

*Research Article*

## Generalizations of the Lax-Milgram Theorem

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We prove a linear and a nonlinear generalization of the Lax-Milgram theorem. In particular, we give sufficient conditions for a real-valued function defined on the product of a reflexive Banach space and a normed space to represent all bounded linear functionals of the latter. We also give two applications to singular differential equations.

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### 1. Introduction

The following generalization of the Lax-Milgram theorem was proved recently by An et al. in [1].

**THEOREM 1.1.** *Let  $X$  be a reflexive Banach space over  $\mathbb{R}$ , let  $\{X_n\}_{n \in \mathbb{N}}$  be an increasing sequence of closed subspaces of  $X$  and  $V = \bigcup_{n \in \mathbb{N}} X_n$ . Suppose that*

$$A : X \times V \longrightarrow \mathbb{R} \tag{1.1}$$

*is a real-valued function on  $X \times V$  for which the following hold:*

- (a)  $A_n = A|_{X_n \times X_n}$  is a bounded bilinear form, for all  $n \in \mathbb{N}$ ;
- (b)  $A(\cdot, v)$  is a bounded linear functional on  $X$ , for all  $v \in V$ ;
- (c)  $A$  is coercive on  $V$ , that is, there exists  $c > 0$  such that

$$A(v, v) \geq c \|v\|^2, \tag{1.2}$$

*for all  $v \in V$ .*

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Then, for each bounded linear functional  $v^*$  on  $V$ , there exists  $x \in X$  such that

$$A(x, v) = \langle v^*, v \rangle, \quad (1.3)$$

for all  $v \in V$ .

In this paper our aim is to prove a linear extension and a nonlinear extension of Theorem 1.1. In the linear case, we use a variant of a theorem due to Hayden [2, 3], and thus manage to substitute the coercivity condition in (c) of the previous theorem with a more general inf-sup condition. In the nonlinear case, we appropriately modify the notion of type  $M$  operator and use a surjectivity result for monotone, hemicontinuous, coercive operators. We also present two examples to illustrate the applicability of our results.

All Banach spaces considered are over  $\mathbb{R}$ . Given a Banach space  $X$ ,  $X^*$  will denote its dual and  $\langle \cdot, \cdot \rangle$  will denote their duality product. Moreover, if  $M$  is a subset of  $X$ , then  $M^\perp$  will denote its annihilator in  $X^*$  and if  $N$  is a subset of  $X^*$ , then  ${}^\perp N$  will denote its preannihilator in  $X$ .

### 2. The linear case

To prove our main result for the linear case, we need the following lemma which is a variant of [2, Theorem 12] and [3, Theorem 1].

LEMMA 2.1. *Let  $X$  be a reflexive Banach space, let  $Y$  be a Banach space and let*

$$A : X \times Y \longrightarrow \mathbb{R} \quad (2.1)$$

*be a bounded, bilinear form satisfying the following two conditions:*

- (a)  *$A$  is nondegenerate with respect to the second variable, that is, for each  $y \in Y \setminus \{0\}$ , there exists  $x \in X$  with  $A(x, y) \neq 0$ ;*
- (b) *there exists  $c > 0$  such that*

$$\sup_{\|y\|=1} |A(x, y)| \geq c\|x\|, \quad (2.2)$$

*for all  $x \in X$ .*

*Then, for every  $y^* \in Y^*$ , there exists a unique  $x \in X$  with*

$$A(x, y) = \langle y^*, y \rangle, \quad (2.3)$$

*for all  $y \in Y$ .*

*Proof.* Let  $T : X \rightarrow Y^*$  with  $\langle Tx, y \rangle = A(x, y)$ , for all  $x \in X$  and all  $y \in Y$ . Obviously,  $T$  is a bounded linear map. Since, by (b),  $\|Tx\| \geq c\|x\|$ , for all  $x \in X$ ,  $T$  is one to one. To complete the proof, we need to show that  $T$  is onto.

