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Research Article

Existence and Uniqueness of Solutions for Singular Higher Order Continuous and Discrete Boundary Value Problems

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By mixed monotone method, the existence and uniqueness are established for singular higher-order continuous and discrete boundary value problems. The theorems obtained are very general and complement previous known results.

1. Introduction

In recent years, the study of higher-order continuous and discrete boundary value problems has been studied extensively in the literature (see [1–17] and their references). Most of the results told us that the equations had at least single and multiple positive solutions.

Recently, some authors have dealt with the uniqueness of solutions for singular higher-order continuous boundary value problems by using mixed monotone method, for example, see [6, 14, 15]. However, there are few works on the uniqueness of solutions for singular discrete boundary value problems.

In this paper, we state a unique fixed point theorem for a class of mixed monotone operators, see [6, 14, 18]. In virtue of the theorem, we consider the existence and uniqueness of solutions for the following singular higher-order continuous and discrete boundary value problems (1.1) and (1.2) by using mixed monotone method. We first discuss the existence and uniqueness of solutions for the following singular higher-order continuous boundary value problem

$$\begin{gathered} y^{(n)}(t)+\lambda q(t)(g(y)+h(y))=0, \quad 0<t<1,\ \lambda>0, \\ y^{(i)}(0)=y^{(n-2)}(1)=0, \quad 0\le i\le n-2, \end{gathered} \tag{1.1}$$

where $n \geq 2$, $q(t) \in C((0,1),(0,+\infty))$, $g : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing; $h : (0,+\infty) \to (0,+\infty)$ is continuous and nonincreasing, and h may be singular at $y = 0$.

Next, we consider the existence and uniqueness of solutions for the following singular higher-order discrete boundary value problem

$$\Delta^n y(i)+\lambda q(i+n-1)(g(y(i+n-1)) +h(y(i+n-1))) = 0, \quad i \in N = \{0,1,2,\dots,T-1\},\ \lambda>0,$$
$$\Delta^k y(0) = \Delta^{n-2} y(T+1) = 0, \quad 0 \leq k \leq n-2, \tag{1.2}$$

where $n \geq 2$, $N^+ = \{0,1,2,\dots,T+n\}$, $q(i) \in C(N^+,(0,+\infty))$, $g : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing; $h : (0,+\infty) \to (0,+\infty)$ is continuous and nonincreasing, and h may be singular at $y = 0$. Throughout this paper, the topology on N^+ will be the discrete topology.

2. Preliminaries

Let P be a normal cone of a Banach space E, and $e \in P$ with $\|e\| \leq 1$, $e \neq \theta$. Define

$$Q_e = \{x \in P \mid x \neq \theta,\ \text{there exist constants } m, M > 0 \text{ such that } me \leq x \leq Me\}. \tag{2.1}$$

Now we give a definition (see [18]).

Definition 2.1 (see [18]). Assume $A : Q_e \times Q_e \to Q_e$. A is said to be mixed monotone if $A(x,y)$ is nondecreasing in x and nonincreasing in y, that is, if $x_1 \leq x_2$ $(x_1,x_2 \in Q_e)$ implies $A(x_1,y) \leq A(x_2,y)$ for any $y \in Q_e$, and $y_1 \leq y_2$ $(y_1,y_2 \in Q_e)$ implies $A(x,y_1) \geq A(x,y_2)$ for any $x \in Q_e$. $x^* \in Q_e$ is said to be a fixed point of A if $A(x^*,x^*) = x^*$.

