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Research Article

On Periodic Solutions of Higher-Order Functional Differential Equations

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For higher-order functional differential equations and, particularly, for nonautonomous differential equations with deviated arguments, new sufficient conditions for the existence and uniqueness of a periodic solution are established.

1. Statement of the main results

1.1. Statement of the problem

Let $n \geq 2$ be a natural number, $\omega > 0$, L_ω the space of ω-periodic and Lebesgue integrable on $[0,\omega]$ functions $u : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|u\|_{L_\omega} = \int_0^\omega |u(s)|\,ds. \tag{1.1}$$

Let C_ω and C_ω^{n-1} be, respectively, the spaces of continuous and $(n-1)$-times continuously differentiable ω-periodic functions with the norms

$$\|u\|_{C_\omega} = \max\{|u(t)| : t \in \mathbb{R}\}, \qquad \|u\|_{C_\omega^{n-1}} = \sum_{k=1}^{n} \|u^{(k-1)}\|_{C_\omega}, \tag{1.2}$$

and let $\widetilde{C}_\omega^{n-1}$ be the space of functions $u \in C_\omega^{n-1}$ for which $u^{(n-1)}$ is absolutely continuous.

We consider the functional differential equation

$$u^{(n)}(t) = f(u)(t), \tag{1.3}$$

whose important particular case is the differential equation with deviated arguments

$$u^{(n)}(t) = g\big(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\big). \tag{1.4}$$

Throughout the paper, it is assumed that $f : C_\omega^{n-1} \to L_\omega$ is a continuous operator satisfying the condition

$$f_r^*(\cdot) = \sup\{|f(u)(\cdot)| : \|u\| \le r\} \in L_\omega \quad \text{for any } r > 0, \tag{1.5}$$

and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function from the Carathéodory class, satisfying the equality

$$g(t+\omega, x_1, \dots, x_n) = g(t, x_1, \dots, x_n) \tag{1.6}$$

for almost all $t \in \mathbb{R}$ and all $(x_1, \dots, x_n) \in \mathbb{R}^n$. As for the functions $\tau_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, n)$, they are measurable on each finite interval and

$$\frac{(\tau_k(t+\omega) - \tau_k(t))}{\omega} \text{ is an integer } (k = 1, \dots, n) \tag{1.7}$$

for almost all $t \in \mathbb{R}$.

A function $u \in \widetilde{C}_\omega^{n-1}$ is said to be an ω-periodic solution of (1.3) or (1.4) if it satisfies this equation almost everywhere on $\mathbb{R}$.

For the case $\tau_k(t) \equiv t (k = 1, \dots, n)$, the problem on the existence and uniqueness of an ω-periodic solution of (1.4) has been investigated in detail (see, e.g., [1–18] and the references therein). For (1.3) and (1.4), where $\tau_k(t) \not\equiv t$ $(k = 1, \dots, n)$, the mentioned problem is studied mainly in the cases $n \in \{1, 2\}$ (see [19–31]), and for the case $n > 2$, the problem remains so far unstudied. The present paper is devoted exactly to this case.

Everywhere below the following notation will be used:

$$\nu_k = \frac{\omega}{2}\left(\frac{\omega}{2\pi}\right)^{n-k-2} \quad (k = 0, \dots, n-2), \quad \nu_{n-1} = 1, \tag{1.8}$$

$$[x]_- = (|x| - x)/2 \quad \text{for } x \in \mathbb{R}, \tag{1.9}$$

$$\mu(u) = \min\{|u(t)| : 0 \le t \le \omega\} \quad \text{for } u \in C_\omega. \tag{1.10}$$

1.2. Existence theorems

The existence of an ω-periodic solution of (1.3) is proved in the cases where the operator f in the space C_ω^{n-1} satisfies the conditions

$$\left(\int_0^\omega f(u)(s)ds\right)\operatorname{sgn}(\sigma u(0)) \ge h(\mu(u)) - \sum_{k=1}^{n-1} \ell_{1k}\|u^{(k)}\|_{C_\omega} - c \quad \text{for } \mu(u) > 0, \tag{1.11}$$

$$\left|\int_t^x f(u)(s)ds\right| \le \ell h(\mu(u)) + \sum_{k=1}^{n-1} \ell_{2k}\|u^{(k)}\|_{C_\omega} + c \quad \text{for } 0 \le t \le x \le \omega, \tag{1.12}$$

or the conditions

$$\left(\int_0^\omega f(u)(s)ds\right)\operatorname{sgn}(\sigma u(0)) \geq 0 \quad \text{for } \mu(u) > c_0, \tag{1.13}$$

$$\left|\int_t^x f(u)(s)ds\right| \leq c_0 + \sum_{k=0}^{n-1} \ell_k \left\|u^{(k)}\right\|_{C_\omega} \quad \text{for } 0 \leq t \leq x \leq \omega. \tag{1.14}$$