Since  $A$  is nondegenerate with respect to the second variable, we have that

$${}^\perp T(X) = \{y \in Y \mid A(x, y) = 0, \forall x \in X\} = \{0\}. \quad (2.4)$$

Hence

$$({}^\perp T(X))^\perp = Y^*, \tag{2.5}$$

and so by [4, Proposition 2.6.6],

$$\overline{T(X)}^{w^*} = Y^*. \tag{2.6}$$

Thus to show that  $T$  maps  $X$  onto  $Y^*$ , we need to prove that  $T(X)$  is  $w^*$ -closed in  $Y^*$ . To see that, let  $\{Tx_\lambda\}_{\lambda \in \Lambda}$  be a net in  $T(X)$  and let  $y^*$  be an element of  $Y^*$  such that

$$Tx_\lambda \xrightarrow{w^*} y^*. \tag{2.7}$$

Without loss of generality, we may assume, using the special case of the Krein-Šmulian theorem on  $w^*$ -closed linear subspaces (see [4, Corollary 2.7.12]), the proof of which is originally due to Banach [5, Theorem 5, page 124] for the separable case and due to Dieudonné [6, Theorem 23] for the general case, that  $\{Tx_\lambda\}_{\lambda \in \Lambda}$  is bounded. Thus, since  $\|Tx\| \geq c\|x\|$  for all  $x \in X$ , the net  $\{x_\lambda\}_{\lambda \in \Lambda}$  is also bounded. Hence, since  $X$  is reflexive, there exist a subnet  $\{x_{\lambda_\mu}\}_{\mu \in M}$  and an element  $x$  of  $X$  such that  $\{x_{\lambda_\mu}\}_{\mu \in M}$  converges weakly to  $x$ . Since  $T$  is  $w - w^*$  continuous,  $Tx_{\lambda_\mu} \xrightarrow{w^*} Tx$ . Hence  $Tx = y^*$ , and so  $T(X)$  is  $w^*$ -closed.  $\square$

*Remark 2.2.* An alternative proof of the previous lemma can be obtained using the closed range theorem.

We are now in a position to prove our main result for the linear case.

**THEOREM 2.3.** *Let  $X$  be a reflexive Banach space, let  $Y$  be a Banach space, let  $\Lambda$  be a directed set, let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be a family of closed subspaces of  $X$ , let  $\{Y_\lambda\}_{\lambda \in \Lambda}$  be an upwards directed family of closed subspaces of  $Y$ , and let  $V = \bigcup_{\lambda \in \Lambda} Y_\lambda$ . Suppose that*

$$A : X \times V \longrightarrow \mathbb{R} \tag{2.8}$$

*is a function for which the following hold:*

- (a)  $A_\lambda = A|_{X_\lambda \times Y_\lambda}$  is a bounded bilinear form, for all  $\lambda \in \Lambda$ ;
- (b)  $A(\cdot, v)$  is a bounded linear functional on  $X$ , for all  $v \in V$ ;
- (c)  $A_\lambda$  is nondegenerate with respect to the second variable, for all  $\lambda \in \Lambda$ ;
- (d) there exists  $c > 0$  such that for all  $\lambda \in \Lambda$ ,

$$\sup_{y \in Y_\lambda, \|y\|=1} |A_\lambda(x, y)| \geq c\|x\|, \tag{2.9}$$

*for all  $x \in X_\lambda$ .*

*Then, for each bounded linear functional  $v^*$  on  $V$ , there exists  $x \in X$  such that*

$$A(x, v) = \langle v^*, v \rangle, \tag{2.10}$$

*for all  $v \in V$ .*

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*Proof.* Let  $v^* \in V^*$ , and for each  $\lambda \in \Lambda$ , let  $v_\lambda^* = v^*|_{Y_\lambda}$ . For all  $\lambda \in \Lambda$ ,  $v_\lambda^*$  is a bounded linear functional on  $Y_\lambda$ . By hypothesis, for all  $\lambda \in \Lambda$ ,  $A_\lambda$  is a bounded bilinear form on  $X_\lambda \times Y_\lambda$  satisfying the two conditions of Lemma 2.1. Since for all  $\lambda \in \Lambda$ ,  $X_\lambda$  is a reflexive Banach space, we get that for each  $\lambda \in \Lambda$ , there exists a unique  $x_\lambda$  such that  $A_\lambda(x_\lambda, y) = \langle v_\lambda^*, y \rangle$ , for all  $y \in Y_\lambda$ . Since  $A$  satisfies condition (d), we get that for all  $\lambda \in \Lambda$ ,

$$c\|x_\lambda\| \leq \sup_{y \in Y_\lambda, \|y\|=1} |A_\lambda(x_\lambda, y)| = \sup_{y \in Y_\lambda, \|y\|=1} |\langle v_\lambda^*, y \rangle| \leq \|v^*\|. \quad (2.11)$$