Theorem 2.2 (see [6, 14]). *Suppose that A: $Q_e \times Q_e \to Q_e$ is a mixed monotone operator and $\exists$ a constant α, $0 \leq \alpha < 1$, such that*

$$A\left(tx, \frac{1}{t}y\right) \geq t^\alpha A(x,y), \quad for\ x,y \in Q_e,\ 0 < t < 1. \tag{2.2}$$

Then A has a unique fixed point $x^* \in Q_e$. Moreover, for any $(x_0,y_0) \in Q_e \times Q_e$,

$$x_n = A(x_{n-1},y_{n-1}), \quad y_n = A(y_{n-1},x_{n-1}), \quad n = 1,2,\dots, \tag{2.3}$$

satisfy

$$x_n \longrightarrow x^*, \qquad y_n \longrightarrow x^*, \tag{2.4}$$

where

$$\|x_n - x^*\| = o(1-r^{\alpha^n}), \qquad \|y_n - x^*\| = o(1-r^{\alpha^n}), \tag{2.5}$$

$0 < r < 1$, r is a constant from (x_0,y_0).

Theorem 2.3 (see [6, 14, 18]). *Suppose that A: $Q_e \times Q_e \to Q_e$ is a mixed monotone operator and $\exists$ a constant $\alpha \in (0,1)$ such that* (2.2) *holds. If x_λ^* is a unique solution of equation*

$$A(x,x) = \lambda x, \quad (\lambda > 0) \tag{2.6}$$

in Q_e, then $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$, $\lambda \to \lambda_0$. If $0 < \alpha < 1/2$, then $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^* \geq x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$, and

$$\lim_{\lambda \to +\infty} \|x_\lambda^*\| = 0, \qquad \lim_{\lambda \to 0^+} \|x_\lambda^*\| = +\infty. \tag{2.7}$$

3. Uniqueness positive solution of differential equations (1.1)

This section discusses singular higher-order boundary value problem (1.1). Throughout this section, we let $G(t,s)$ be the Green's function to $-y''=0$, $y(0)=y(1)=0$, we note that

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1,\end{cases} \tag{3.1}$$

and one can show that

$$G(t,t)G(s,s)\le G(t,s)\le G(t,t),\quad \text{for } G(t,s)\le G(s,s),\ (t,s)\in[0,1]\times[0,1]. \tag{3.2}$$

Suppose that y is a positive solution of (1.1). Let

$$x(t)=y^{(n-2)}(t), \tag{3.3}$$

from $y^{(i)}(0)=y^{(n-2)}(1)=0, 0\le i\le n-2$, and Taylor Formula, we define operator $T: C^{(2)}[0,1]\to C^{(n)}[0,1]$, by

$$\begin{gathered} y(t)=Tx(t)=\int_0^t \frac{(t-s)^{n-3}}{(n-3)!}x(s)ds,\quad \text{for } 3\le n,\\ y(t)=Tx(t)=x(t),\quad \text{for } n=2.. \end{gathered} \tag{3.4}$$

Then we have

$$\begin{gathered} x^{(2)}(t)+\lambda f(t,Tx(t))=0,\quad 0<t<1,\ \lambda>0,\\ x(0)=x(1)=0. \end{gathered} \tag{3.5}$$

Then from (3.4), we have the next lemma.

Lemma 3.1. *If $x(t)$ is a solution of* (3.5)*, then $y(t)$ is a solution of* (1.1).

Further, if $y(t)$ is a solution of (1.1), imply that $x(t)$ is a solution of (3.5).

Let $P=\{x\in C[0,1]\mid x(t)\ge 0,\ \text{for all}\, t\in[0,1]\}$. Obviously, P is a normal cone of Banach space $C[0,1]$.

Theorem 3.2. *Suppose that there exists $\alpha\in(0,1)$ such that*

$$g(tx)\ge t^{\alpha}g(x), \tag{3.6}$$

$$h(t^{-1}x)\ge t^{\alpha}h(x), \tag{3.7}$$

for any $t\in(0,1)$ and $x>0$, and $q\in C((0,1),(0,\infty))$ satisfies

$$\int_0^1 \left[s^{n-1}(n-2s)\right]^{-\alpha}q(s)ds<+\infty. \tag{3.8}$$

Then (1.1) *has a unique positive solution $y_\lambda^*(t)$. And moreover, $0<\lambda_1<\lambda_2$ implies $y_{\lambda_1}^*\le y_{\lambda_2}^*$, $y_{\lambda_1}^*\ne y_{\lambda_2}^*$. If $\alpha\in(0,1/2)$, then*

$$\lim_{\lambda\to 0^+}\|y_\lambda^*\|=0,\qquad \lim_{\lambda\to+\infty}\|y_\lambda^*\|=+\infty. \tag{3.9}$$