Theorem 1.1. *Let there exist an increasing function $h : [0,+\infty[\to [0,+\infty[$ and constants $c \geq 0$, $\ell_{ik} \geq 0$ $(i = 1,2;\ k = 1,\dots,n-1)$, $\ell \geq 1$, and $\sigma \in \{-1,1\}$ such that $h(x) \to +\infty$ as $x \to +\infty$,*

$$\sum_{k=1}^{n-1} (\ell \ell_{1k} + \ell_{2k}) \nu_k < 1, \tag{1.15}$$

and inequalities (1.11) *and* (1.12) *are satisfied in the space C_ω^{n-1}. Then* (1.3) *has at least one ω-periodic solution.*

Theorem 1.2. *Let there exist constants $c_0 \geq 0$, $\ell_k \geq 0$ $(k = 0,\dots,n-1)$, and $\sigma \in \{-1,1\}$ such that*

$$\sum_{k=0}^{n-1} \ell_k \nu_k < 1, \tag{1.16}$$

and inequalities (1.13) *and* (1.14) *are satisfied in the space C_ω^{n-1}. Then* (1.3) *has at least one ω-periodic solution.*

Theorems 1.1 and 1.2 imply the following propositions.

Corollary 1.3. *Let there exist constants $\lambda > 0$, $\sigma \in \{-1,1\}$, and functions $p_{ik} \in L_\omega$ $(i,k = 1,\dots,n)$, $q \in L_\omega$ such that the inequalities*

$$\begin{gathered} g(t,x_1,\dots,x_n)\operatorname{sgn}(\sigma x_1) \geq p_{11}(t)|x_1|^\lambda - \sum_{k=2}^{n} p_{1k}(t)|x_k| - q(t), \\ |g(t,x_1,\dots,x_n)| \leq p_{21}(t)|x_1|^\lambda + \sum_{k=2}^{n} p_{2k}(t)|x_k| + q(t) \end{gathered} \tag{1.17}$$

hold on the set $\mathbb{R} \times \mathbb{R}^n$. Let, moreover,

$$\int_0^\omega p_{11}(t)dt > 0, \tag{1.18}$$

and either $\lambda < 1$ and

$$\sum_{k=2}^{n} \nu_{k-1} \int_0^\omega (\ell p_{1k}(s) + p_{2k}(s))ds < 1, \tag{1.19}$$

or $\lambda = 1$ and

$$\nu_0 \int_0^\omega (\ell[p_{11}(s)]_- + p_{21}(s))ds + \sum_{k=2}^{n} \nu_{k-1} \int_0^\omega (\ell p_{1k}(s) + p_{2k}(s))ds < 1, \tag{1.20}$$

where $\ell = \int_0^\omega p_{21}(t)dt / \int_0^\omega p_{11}(t)dt$. Then (1.4) *has at least one ω-periodic solution.*

Corollary 1.4. *Let there exist constants* $c_0 \geq 0$, $\sigma \in \{-1,1\}$, *and functions* $g_0 \in L_\omega$, $p_k \in L_\omega$ $(k = 1,\ldots,n)$, $q \in L_\omega$ *such that*

$$\int_0^\omega g_0(s)ds = 0 \tag{1.21}$$

and the inequalities

$$\begin{gathered}(g(t,x_1,\ldots,x_n) - g_0(t))\operatorname{sgn}(\sigma x_1) \geq 0 \quad \text{for } |x_1| > c_0,\\ |g(t,x_1,\ldots,x_n)| \leq \sum_{k=1}^n p_k(t)|x_k| + q(t)\end{gathered} \tag{1.22}$$

hold on the set $\mathbb{R} \times \mathbb{R}^n$. *If, moreover,*

$$\sum_{k=1}^n \nu_{k-1}\int_0^\omega p_k(s)ds < 1, \tag{1.23}$$

then (1.4) *has at least one* ω*-periodic solution.*

1.3. Uniqueness theorems

The unique solvability of a periodic problem for (1.3) is proved in the cases where the operator f, for any u and $v \in C_\omega^{n-1}$, satisfies the conditions:

$$\left(\int_0^\omega (f(u+v)(s) - f(v)(s))ds\right)\operatorname{sgn}(\sigma u(0)) \geq \ell_{10}\mu(u) - \sum_{k=1}^{n-1}\ell_{1k}\|u^{(k)}\|_{C_\omega} \quad \text{for } \mu(u) > 0, \tag{1.24}$$

$$\left|\int_t^x (f(u+v)(s) - f(v)(s))ds\right| \leq \ell_{20}\mu(u) + \sum_{k=1}^{n-1}\ell_{2k}\|u^{(k)}\|_{C_\omega} \quad \text{for } 0 \leq t \leq x \leq \omega, \tag{1.25}$$

or the conditions

$$\left(\int_0^\omega (f(u+v)(s) - f(v)(s))ds\right)\operatorname{sgn}(\sigma u(0)) > 0 \quad \text{for } \mu(u) > 0, \tag{1.26}$$

$$\left|\int_t^x (f(u+v)(s) - f(v)(s))ds\right| \leq \ell_0\|u\|_{C_\omega} \quad \text{for } 0 \leq t \leq x \leq \omega. \tag{1.27}$$

Theorem 1.5. *Let there exist constants* $\ell_{20} \geq \ell_{10} > 0$, $\ell_{ik} \geq 0$ $(i = 1,2;\ k = 1,\ldots,n-1)$, *and* $\sigma \in \{-1,1\}$ *such that for arbitrary* $u, v \in C_\omega^{n-1}$ *the operator* f *satisfies inequalities* (1.24) *and* (1.25). *If, moreover, inequality* (1.15) *holds, where* $\ell = \ell_{20}/\ell_{10}$, *then* (1.3) *has one and only one* ω*-periodic solution.*

Theorem 1.6. *Let there exist constants* $\ell_0 > 0$ *and* $\sigma \in \{-1,1\}$ *such that for arbitrary* $u, v \in C_\omega^{n-1}$ *an operator* f *satisfies conditions* (1.26) *and* (1.27). *If, moreover,*

$$\int_0^\omega f(0)(s)ds = 0, \qquad \ell_0\nu_0 < 1, \tag{1.28}$$

then (1.3) *has one and only one* ω*-periodic solution.*

From Theorem 1.5, the following corollary holds.

Corollary 1.7. *Let there exist a constant $\sigma \in \{-1,1\}$ and functions $p_{ik} \in L_\omega$ $(i = 1,2;\ k = 1,\dots,n)$ such that for almost all $t \in \mathbb{R}$ and all $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n) \in \mathbb{R}^n$ the conditions*

$$\begin{gathered}\big(g(t,x_1,\dots,x_n) - g(t,y_1,\dots,y_n)\big)\operatorname{sgn}\big(\sigma(x_1-y_1)\big) \ge p_{11}(t)\left|x_1-y_1\right| - \sum_{k=2}^{n} p_{1k}(t)\left|x_k-y_k\right|, \\ \left|g(t,x_1,\dots,x_n) - g(t,y_1,\dots,y_n)\right| \le \sum_{k=1}^{n} p_{2k}(t)\left|x_k-y_k\right|\end{gathered} \tag{1.29}$$

are satisfied. If, moreover, inequalities (1.18) *and* (1.20) *hold, where $\ell = \int_0^\omega p_{21}(s)ds / \int_0^\omega p_{11}(s)ds$, then* (1.4) *has one and only one ω-periodic solution.*

Note that the functions p_{1k} $(k = 2,\dots,n)$ and p_{2k} $(k = 1,\dots,n)$ in this corollary as in Corollary 1.3 are nonnegative, and p_{11} may change its sign.

Consider now the equation

$$u^{(n)}(t) = g\big(t, u(\tau(t))\big), \tag{1.30}$$

which is derived from (1.4) in the case where $g(t,x_1,\dots,x_n) \equiv g(t,x_1)$ and $\tau_1(t) \equiv \tau(t)$. As above, we will assume that the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to the Carathéodory class and

$$g(t+\omega, x) = g(t,x) \tag{1.31}$$

for almost all $t \in \mathbb{R}$ and all $x \in \mathbb{R}$. As for the function $\tau : \mathbb{R} \to \mathbb{R}$, it is measurable on each finite interval and

$$\frac{\big(\tau(t+\omega) - \tau(t)\big)}{\omega} \text{ is an integer} \tag{1.32}$$

for almost all $t \in \mathbb{R}$.