So  $\{x_\lambda\}_{\lambda \in \Lambda}$  is a bounded net in  $X$ . Since  $X$  is reflexive, there exist a subnet  $\{x_{\lambda_\mu}\}_{\mu \in M}$  of  $\{x_\lambda\}_{\lambda \in \Lambda}$  and  $x$  in  $X$  such that  $\{x_{\lambda_\mu}\}_{\mu \in M}$  converges weakly to  $x$ .

We are going to prove that  $A(x, v) = \langle v^*, v \rangle$ , for all  $v \in V$ . Take  $v \in V$ . Then there exists some  $\lambda_0 \in \Lambda$  with  $v \in Y_{\lambda_0}$ . Since  $\{x_{\lambda_\mu}\}_{\mu \in M}$  is a subnet of  $\{x_\lambda\}_{\lambda \in \Lambda}$ , there exists some  $\mu_0 \in M$  with  $\lambda_{\mu_0} \geq \lambda_0$ . Hence, since the family  $\{Y_\lambda\}_{\lambda \in \Lambda}$  is upwards directed,

$$v \in Y_{\lambda_\mu}, \quad (2.12)$$

for all  $\mu \geq \mu_0$ . Thus, for all  $\mu \geq \mu_0$ ,

$$A_{\lambda_\mu}(x_{\lambda_\mu}, v) = \langle v_{\lambda_\mu}^*, v \rangle. \quad (2.13)$$

Therefore

$$\lim_{\mu \in M} A(x_{\lambda_\mu}, v) = \langle v^*, v \rangle. \quad (2.14)$$

Since  $A(\cdot, v)$  is a bounded linear functional on  $X$ ,

$$\lim_{\mu \in M} A(x_{\lambda_\mu}, v) = A(x, v). \quad (2.15)$$

Hence  $A(x, v) = \langle v^*, v \rangle$ . □

The following example illustrates the possible applicability of Theorem 2.3.

*Example 2.4.* Let  $a \in C^1(0, 1)$  be a decreasing function with  $\lim_{t \rightarrow 0} a(t) = \infty$  and  $a(t) \geq 0$ , for all  $t \in (0, 1)$ . We will establish the existence of a solution for the following Cauchy problem:

$$\begin{aligned} u' + a(t)u &= f \quad \text{a.e. on } (0, 1), \\ u(0) &= 0, \end{aligned} \quad (2.16)$$

where  $f \in L^2(0, 1)$ .

Let  $X = \{u \in H^1(0, 1) \mid u(0) = 0\}$  be equipped with the norm  $\|u\| = (\int_0^1 |u'|^2 dt)^{1/2}$ , which is equivalent to the original Sobolev norm, and  $Y = L^2(0, 1)$ . Note that  $X$  is a reflexive Banach space, being a closed subspace of  $H^1(0, 1)$ . Let  $\{\alpha_n\}_{n \in \mathbb{N}}$  be a decreasing sequence in  $(0, 1)$  with  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Define

$$X_n = \{u \in H^1(\alpha_n, 1) \mid u(\alpha_n) = 0\}, \quad Y_n = L^2(\alpha_n, 1) \quad (2.17)$$

(we can consider  $X_n$  and  $Y_n$  as closed subspaces of  $X$  and  $Y$ , resp., by extending their elements by zero outside  $(\alpha_n, 1)$ ). Also let  $V = \bigcup_{n=1}^{\infty} Y_n$ .

Let  $A : X \times V \rightarrow \mathbb{R}$  be the bilinear map defined by

$$A(u, v) = \int_0^1 u' v dt + \int_0^1 a(t) u v dt. \tag{2.18}$$

$A$  is well defined and  $A(\cdot, v)$  is a bounded linear functional on  $X$  for any  $v \in V$ .