Proof. Since (3.7) holds, let $t^{-1}x = y$, one has

$$h(y) \geq t^{\alpha} h(ty). \tag{3.10}$$

Then

$$h(ty) \leq \frac{1}{t^{\alpha}} h(y), \quad \text{for } t \in (0,1),\ y > 0. \tag{3.11}$$

Let $y = 1$. The above inequality is

$$h(t) \leq \frac{1}{t^{\alpha}} h(1), \quad \text{for } t \in (0,1). \tag{3.12}$$

From (3.7), (3.11), and (3.12), one has

$$h(t^{-1}x) \geq t^{\alpha} h(x), \quad h\left(\frac{1}{t}\right) \geq t^{\alpha} h(1), \quad h(tx) \leq \frac{1}{t^{\alpha}} h(x), \quad h(t) \leq \frac{1}{t^{\alpha}} h(1), \text{for } t \in (0,1),\ x > 0. \tag{3.13}$$

Similarly, from (3.6), one has

$$g(tx) \geq t^{\alpha} g(x), \quad g(t) \geq t^{\alpha} g(1), \quad \text{for } t \in (0,1),\ x > 0. \tag{3.14}$$

Let $t = 1/x$, $x > 1$, one has

$$g(x) \leq x^{\alpha} g(1), \quad \text{for } x \geq 1. \tag{3.15}$$

Let $e(t) = G(t,t) = t(1-t)$, and we define

$$Q_e = \left\{ x \in C[0,1] \mid \frac{1}{M} G(t,t) \leq x(t) \leq MG(t,t),\ t \in [0,1] \right\}, \tag{3.16}$$

where $M > 1$ is chosen such that

$$\begin{aligned} M > \max \Bigg\{ &\left[\lambda g(1) \int_0^1 q(s) ds + \lambda h(1) \int_0^1 \left(\frac{s^{n-1}(n-2s)}{n!} \right)^{-\alpha} q(s) ds \right]^{1/(1-\alpha)}, \\ &\left[\lambda g(1) \int_0^1 G(s,s) \left(\frac{s^{n-1}(n-2s)}{n!} \right)^{\alpha} q(s) ds + \lambda h(1) \int_0^1 G(s,s) q(s) ds \right]^{-1/(1-\alpha)} \Bigg\}. \end{aligned} \tag{3.17}$$

First, from (3.4) and (3.16), for any $x \in Q_e$, we have the following.
When $3 \leq n$,

$$\begin{aligned} \frac{1}{M} \frac{t^{n-1}(n-2t)}{n!} &\leq \int_0^t \frac{1}{M} G(s,s) \frac{(t-s)^{n-3}}{(n-3)!} ds \leq Tx(t) \\ &\leq \int_0^t MG(s,s) \frac{(t-s)^{n-3}}{(n-3)!} ds \leq M \frac{t^{n-1}(n-2t)}{n!} \leq M, \quad \text{for } t \in [0,1], \end{aligned} \tag{3.18}$$

when $n = 2$,

$$\frac{1}{M}\frac{t^{n-1}(n-2t)}{n!} \le Tx(t) = x(t) \le M\frac{t(n-2t)}{n!} \le M, \quad \text{for } t \in [0,1], \tag{3.19}$$

then

$$\frac{1}{M}\frac{t^{n-1}(n-2t)}{n!} \le Tx(t) \le M\frac{t^{n-1}(n-2t)}{n!} \le M, \quad \text{for } t \in [0,1]. \tag{3.20}$$

For any $x, y \in Q_e$, we define

$$A_\lambda(x,y)(t) = \lambda\int_0^1 G(t,s)q(s)[g(Tx(s)) + h(Ty(s))]ds, \quad \text{for } t \in [0,1]. \tag{3.21}$$

First, we show that $A_\lambda : Q_e \times Q_e \to Q_e$.