Theorem 1.6 yields the following corollary.

Corollary 1.8. *Let there exist a constant $\sigma \in \{-1,1\}$ and a function $p \in L_\omega$ such that the condition*

$$0 < \big(g(t,x) - g(t,y)\big)\operatorname{sgn}\big(\sigma(x-y)\big) \le p(t)|x-y| \tag{1.33}$$

holds for almost all $t \in \mathbb{R}$ and all $x \ne y$. If, moreover,

$$\int_0^\omega g(s,0)ds = 0, \qquad \nu_0 \int_0^\omega p(s)ds < 1, \tag{1.34}$$

then (1.30) *has one and only one ω-periodic solution.*

2. Auxiliary propositions

2.1. Lemmas on a priori estimates

Everywhere in this section, we will assume that ν_k $(k = 0, \ldots, n-1)$ are numbers given by (1.13).

Lemma 2.1. *If* $u \in C_\omega^{n-1}$, *then*

$$\|u\|_{C_\omega} \le \mu(u) + \nu_0 \|u^{(n-1)}\|_{C_\omega}, \tag{2.1}$$

$$\|u^{(k)}\|_{C_\omega} \le \nu_k \|u^{(n-1)}\|_{C_\omega} \quad (k = 1, \ldots, n-1). \tag{2.2}$$

Proof. We choose $t_0 \in [0, \omega]$ so that

$$u(t_0) = \mu(u), \tag{2.3}$$

and suppose

$$v(t) = u(t) - u(t_0). \tag{2.4}$$

Then $v(t_0) = v(t_0 + \omega) = 0$. Thus

$$|v(t)| = \left| \int_{t_0}^{t} v'(s)ds \right| \le \int_{t_0}^{t} |v'(s)|ds, \quad |v(t)| = \left| \int_{t}^{t_0+\omega} v'(s)ds \right| \le \int_{t}^{t_0+\omega} |v'(s)|ds \quad \text{for } 0 \le t \le \omega. \tag{2.5}$$

If we sum up these two inequalities, we obtain

$$2|v(t)| \le \int_{t_0}^{t_0+\omega} |v'(s)|ds \quad \text{for } 0 \le t \le \omega. \tag{2.6}$$

Consequently,

$$\|v\|_{C_\omega} \le \frac{1}{2} \int_{t_0}^{t_0+\omega} |v'(s)|ds. \tag{2.7}$$

However,

$$\|u\|_{C_\omega} \le \mu(u) + \|v\|_{C_\omega}, \qquad \int_{t_0}^{t_0+\omega} |v'(s)|ds = \int_0^\omega |u'(s)|ds, \tag{2.8}$$

which together with the previous inequality yields

$$\|u\|_{C_\omega} \le \mu(u) + \frac{1}{2} \int_0^\omega |u'(s)|ds \le \mu(u) + \frac{1}{2} \omega^{1/2} \left(\int_0^\omega |u'(s)|^2 ds \right)^{1/2}. \tag{2.9}$$

On the other hand, by the Wirtinger inequality (see [32, Theorem 258] and [13, Lemma 1.1]), we have

$$\int_0^\omega |u'(s)|^2 ds \le \left(\frac{\omega}{2\pi} \right)^{2n-4} \int_0^\omega |u^{(n-1)}(s)|^2 ds \le \omega \left(\frac{\omega}{2\pi} \right)^{2n-4} \|u^{(n-1)}\|_{C_\omega}^2. \tag{2.10}$$

Consequently, estimate (2.1) is valid.

