall (x,t), \tag{1.2}$$

where C denotes a generic positive constant. Assuming

(f_1) $\exists \delta > 0$ such that $F(x,t) := \int_0^t f(x,\tau)d\tau \leq 0$ when $t \leq \delta$,

(f_2) $\exists t_0 > 0$ such that $F(x,t_0) > 0$,

(f_3) $\limsup_{t\to\infty}(F(x,t)/t^p) \leq 0$ uniformly in x

and using variational methods, the author proved that there are $\underline{\lambda} < \overline{\lambda}$ such that (1.1) has no positive solution for $\lambda < \underline{\lambda}$ and at least two positive solutions $u_1 > u_2$ for $\lambda \geq \overline{\lambda}$. A similar result for the semilinear case $p = 2$ was proved by Maya and Shivaji [2].

In the present paper we consider the corresponding (p,q)-Laplacian system

$$\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v) \quad \text{in } \Omega, \\ -\Delta_q v &= \lambda F_v(x,u,v) \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $1 < p,\, q < \infty$ and F is a C^1-function on $\Omega \times [0,\infty) \times [0,\infty)$ satisfying

$$|F_t(x,t,s)| \le C t^{\alpha} s^{\beta+1}, \qquad |F_s(x,t,s)| \le C t^{\alpha+1} s^{\beta} \quad \forall (x,t,s) \tag{1.4}$$

for some $\alpha,\beta > 0$ with $(\alpha+1)/p + (\beta+1)/q = 1$. We will extend the results of Perera [1] to this system as follows.

Theorem 1.1. *There is a $\underline{\lambda}$ such that (1.3) has no positive solution for $\lambda < \underline{\lambda}$.*

Theorem 1.2. *Assume*
(F_1) $\exists \delta > 0$ *such that* $F(x,t,s) \le 0$ *when* $t^p + s^q \le \delta$*;*
(F_2) $\exists t_0, s_0 > 0$ *such that* $F(x,t_0,s_0) > 0$*;*
(F_3) $\limsup_{\substack{|(t,s)| \to \infty \\ t,s>0}} (F(x,t,s)/t^{\alpha+1}s^{\beta+1}) \le 0$ *uniformly in* x.
Then there is a $\overline{\lambda}$ such that (1.3) has at least two positive solutions for $\lambda \ge \overline{\lambda}$.

2. Proofs of Theorems 1.1 and 1.2

The first eigenvalue of the problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{\alpha-1} u |v|^{\beta+1} \quad \text{in } \Omega, \\ -\Delta_q v &= \lambda |u|^{\alpha+1} |v|^{\beta-1} v \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\alpha,\beta > 0$ with $(\alpha+1)/p + (\beta+1)/q = 1$ is positive and is given by

$$\begin{aligned} \lambda_1 = \inf \Big\{ &\int_\Omega \frac{\alpha+1}{p} |\nabla u|^p + \frac{\beta+1}{q} |\nabla v|^q : (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \\ &\int_\Omega |u|^{\alpha+1} |v|^{\beta+1} = 1 \Big\} \end{aligned} \tag{2.2}$$

(see de Thélin [3]). If (1.3) has a positive solution (u,v), testing the two equations in (1.3) by u and v, respectively, and using (1.4) give

$$\begin{aligned} \int_\Omega |\nabla u|^p &= \lambda \int_\Omega F_u(x,u,v) u \le \lambda C \int_\Omega |u|^{\alpha+1} |v|^{\beta+1}, \\ \int_\Omega |\nabla v|^q &= \lambda \int_\Omega F_v(x,u,v) v \le \lambda C \int_\Omega |u|^{\alpha+1} |v|^{\beta+1}, \end{aligned} \tag{2.3}$$

so

$$\int_\Omega \frac{\alpha+1}{p} |\nabla u|^p + \frac{\beta+1}{q} |\nabla v|^q \le \lambda C \int_\Omega |u|^{\alpha+1} |v|^{\beta+1} \tag{2.4}$$

and hence $\lambda \geq \lambda_1/C$ by (2.2), proving Theorem 1.1.

To prove Theorem 1.2, set $F(x,t,s) = 0$ if $t < 0$ or $s < 0$, and consider the C^1-functional

$$\Phi_\lambda(u,v) = \int_\Omega \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q - \lambda F(x,u,v) \tag{2.5}$$

on the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with the norm

$$||(u,v)|| = \|u\|_1 + \|v\|_2, \tag{2.6}$$

where

$$\|u\|_1 = \left(\int_\Omega |\nabla u|^p\right)^{1/p}, \qquad \|v\|_2 = \left(\int_\Omega |\nabla v|^q\right)^{1/q}. \tag{2.7}$$

If (u,v) is a critical point of Φ_λ, denoting by u^- and v^- the negative parts of u and v, respectively, we have

$$\begin{aligned} 0 &= (\Phi_{\lambda'}(u,v),(u^-,v^-)) \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u^- + |\nabla v|^{q-2}\nabla v\cdot\nabla v^- \\ &\quad - \lambda(F_u(x,u,v)u^- + F_v(x,u,v)v^-) = \|u^-\|_1^p + \|v^-\|_2^q, \end{aligned} \tag{2.8}$$

so $u,v \geq 0$. Furthermore, $u,v \in L^\infty(\Omega) \cap C^1(\Omega)$ by Anane [4] and DiBenedetto [5], so it follows from the Harnack inequality that either $u,v > 0$ or $u,v \equiv 0$ (see Trudinger [6]). Thus, nontrivial critical points of Φ_λ are positive solutions of (1.3).