Let $x, y \in Q_e$, from (3.14), (3.15), and (3.20), we have

$$g(Tx(t)) \le g(M) \le M^\alpha g(1), \quad \text{for } t \in (0,1), \tag{3.22}$$

and from (3.13), we have

$$\begin{aligned} h(Ty(t)) &\le h\left(\frac{1}{M}\frac{t^{n-1}(n-2t)}{n!}\right) \le \left[\frac{t^{n-1}(n-2t)}{n!}\right]^{-\alpha} h\left(\frac{1}{M}\right) \\ &\le M^\alpha\left[\frac{t^{n-1}(n-2t)}{n!}\right]^{-\alpha} h(1), \quad \text{for } t \in (0,1). \end{aligned} \tag{3.23}$$

Then, from (3.2), (3.21), (3.22) and (3.23), we have

On the other hand, for any $x, y \in Q_e$, from (3.13) and (3.14), we have

$$\begin{gathered} g(Tx(t)) \ge g\left(\frac{1}{M}\frac{t^{n-1}(n-2t)}{n!}\right) \ge \left(\frac{t^{n-1}(n-2t)}{n!}\right)^\alpha g\left(\frac{1}{M}\right) \ge \left(\frac{t^{n-1}(n-2t)}{n!}\right)^\alpha \frac{1}{M^\alpha}g(1), \\ h(Ty(t)) \ge h(M) = h\left(\frac{1}{1/M}\right) \ge \frac{1}{M^\alpha}h(1), \quad \text{for } t \in (0,1). \end{gathered} \tag{3.24}$$

Thus, from (3.2), (3.21) and (3.24), we have

$$\begin{aligned} &A_\lambda(x,y)(t) \\ &\quad \ge \lambda G(t,t)\left\{\int_0^1 G(s,s)q(s)M^{-\alpha}\left[\frac{s^{n-1}(n-2s)}{n!}\right]^\alpha g(1)ds + \int_0^1 G(s,s)q(s)M^{-\alpha}h(1)ds\right. \\ &\quad \ge \frac{1}{M}G(t,t), \quad \text{for } t \in [0,1]. \end{aligned} \tag{3.25}$$

So, A_λ is well defined and $A_\lambda(Q_e \times Q_e) \subset Q_e$.

Next, for any $l \in (0,1)$, one has

$$\begin{aligned} A_\lambda(lx, l^{-1}y)(t) &= \lambda \int_0^1 G(t,s)q(s)\left[g(lTx(s)) + h\left(l^{-1}Ty(s)\right)\right]ds \\ &\geq \lambda \int_0^1 G(t,s)q(s)\left[l^\alpha g(Tx(s)) + l^\alpha h(Ty(s))\right]ds \\ &= l^\alpha A_\lambda(x,y)(t), \quad \text{for } t \in [0,1]. \end{aligned} \tag{3.26}$$

So the conditions of Theorems 2.2 and 2.3 hold. Therefore, there exists a unique $x_\lambda^* \in Q_e$ such that $A_\lambda(x^*, x^*) = x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (3.5) for given $\lambda > 0$. Moreover, Theorem 2.3 means that if $0 < \lambda_1 < \lambda_2$, then $x_{\lambda_1}^*(t) \leq x_{\lambda_2}^*(t)$, $x_{\lambda_1}^*(t) \not\equiv x_{\lambda_2}^*(t)$ and if $\alpha \in (0, 1/2)$, then

$$\lim_{\lambda \to 0^+} \left\| x_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| x_\lambda^* \right\| = +\infty. \tag{3.27}$$