Let  $A_n = A|_{X_n \times Y_n}$ .  $A_n$  be a bounded bilinear form since

$$|A_n(u, v)| \leq (1 + M_n) \|u\|_{X_n} \|v\|_{Y_n}, \tag{2.19}$$

where  $M_n$  is the bound of  $a$  on  $[\alpha_n, 1]$ . It should be noted that  $A$  is not bounded on the whole of  $X \times V$ .

To show that  $A_n$  is nondegenerate, let  $v \in Y_n$  and assume that  $A_n(u, v) = 0$  for all  $u \in X_n$ , that is,

$$\int_{\alpha_n}^1 (u' + a(t)u) v dt = 0, \quad \forall u \in X_n. \tag{2.20}$$

It is easy to see that the above implies that

$$\int_{\alpha_n}^1 w v dt = 0, \tag{2.21}$$

for any continuous function  $w$ , and therefore  $v = 0$ .

We next show that

$$\sup_{\|v\|=1, v \in Y_n} |A_n(u, v)| \geq \|u\|_{X_n}. \tag{2.22}$$

Define  $T_n : X_n \rightarrow Y_n^*$  by  $\langle T_n u, v \rangle = A_n(u, v)$ .  $T_n$  is a well-defined bounded linear operator and  $T_n u = u' + a(t)u$ . Hence

$$\begin{aligned} \|T_n u\|^2 &= \int_{\alpha_n}^1 |u' + a(t)u|^2 dt \\ &= \int_{\alpha_n}^1 |u'|^2 dt + \int_{\alpha_n}^1 a^2(t) |u|^2 dt + \int_{\alpha_n}^1 a(t) (u^2)' dt \\ &= \int_{\alpha_n}^1 |u'|^2 dt + \int_{\alpha_n}^1 (a^2(t) - a'(t)) |u|^2 dt + a(1)u^2(1) \geq \|u\|_{X_n}^2, \end{aligned} \tag{2.23}$$

since  $u(\alpha_n) = 0$ ,  $a$  is decreasing and  $a(t) \geq 0$  for all  $t \in (0, 1)$ .

All the hypotheses of Theorem 2.3 are hence satisfied and so if  $F \in V^*$  is defined by  $F(v) = \int_0^1 f v dt$ , then there exists  $u \in X$  such that

$$A(u, v) = F(v), \quad \forall v \in V. \tag{2.24}$$

Thus  $u$  satisfies (2.16).

### 3. The nonlinear case

We start by recalling some well-known definitions.

*Definition 3.1.* Let  $T : X \rightarrow X^*$  be an operator. Then  $T$  is said to be

- (i) monotone if  $\langle Tx - Ty, x - y \rangle \geq 0$ , for all  $x, y \in X$ ;
- (ii) hemicontinuous if for all  $x, y \in X$ ,  $T(x + ty) \xrightarrow{w} Tx$  as  $t \rightarrow 0^+$ ;
- (iii) coercive if

$$\lim_{\|x\| \rightarrow \infty} \frac{\langle Tx, x \rangle}{\|x\|} = \infty. \quad (3.1)$$

We also need the following generalization of the notion of type  $M$  operator (for the classical definition, see [7] or [8]).

*Definition 3.2.* Let  $X$  be a Banach space, let  $V$  be a linear subspace of  $X$ , and let

$$A : X \times V \longrightarrow \mathbb{R} \quad (3.2)$$

be a function. Then  $A$  is said to be of type  $M$  with respect to  $V$  if for any net  $\{v_\lambda\}_{\lambda \in \Lambda}$  in  $V$ ,  $x \in X$  and  $v^* \in V^*$ ;

- (a)  $v_\lambda \xrightarrow{w} x$ ;
  - (b)  $A(v_\lambda, v) \rightarrow \langle v^*, v \rangle$ , for all  $v \in V$ ;
  - (c)  $A(v_\lambda, v_\lambda) \rightarrow \langle \hat{v}^*, x \rangle$ , where  $\hat{v}^*$  is the extension of  $v^*$  on the closure of  $V$ ,
- imply that  $A(x, v) = \langle v^*, v \rangle$ , for all  $v \in V$ .

Our result is the following.