If now we take into account that $u^{(k)} \in C_\omega^{n-1-k}$ and $\mu(u^{(k)}) = 0$ $(k = 1,\dots,m)$, then the validity of estimates (2.2) becomes evident. □

Lemma 2.2. *Let* $u \in C_\omega^{n-1}$ *and*

$$\|u^{(n-1)}\|_{C_\omega} \le c_0 + \sum_{k=0}^{n-1} \ell_k \|u^{(k)}\|_{C_\omega}, \tag{2.11}$$

where c_0 *and* ℓ_k $(k = 0,\dots,n-1)$ *are nonnegative constants. If, moreover,*

$$\delta = \sum_{k=0}^{n-1} \ell_k \nu_k < 1, \tag{2.12}$$

then

$$\|u^{(n-1)}\|_{C_\omega} \le (1-\delta)^{-1}\big(c_0 + \ell_0 \mu(u)\big), \tag{2.13}$$

$$\|u\|_{C_\omega^{n-1}} \le \mu(u) + (1-\delta)^{-1}\big(c_0 + \ell_0\mu(u)\big)\sum_{k=0}^{n-1}\nu_k. \tag{2.14}$$

Proof. By Lemma 2.1, the function u satisfies inequalities (2.1) and (2.2). In view of these inequalities from (2.11) we find

$$\|u^{(n-1)}\|_{C_\omega} \le c_0 + \ell_0\mu(u) + \left(\sum_{k=0}^{n-1}\ell_k\nu_k\right)\|u^{(n-1)}\|_{C_\omega}. \tag{2.15}$$

Hence, by virtue of condition (2.12), we have estimate (2.13). On the other hand, according to (2.13), inequalities (2.1) and (2.2) result in (2.14). □

Lemma 2.3. *Let* $u \in C_\omega^{n-1}$ *and*

$$\mu(u) \le \varphi\big(\|u^{(n-1)}\|_{C_\omega}\big), \qquad \|u^{(n-1)}\|_{C_\omega} \le c_0 + \sum_{k=1}^{n-1}\ell_k\|u^{(k)}\|_{C_\omega}, \tag{2.16}$$

where $\varphi : [0,+\infty[\to [0,+\infty[$ *is a nondecreasing function,* $c_0 \ge 0$, $\ell_k \ge 0$ $(k = 1,\dots,n-1)$, *and*

$$\delta = \sum_{k=1}^{n-1}\ell_k\nu_k < 1. \tag{2.17}$$

Then

$$\|u\|_{C_\omega^{n-1}} \le r_0, \tag{2.18}$$

where

$$r_0 = \varphi\big((1-\delta)^{-1}c_0\big) + (1-\delta)^{-1}c_0\sum_{k=0}^{n-1}\nu_k. \tag{2.19}$$

Proof. Inequalities (2.16) and (2.17) imply inequalities (2.11) and (2.12), where $\ell_0 = 0$. However, by Lemma 2.2, these inequalities guarantee the validity of the estimates

$$\|u^{(n-1)}\|_{C_\omega} \le (1-\delta)^{-1}c_0, \qquad \|u\|_{C_\omega^{n-1}} \le \mu(u) + (1-\delta)^{-1}c_0\sum_{k=0}^{n-1}\nu_k. \tag{2.20}$$

On the other hand, according to the first inequality in (2.16), we have

$$\mu(u) \le \varphi\big((1-\delta)^{-1}c_0\big). \tag{2.21}$$

Consequently, estimate (2.18) is valid, where r_0 is a number given by equality (2.19). □

Analogously, from Lemma 2.2, the following hold.

Lemma 2.4. *Let $u \in C_\omega^{n-1}$ and*

$$\mu(u) \le c_0, \qquad \left\|u^{(n-1)}\right\|_{C_\omega} \le c_0 + \sum_{k=0}^{n-1} \ell_k \left\|u^{(k)}\right\|_{C_\omega}, \tag{2.22}$$

where $c_0 \ge 0$, $\ell_k \ge 0$ $(k = 0,\dots,n-1)$. If, moreover, inequality (2.12) *holds, then estimate* (2.18) *is valid, where*

$$r_0 = \left[1 + (1-\delta)^{-1}\big(1+\ell_0\big)\sum_{k=0}^{n-1}\nu_k\right]c_0. \tag{2.23}$$

2.2. Lemma on the solvability of a periodic problem

Below, by $C^{n-1}([0,\omega])$ we denote the space of $(n-1)$-times continuously differentiable functions $u : [0,\omega] \to \mathbb{R}$ with the norm

$$\|u\|_{C^{n-1}([0,\omega])} = \sum_{k=1}^{n} \max\left\{\left|u^{(k-1)}(t)\right| : 0 \le t \le \omega\right\}, \tag{2.24}$$

and by $L([0,\omega])$ we denote the space of Lebesgue integrable functions $u : [0,\omega] \to \mathbb{R}$ with the norm