Next, from Lemma 3.1 and (3.4), we get that $y_\lambda^* = Tx_\lambda^*$ is a unique positive solution of (1.1) for given $\lambda > 0$. Moreover, if $0 < \lambda_1 < \lambda_2$, then $y_{\lambda_1}^*(t) \leq y_{\lambda_2}^*(t)$, $y_{\lambda_1}^*(t) \not\equiv y_{\lambda_2}^*(t)$ and if $\alpha \in (0, 1/2)$, then

$$\lim_{\lambda \to 0^+} \left\| y_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| y_\lambda^* \right\| = +\infty. \tag{3.28}$$

This completes the proof. □

Example 3.3. Consider the following singular boundary value problem:

$$\begin{gathered} y^{(n)}(t) + \lambda\left(\mu y^a(t) + y^{-b}(t)\right) = 0, \quad t \in [0,1], \\ y^{(i)}(0) = y^{(n-2)}(1) = 0, \quad 0 \leq i \leq n-2, \end{gathered} \tag{3.29}$$

where $\lambda, a, b > 0$, $\mu \geq 0$, $\max\{a,b\} < 1/(n-1)$.

Applying Theorem 3.2, let $\alpha = \max\{a,b\} < 1/(n-1)$, $q(t) = 1$, $g(y) = \mu y^a$, $h(y) = y^{-b}$, then

$$\begin{gathered} g(ty) \geq t^\alpha g(y), \qquad h\left(t^{-1}\right) \geq t^\alpha h(y), \\ \textstyle\int_0^1 \left[s^{n-1}(n-2s)\right]^{-\alpha} ds < +\infty. \end{gathered} \tag{3.30}$$

Thus all conditions in Theorem 3.2 are satisfied. We can find (3.29) has a unique positive solution $y_\lambda^*(t)$. In addition, $0 < \lambda_1 < \lambda_2$ implies $y_{\lambda_1}^* \leq y_{\lambda_2}^*$, $y_{\lambda_1}^* \not\equiv y_{\lambda_2}^*$. If $\alpha = \max\{a,b\} \in (0, 1/2)$, then

$$\lim_{\lambda \to 0^+} \left\| y_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| y_\lambda^* \right\| = +\infty. \tag{3.31}$$

4. Uniqueness positive solution of difference equations (1.2)

This section discusses singular higher-order boundary value problem (1.2). Throughout this section, we let $K(i,j)$ be Green's function to $-\Delta^2 y(i) + u(i+1) = 0$, $i \in N$, $y(0) = y(T+1) = 0$, we note that

$$K(i,j) = \begin{cases} \dfrac{j(T+1-i)}{T+1}, & 0 \le j \le i-1, \\ \dfrac{i(T+1-j)}{T+1}, & i \le j \le T+1, \end{cases} \tag{4.1}$$

and one can show that

$$K(i,i) \ge K(i,j), \quad K(j,j) \ge K(i,j), \quad K(i,j) \ge \frac{K(i,i)}{T+1}, \quad \text{for } 0 \le i \le T+1,\ 1 \le j \le T. \tag{4.2}$$

Suppose that y is a positive solution of (1.2). Let

$$x(i) = \Delta^{n-2} y(i), \quad \text{for } 0 \le i \le T+1. \tag{4.3}$$

From $\Delta^i y(0) = \Delta^{n-2} y(T+1) = 0$, $0 \le i \le n-2$, and $\Delta^m y(i-1) = \Delta^{m-1} y(i) - \Delta^{m-1} y(i-1)$, so we define operator T, by

$$Tx(i) = y(i+n-1) = \sum_{l=1}^{i+1} C_{i-l+n-1}^{n-2} x(l), \quad \text{for } 0 \le i \le T. \tag{4.4}$$

Then

$$\begin{gathered} \Delta^2 x(i) + \lambda F(i+n-1, Tx(i)) = 0, \quad 0 \le i \le T-1,\ \lambda > 0, \\ x(0) = x(T+1) = 0. \end{gathered} \tag{4.5}$$

Lemma 4.1. *If $x(i)$ is a solution of* (4.5), *then $y(i)$ is a solutionn of* (1.2).