**THEOREM 3.3.** *Let  $X$  be a reflexive Banach space, let  $\Lambda$  be a directed set, let  $\{X_\lambda\}_{\lambda \in \Lambda}$  be an upwards directed family of closed subspaces of  $X$ , and let  $V = \bigcup_{\lambda \in \Lambda} X_\lambda$ . Suppose that*

$$A : X \times V \longrightarrow \mathbb{R} \quad (3.3)$$

*is a function for which the following hold:*

- (a)  *$A$  is of type  $M$  with respect to  $V$ ;*
- (b)  $\lim_{\|x\| \rightarrow \infty} A(x, x)/\|x\| = \infty$ ;
- (c)  $A_\lambda(x, \cdot) \in X_\lambda^*$ , for all  $\lambda \in \Lambda$  and all  $x \in X_\lambda$ , where  $A_\lambda$  is the restriction of  $A$  on  $X_\lambda \times X_\lambda$ ;
- (d) *the operator  $T_\lambda : X_\lambda \rightarrow X_\lambda^*$ , defined by  $\langle T_\lambda x, y \rangle = A_\lambda(x, y)$  for all  $x, y \in X_\lambda$ , is monotone and hemicontinuous for all  $\lambda \in \Lambda$ .*

*Then for each  $v^* \in V^*$ , there exists  $x \in X$  such that*

$$A(x, v) = \langle v^*, v \rangle, \quad (3.4)$$

*for all  $v \in V$ .*

*Proof.* As in the proof of Theorem 2.3, for each  $\lambda \in \Lambda$ , let  $v_\lambda^* = v^*|_{X_\lambda}$ . By the Browder-Minty theorem (see [8, Theorem 26.A]), a monotone, coercive, and hemicontinuous operator, from a real reflexive Banach space into its dual, is onto. Thus, by (b) and (d), for

each  $\lambda \in \Lambda$ , the operator  $T_\lambda$  is onto and so there exists  $x_\lambda \in X_\lambda$  such that

$$A_\lambda(x_\lambda, y) = \langle v_\lambda^*, y \rangle, \quad (3.5)$$

for all  $y \in X_\lambda$ . In particular  $A_\lambda(x_\lambda, x_\lambda) = \langle v_\lambda^*, x_\lambda \rangle$ , and hence by (b), we get that the net  $\{x_\lambda\}_{\lambda \in \Lambda}$  is bounded. Continuing as in the proof of Theorem 2.3 and applying the fact that  $A$  is of type  $M$  with respect to  $V$ , we get the required result.  $\square$

*Remark 3.4.* It should be noted that since a crucial point in the above proof is the existence and boundedness of the net  $\{x_\lambda\}_{\lambda \in \Lambda}$ , variants of the previous theorem could be obtained using in (b) and (d) alternative conditions corresponding to other surjectivity results.

We now apply Theorem 3.3 to a singular Dirichlet problem.

*Example 3.5.* Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . We consider the Dirichlet problem

$$\begin{aligned} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u}{\partial x_i} \right) + f(x, u) &= 0 \quad \text{a.e. on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (3.6)$$

where  $a \in L_{\text{loc}}^\infty(\Omega)$  and there exists  $c_1 > 0$  such that  $a(x) \geq c_1$  a.e. on  $\Omega$ , and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a monotone increasing (with respect to its second variable for each fixed  $x \in \Omega$ ) Carathéodory function, for which there exist  $h \in L^2(\Omega)$  and  $c_2 > 0$  such that

$$|f(x, u)| \leq h(x) + c_2|u|, \quad \forall x \in \Omega, u \in \mathbb{R}. \quad (3.7)$$

We will show that if the above hypotheses on  $a$  and  $f$  hold, then problem (3.6) has a weak solution, that is, that there exists a function  $u \in H_0^1(\Omega)$  with

$$\int_{\Omega} a(x) \nabla u \nabla v \, dx + \int_{\Omega} f(x, u) v \, dx = 0, \quad \forall v \in C_0^\infty(\Omega). \quad (3.8)$$

To this end, let  $X = H_0^1(\Omega)$ , let  $\{\Omega_n\}_{n \in \mathbb{N}}$  be an increasing sequence of open subsets of  $\Omega$  such that  $\overline{\Omega_n} \subseteq \Omega_{n+1}$  and