$$\|u\|_{L([0,\omega])} = \int_0^\omega \left|u(t)\right| dt. \tag{2.25}$$

Consider the differential equation

$$u^{(n)}(t) = \overline{f}(u)(t) \tag{2.26}$$

with the periodic boundary conditions

$$u^{(i-1)}(0) = u^{(i-1)}(\omega) \quad (i = 1,\dots,n), \tag{2.27}$$

where $\overline{f} : C^{n-1}([0,\omega]) \to L([0,\omega])$ is a continuous operator such that

$$\overline{f}_r(\cdot) = \sup\left\{\left|\overline{f}(u)(\cdot)\right| : \|u\|_{C^{n-1}([0,\omega])} \le r\right\} \in L\big([0,\omega]\big) \tag{2.28}$$

for any $r > 0$. The following lemma is valid.

Lemma 2.5. *Let there exist a linear, bounded operator* $p : C^{n-1}([0,\omega]) \to L([0,\omega])$ *and a positive constant* r_0 *such that the linear differential equation*

$$u^{(n)}(t) = p(u)(t) \tag{2.29}$$

with the periodic conditions (2.27) *has only a trivial solution and for an arbitrary* $\lambda \in]0,1[$ *every solution of the differential equation*

$$u^{(n)}(t) = \lambda p(u)(t) + (1-\lambda) f(u)(t), \tag{2.30}$$

satisfying condition (2.27)*, admits the estimate*

$$\|u\|_{C^{n-1}([0,\omega])} \le r_0. \tag{2.31}$$

Then problem (2.26), (2.27) *has at least one solution.*

For the proof of this lemma see [33, Corollary 2].

Lemma 2.6. *Let* $f : C_\omega^{n-1} \to L_\omega$ *be a continuous operator satisfying condition* (1.5) *for any* $r > 0$*. Let, moreover, there exist constants* $\alpha \neq 0$ *and* $r_0 > 0$ *such that for an arbitrary* $\lambda \in]0,1[$*, every* ω*-periodic solution of the functional differential equation*

$$u^{(n)}(t) = \lambda \alpha u(0) + (1-\lambda) f(u)(t) \tag{2.32}$$

admits estimate (2.18)*. Then* (1.3) *has at least one* ω*-periodic solution.*

Proof. Let $c_1, \ldots, c_n$ be arbitrary constants. Then the problem

$$y^{(2n)}(t) = 0, \quad y^{(i-1)}(0) = 0, \quad y^{(i-1)}(\omega) = c_i \quad (i = 1, \ldots, n) \tag{2.33}$$

has a unique solution. Let us denote by $y(t; c_1, \ldots, c_n)$ the solution of that problem.

For any $u \in C^{n-1}([0,\omega])$, we set

$$z(u)(t) = u(t) - y\big(t; u(\omega) - u(0), \ldots, u^{(n-1)}(\omega) - u^{(n-1)}(0)\big) \quad \text{for } 0 \le t \le \omega, \tag{2.34}$$

and extend $z(u)(\cdot)$ to $\mathbb{R}$ periodically with a period ω. Then, it is obvious that $z : C^{n-1}([0,\omega]) \to C_\omega^{n-1}$ is a linear, bounded operator.

Suppose

$$\overline{f}(u)(t) = f\big(z(u)\big)(t). \tag{2.35}$$

Consider the boundary value problem (2.26), (2.27). If the function u is an ω-periodic solution of (1.3), then its restriction to $[0,\omega]$ is a solution of problem (2.26), (2.27), and vice versa, if u is a solution of problem (2.26), (2.27), then its periodic extension to $\mathbb{R}$ with a period ω is an ω-periodic solution of (1.3). Thus to prove the lemma, it suffices to state that problem (2.26), (2.27) has at least one solution.

By virtue of equalities (2.34), (2.35) and condition (1.5), $\overline{f} : C^{n-1}([0,\omega]) \to L([0,\omega])$ is a continuous operator, satisfying condition (2.28) for any $r > 0$. On the other hand, it is evident that if $p(u)(t) \equiv \alpha u(0)$, then problem (2.29), (2.27) has only a trivial solution. By these conditions and Lemma 2.5, problem (2.26), (2.27) is solvable if for any $\lambda \in]0,1[$ every solution u of problem (2.30), (2.27), where $p(u)(t) \equiv \alpha u(0)$, admits estimate (2.31).