Proof. Since we remark that $x(i)$ is a solution of (4.5), if and only if

$$x(i) = \sum_{j=1}^{T} K(i,j) \lambda F(j+n-1, Tx(j)), \quad \text{for } 0 \le i \le T+1. \tag{4.6}$$

Let

$$Tx(i) = y(i+n-1), \quad \text{for } 0 \le i \le T. \tag{4.7}$$

From (4.4) we find $\Delta^i y(0) = \Delta^{n-2} y(T+1) = 0$, $0 \le i \le n-2$, and $x(i) = \Delta^{n-2} y(i)$, so that $y(i)$ is a solution of (1.2).

Further, if $y(i)$ is a solution of (1.2), imply that $x(i)$ is a solution of (4.5).

Let $P = \{x \in C(N^+, [0,+\infty)) \mid x(i) \ge 0, \text{ for all } i \in N^+\}$. Obviously, P is a normal cone of Banach space $C(N^+, [0,+\infty))$. □

Theorem 4.2. *Suppose that there exists $\alpha \in (0,1)$ such that*

$$
\begin{aligned}
g(tx) &\geq t^{\alpha} g(x), \\
h(t^{-1}x) &\geq t^{\alpha} g(x),
\end{aligned}
\tag{4.8}
$$

for any $t \in (0,1)$ and $x > 0$, and $q \in C(N^{+},(0,\infty))$.

Then (1.2) *has a unique positive solution $y_{\lambda}^{*}(i)$. And moreover, $0 < \lambda_1 < \lambda_2$ implies $y_{\lambda_1}^{*} \leq y_{\lambda_2}^{*}$, $y_{\lambda_1}^{*} \neq y_{\lambda_2}^{*}$. If $\alpha \in (0,1/2)$, then*

$$
\lim_{\lambda \to 0^{+}} \|y_{\lambda}^{*}\| = 0, \qquad \lim_{\lambda \to +\infty} \|y_{\lambda}^{*}\| = +\infty. \tag{4.9}
$$

Proof. The proof is the same as that of Theorem 3.2, from (4.12) and (4.13), one has

$$
\begin{gathered}
h(t^{-1}x) \geq t^{\alpha} h(x), \quad h\left(\frac{1}{t}\right) \geq t^{\alpha} h(1), \quad h(tx) \leq \frac{1}{t^{\alpha}} h(x), \quad h(t) \leq \frac{1}{t^{\alpha}} h(1), \quad \text{for } t \in (0,1),\ x > 0; \\
g(tx) \geq t^{\alpha} g(x),\ g(t) \geq t^{\alpha} g(1), \quad \text{for } t \in (0,1),\ x > 0.
\end{gathered}
\tag{4.10}
$$

$$
g(tx) \geq t^{\alpha} g(x),\ g(t) \geq t^{\alpha} g(1), \quad \text{for } t \in (0,1),\ x > 0. \tag{4.11}
$$

Let $t = 1/x$, $x > 1$, one has

$$
g(x) \leq x^{\alpha} g(1), \quad \text{for } x \geq 1. \tag{4.12}
$$

Let $e(i) = K(i,i)/(T+1)$, and we define

$$
Q_e = \left\{ x \in P \mid \frac{1}{M} e(i) \leq x(i) \leq M e(i),\ \text{for } 0 \leq i \leq T+1 \right\}, \tag{4.13}
$$

where $M > 1$ is chosen such that

$$
\begin{aligned}
M > \max \Bigg\{ &\Bigg[\lambda (T+1) g(1) \sum_{j=1}^{T} q(j+n-1) \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^{\alpha} \\
&\quad + \lambda (T+1)^{1+\alpha} h(1) \sum_{j=1}^{T} K^{-\alpha}(j,j) q(j+n-1) \Bigg]^{1/(1-\alpha)}; \\
&\Bigg[\lambda g(1) \sum_{j=1}^{T} q(j+n-1) \left(\frac{K(j,j)}{T+1} \right)^{\alpha} + \lambda h(1) \sum_{j=1}^{T} q(j+n-1) \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^{-\alpha} \Bigg]^{-1/(1-\alpha)} \Bigg\}.
\end{aligned}
\tag{4.14}
$$