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \quad (3.9)$$

and  $X_n = H_0^1(\Omega_n)$ , for each  $n \in \mathbb{N}$ . Observe that we can consider each  $X_n$  as a closed subspace of  $X$  by extending its elements by zero outside  $\Omega_n$  and let

$$V = \bigcup_{n=1}^{\infty} X_n. \quad (3.10)$$

Finally, let

$$A : X \times V \longrightarrow \mathbb{R} \quad (3.11)$$

be the function defined by

$$A(u, v) = \int_{\Omega} a(x) \nabla u \nabla v \, dx + \int_{\Omega} f(x, u) v \, dx. \quad (3.12)$$

By  $a(x) \geq c_1$  a.e. on  $\Omega$ , the monotonicity of  $f$ , and the growth condition (3.7), we have

$$\begin{aligned} A(u, u) &= \int_{\Omega} a(x) |\nabla u|^2 \, dx + \int_{\Omega} f(x, u) u \, dx \\ &= \int_{\Omega} a(x) |\nabla u|^2 \, dx + \int_{\Omega} (f(x, u) - f(x, 0)) u \, dx + \int_{\Omega} f(x, 0) u \, dx \\ &\geq c_1 \|\nabla u\|_{L^2(\Omega)}^2 - \|h\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)}. \end{aligned} \quad (3.13)$$

Since by the Poincaré inequality  $\|\nabla u\|_{L^2(\Omega)}$  is equivalent to the norm of  $X$ , it follows that  $A$  is coercive.

Let  $A_n = A|_{X_n \times X_n}$ . Then, since  $a \in L_{\text{loc}}^{\infty}(\Omega)$ , it follows that  $a \in L^{\infty}(\Omega_n)$ , for all  $n \in \mathbb{N}$ . Combining this with (3.7), we have that

$$|A_n(u, v)| \leq c(u, n) \|v\|_{X_n}, \quad (3.14)$$

where  $c(u, n)$  is a positive constant depending on  $n$  and  $u$ . So the operator

$$T_n : X_n \longrightarrow X_n^*, \quad (3.15)$$

with  $\langle T_n u, v \rangle_{X_n} = A_n(u, v)$ , is well defined for all  $n \in \mathbb{N}$ . Let

$$T_{1,n}, T_{2,n} : X_n \longrightarrow X_n^* \quad (3.16)$$

be the operators defined by

$$\langle T_{1,n} u, v \rangle_{X_n} = \int_{\Omega_n} a(x) \nabla u \nabla v \, dx, \quad \langle T_{2,n} u, v \rangle_{X_n} = \int_{\Omega_n} f(x, u) v \, dx. \quad (3.17)$$

Then  $T_{1,n}$  is a monotone bounded linear operator. Using the monotonicity of  $f$ , it is easy to see that  $T_{2,n}$  is monotone. Finally, recalling that the Nemytskii operator corresponding to  $f$  is continuous (see, e.g., [8, Proposition 26.7]) and that the embedding of  $X_n$  into  $L^2(\Omega_n)$  is compact, we have that  $T_{2,n}$  is hemicontinuous. Thus  $T_n = T_{1,n} + T_{2,n}$  is monotone and hemicontinuous for all  $n \in \mathbb{N}$ .

To finish the proof, let  $u_n \xrightarrow{w} u$  in  $X$ . Then since for all  $v \in V$ ,

$$u \longmapsto \int_{\Omega} a(x) \nabla u \nabla v \, dx \quad (3.18)$$

is a bounded linear functional and, by the continuity of the Nemytskii operator and the compactness of the embedding of  $X$  into  $L^2(\Omega)$ ,

$$\int_{\Omega} f(x, u_n) v \, dx \longrightarrow \int_{\Omega} f(x, u) v \, dx, \quad (3.19)$$



for all  $v \in V$ , we get that

$$A(u_n, v) \rightarrow A(u, v), \quad \forall v \in V. \quad (3.20)$$

Thus  $A$  is of type  $M$  with respect to  $V$ . Applying now Theorem 3.3 we get that there exists  $u \in X$  such that  $A(u, v) = 0$  for all  $v \in V$ . Observing that  $C_0^\infty(\Omega)$  is contained in  $V$ , we get that  $u$  is the required weak solution of (3.6).

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