By (1.4),

$$|F(x,t,s)| \leq C|t|^{\alpha+1}|s|^{\beta+1} \quad \forall (x,t,s) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{2.9}$$

Let $\gamma = 1/(\max\{\alpha,\beta\}+1)$. By (F_3), there is an $M_\lambda > 0$ such that

$$|(t,s)| \geq M_\lambda \Longrightarrow F(x,t,s) \leq \frac{\gamma\lambda_1}{2\lambda}|t|^{\alpha+1}|s|^{\beta+1}. \tag{2.10}$$

Combining (2.9) and (2.10) gives

$$\lambda F(x,t,s) \leq \frac{\gamma\lambda_1}{2}|t|^{\alpha+1}|s|^{\beta+1} + C_\lambda \quad \forall (x,t,s) \tag{2.11}$$

for some $C_\lambda > 0$. Hence,

$$\begin{aligned} \Phi_\lambda(u,v) &\geq \int_\Omega \gamma\left(\frac{\alpha+1}{p}|\nabla u|^p + \frac{\beta+1}{q}|\nabla v|^q - \frac{\lambda_1}{2}|u|^{\alpha+1}|v|^{\beta+1}\right) - C_\lambda \\ &\geq \delta(\|u\|_1^p + \|v\|_2^q) - C_\lambda\mu(\Omega), \end{aligned} \tag{2.12}$$

where $\delta = \min\{(\alpha+1)/p,(\beta+1)/q\}\gamma/2$ and μ denotes the Lebesgue measure in $\mathbb{R}^n$. So Φ_λ is bounded from below and coercive. This yields a global minimizer (u_1,v_1) since Φ_λ is weakly lower semicontinuous.

Lemma 2.1. *There is a* $\bar{\lambda}$ *such that* $\inf \Phi_\lambda < 0$, *and hence* $(u_1, v_1) \neq (0,0)$, *for* $\lambda \geq \bar{\lambda}$.

Proof. Taking a sufficiently large compact subset Ω' of Ω and $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that $u_0 = t_0$, $v_0 = s_0$ on Ω' and $0 \leq u_0 \leq t_0, 0 \leq v_0 \leq s_0$ on $\Omega \setminus \Omega'$, where t_0, s_0 are as in (F_2), we have

$$\int_\Omega F(x, u_0, v_0) \geq \int_{\Omega'} F(x, t_0, s_0) - C t_0^{\alpha+1} s_0^{\beta+1} \mu(\Omega \setminus \Omega') > 0, \tag{2.13}$$

so $\Phi_\lambda(u_0, v_0) < 0$ for λ large enough. □

Now we fix $\lambda \geq \bar{\lambda}$ and obtain a critical point (u_2, v_2) with $\Phi_\lambda(u_2, v_2) > 0$ via the mountain pass lemma, which will complete the proof since $\Phi_\lambda(0,0) = 0 > \Phi_\lambda(u_1, v_1)$.

Lemma 2.2. *The origin is a strict local minimizer of* Φ_λ.

Proof. Set $\Omega_{u,v} = \{x \in \Omega : |u(x)|^p + |v(x)|^q > \delta\}$. By (F_1), $F(x, u, v) \leq 0$ on $\Omega \setminus \Omega_{u,v}$ and hence

$$\Phi_\lambda(u, v) \geq \frac{1}{p} \|u\|_1^p + \frac{1}{q} \|v\|_2^q - \lambda \int_{\Omega_{u,v}} F(x, u, v). \tag{2.14}$$

By (2.9), Young's and Hölder's inequalities, and the Sobolev imbedding,

$$\begin{aligned} \int_{\Omega_{u,v}} F(x, u, v) &\leq C \int_{\Omega_{u,v}} |u|^{\alpha+1} |v|^{\beta+1} \leq C \int_{\Omega_{u,v}} \frac{\alpha+1}{p} |u|^p + \frac{\beta+1}{q} |v|^q \\ &\leq C \left(\mu(\Omega_{u,v})^{1-(p/r)} \|u\|_1^p + \mu(\Omega_{u,v})^{1-(q/s)} \|v\|_2^q \right), \end{aligned} \tag{2.15}$$

where $r = np/(n-p)$ if $p < n$, $r > p$ if $p \geq n$ and $s = nq/(n-q)$ if $q < n$, $s > q$ if $q \geq n$. Since

$$\mu(\Omega_{u,v}) \leq \frac{1}{\delta} \int_{\Omega_{u,v}} |u|^p + |v|^q \leq C \left(\|u\|_1^p + \|v\|_2^q \right) \longrightarrow 0 \quad \text{as } \|(u, v) \longrightarrow 0, \tag{2.16}$$

the conclusion follows from (2.14) and (2.15). □

Since Φ_λ is coercive, every Palais-Smale sequence is bounded and hence contains a convergent subsequence as usual. So the mountain pass lemma now gives a critical point (u_2, v_2) of Φ_λ at the level

$$c := \inf_{\gamma \in \Gamma} \max_{(u,v) \in \gamma([0,1])} \Phi_\lambda(u, v) > 0, \tag{2.17}$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)) : \gamma(0) = (0,0), \gamma(1) = (u_1, v_1)\}$ is the class of paths joining the origin to (u_1, v_1) (see Rabinowitz [7]).

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Said El Manouni: Department of Mathematics, Faculty of Sciences, Al-Imam University,
P.O. Box 90950, Riyadh 11623, Saudi Arabia
Email address: samanouni@imamu.edu.sa

Kanishka Perera: Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901, USA
Email address: kperera@fit.edu