From (4.4) and (4.13), for any $x \in Q_e$, we have

$$
\frac{1}{M} e(j) \leq Tx(j) = \sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} x(l) \leq M e(j) \sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2}, \quad \text{for } 0 \leq j \leq T. \tag{4.15}
$$

For any $x, y \in Q_e$, we define

$$A_\lambda(x,y)(i) = \lambda \sum_{j=1}^{T} K(i,j) q(j+n-1)[g(Tx(j)) + h(Ty(j))], \quad \text{for } 0 \le i \le T+1. \tag{4.16}$$

First we show that $A_\lambda : Q_e \times Q_e \to Q_e$.

Let $x, y \in Q_e$, from (4.11) and (4.12), we have

$$g(Tx(j)) \le g\left(Me(j) \sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right) \le g\left(M \sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right) \le M^\alpha \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^\alpha g(1), \quad \text{for } 1 \le j \le T, \tag{4.17}$$

and from (4.10), we have

$$h(Ty(j)) \le h\left(\frac{1}{M} e(j) \right) \le e^{-\alpha}(j) h\left(\frac{1}{M} \right) \le M^\alpha e^{-\alpha}(j) h(1), \quad \text{for } 1 \le j \le T. \tag{4.18}$$

Then, from (4.2) and the above, we have

$$\begin{aligned}
A_\lambda(x,y)(i) &\le \lambda K(i,i) \sum_{j=1}^{T} q(j+n-1)[g(Tx(j)) + (T+1) h(Ty(j))] \\
&\le e(i) M^\alpha \lambda (T+1) \left[g(1) \sum_{j=1}^{T} q(j+n-1) \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^\alpha \right. \\
&\qquad \left. + h(1) \sum_{j=1}^{T} e^{-\alpha}(j) q(j+n-1) \right] \\
&\le e(i) M^\alpha \lambda (T+1) \left[g(1) \sum_{j=1}^{T} q(j+n-1) \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^\alpha \right. \\
&\qquad \left. + h(1) \sum_{j=1}^{T} \left(\frac{K(j,j)}{T+1} \right)^{-\alpha} q(j+n-1) \right] \\
&\le Me(i), \quad \text{for } 0 \le i \le T+1.
\end{aligned} \tag{4.19}$$

On the other hand, for any $x, y \in Q_e$, from (4.10) and (4.12), we have

$$g(x(j)) \ge g\left(\frac{1}{M} e(j) \right) \ge e^\alpha(j) \frac{1}{M^\alpha} g(1), \quad \text{for } 1 \le j \le T, \tag{4.20}$$

$$h(y(j)) \ge h\left(Me(j) \sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right) \ge M^{-\alpha} \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^{-\alpha} h(1), \quad \text{for } 1 \le j \le T. \tag{4.21}$$

Thus, from (4.2) and (4.16), we have

$$\begin{aligned}
A_\lambda(x,y)(i) &\ge \lambda e(i) \left[\sum_{j=0}^{T} q(j+n-1) g(Tx(j)) + \sum_{j=0}^{T} q(j+n-1) h(Ty(j)) \right] \\
&\ge \lambda e(i) M^{-\alpha} \left[g(1) \sum_{j=0}^{T} q(j) \left(\frac{K(i,i)}{T+1} \right)^\alpha + h(1) \sum_{j=0}^{T} q(j+n-1) \left(\sum_{l=1}^{j+1} C_{j-l+n-1}^{n-2} \right)^{-\alpha} \right] \\
&\ge \frac{1}{M} e(i), \quad \text{for } 0 \le i \le T+1.
\end{aligned} \tag{4.22}$$

So, A_λ is well defined and $A_\lambda(Q_e \times Q_e) \subset Q_e$.

Next, for any $l \in (0,1)$, one has

$$\begin{aligned} A_\lambda(lx, l^{-1}y)(i) &= \lambda \sum_{j=1}^{T} K(i,j) q(j+n-1) \left[g(T(lx)(j)) + h\left(T\left(l^{-1}\right)y(j)\right)\right] \\ &= \lambda \sum_{j=1}^{T} K(i,j) q(j+n-1) \left[g(lTx(j)) + h\left(l^{-1}Ty(j)\right)\right] \\ &\geq \lambda \sum_{j=1}^{T} K(i,j) q(j+n-1) \left[l^{\alpha} g(Tx(j)) + l^{\alpha} h(Ty(j))\right] ds \\ &= l^{\alpha} A_\lambda(x,y)(i), \quad \text{for } 0 \leq i \leq T+1. \end{aligned} \tag{4.23}$$

So the conditions of Theorems 2.2 and 2.3 hold. Therefore, there exists a unique $x_\lambda^* \in Q_e$ such that $A_\lambda(x^*, x^*) = x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (4.5) for given $\lambda > 0$. Moreover, Theorem 2.3 means that if $0 < \lambda_1 < \lambda_2$, then $x_{\lambda_1}^*(t) \leq x_{\lambda_2}^*(t)$, $x_{\lambda_1}^*(t) \neq x_{\lambda_2}^*(t)$ and if $\alpha \in (0, 1/2)$, then

$$\lim_{\lambda \to 0^+} \left\| x_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| x_\lambda^* \right\| = +\infty. \tag{4.24}$$

Next, on using Lemma 3.1, from (4.5), we get that $y_\lambda^* = Tx_\lambda^*$ is a unique positive solution of (1.2) for given $\lambda > 0$. Moreover, if $0 < \lambda_1 < \lambda_2$, then $y_{\lambda_1}^*(t) \leq y_{\lambda_2}^*(t)$, $y_{\lambda_1}^*(t) \neq y_{\lambda_2}^*(t)$ and if $\alpha \in (0, 1/2)$, then

$$\lim_{\lambda \to 0^+} \left\| y_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| y_\lambda^* \right\| = +\infty. \tag{4.25}$$

This completes the proof. □

Example 4.3. Consider the following singular boundary value problem:

$$\begin{gathered} \Delta^n y(i-1) + \lambda\left(\mu y^a(i) + y^{-b}(i)\right) = 0, \quad i \in N, \\ \Delta^i y(0) = \Delta^{n-2} y(1) = 0, \quad 0 \leq i \leq n-2, \end{gathered} \tag{4.26}$$

where $\lambda, a, b > 0$, $\mu \geq 0$, $\max\{a, b\} < 1$.

Let $q(i) = 1$, $g(y) = \mu y^a$, $h(y) = y^{-b}$, $\alpha = \max\{a, b\} < 1$, then

$$g(ty) \geq t^{\alpha} g(y), \qquad h\left(t^{-1}y\right) \geq t^{\alpha} h(y), \tag{4.27}$$

thus all conditions in Theorem 4.2 are satisfied. We can find (4.26) has a unique positive solution $y_\lambda^*(t)$. In addition, $0 < \lambda_1 < \lambda_2$ implies $y_{\lambda_1}^* \leq y_{\lambda_2}^*$, $y_{\lambda_1}^* \neq y_{\lambda_2}^*$. If $\alpha = \max\{a, b\} \in (0, 1/2)$, then

$$\lim_{\lambda \to 0^+} \left\| y_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| y_\lambda^* \right\| = +\infty. \tag{4.28}$$

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