Let u be a solution of problem (2.30), (2.27) for some $\lambda \in]0,1[$. Then its periodic extension to $\mathbb{R}$ with a period ω is a solution of (2.32), and according to one of the conditions of the lemma, admits estimate (2.18). Therefore, estimate (2.31) is valid. □

3. Proof of the main results

Proof of Theorem 1.1. Without loss of generality, it can be assumed that $h(0) = 0$. On the other hand, according to condition (1.15), we can choose a constant a so that $\sigma a > 0$ and the numbers

$$\ell_k = \ell_{1k} + \ell\ell_{2k} \quad (k = 1, \ldots, n-2), \qquad \ell_{n-1} = \ell_{1n-1} + \ell\ell_{2n-1} + \omega\nu_0|a| \tag{3.1}$$

satisfy inequality (2.17).

Let

$$h_0(x) = \min\{|a|\omega x, h(x)\}, \tag{3.2}$$

let h_0^{-1} be a function, inverse to h_0,

$$\varphi(x) = h_0^{-1}\left(\left(\sum_{k=1}^{n-1}\ell_{1k}\nu_k\right)x + c\right), \quad c_0 = 2c, \tag{3.3}$$

and let r_0 be a number given by equality (2.19). By virtue of Lemma 2.6, to prove the theorem, it suffices to state that for any $\lambda \in]0,1[$ every ω-periodic solution of (2.32) admits estimate (2.18).

Due to condition (1.12), from (2.32), we find

$$\begin{aligned} \|u^{(n-1)}\|_{C_\omega} &\le \max\left\{\left|\int_t^x u^{(n)}(s)ds\right| : 0 \le t \le x \le \omega\right\} \\ &\le \lambda\omega|a||u(0)| + (1-\lambda)\ell h(\mu(u)) + \sum_{k=1}^{n-1}\ell_{2k}\|u^{(k)}\|_{C_\omega} + c. \end{aligned} \tag{3.4}$$

On the other hand, if $\mu(u) > 0$, then by condition (1.11) we have

$$0 = \left(\int_0^\omega u^{(n)}(s)ds\right)\operatorname{sgn}(\sigma u(0)) \ge \lambda\omega|a||u(0)| + (1-\lambda)h(\mu(u)) - \sum_{k=1}^{n-1}\ell_{1k}\|u^{(k)}\|_{C_\omega} - c, \tag{3.5}$$

and consequently,

$$\lambda\omega|a||u(0)| + (1-\lambda)h(\mu(u)) \le \sum_{k=1}^{n-1}\ell_{1k}\|u^{(k)}\|_{C_\omega} + c. \tag{3.6}$$

If $\mu(u) > 0$, then by Lemma 2.1 and notations (3.1)–(3.3), from (3.4) and (3.6), inequalities (2.16) hold. And if $\mu(u) = 0$, then by Lemma 2.1,

$$|u(0)| \le \nu_0\|u^{(n-1)}\|_{C_\omega}. \tag{3.7}$$

On the other hand, $h(\mu(u)) = h(0) = 0$. Thus from (3.4) we obtain

$$\|u^{(n-1)}\|_{C_\omega} \le \omega\nu_0|a|\|u^{(n-1)}\|_{C_\omega} + \sum_{k=1}^{n-1}\ell_{2k}\|u^{(k)}\|_{C_\omega} + c. \tag{3.8}$$

If along with this we take into account notations (3.1) and (3.3), then it becomes evident that inequalities (2.16) are fulfilled also in the case $\mu(u) = 0$.

However, by Lemma 2.3, inequalities (2.16) and (2.17) guarantee the validity of estimate (2.18). □

Proof of Theorem 1.2. Due to (1.16), inequality (2.12) holds. Let r_0 be a number given by equality (2.23) and

$$a = \frac{(\sigma\ell_0)}{\omega}. \tag{3.9}$$

By Lemma 2.6, to prove the theorem, it suffices to state that for any $\lambda \in]0,1[$ every ω-periodic solution u of (2.32) admits estimate (2.18).

If we suppose

$$\mu(u) > c_0, \tag{3.10}$$

then in view of (1.13) and (3.9) from (2.32) we find

$$0 = \left(\int_0^{\omega} u^{(n)}(s)ds\right)\operatorname{sgn}(\sigma u(0)) = \lambda\ell_0|u(0)| + (1-\lambda)\left(\int_0^{\omega} f(u)(s)ds\right)\operatorname{sgn}(\sigma u(0)) > 0. \tag{3.11}$$

The contradiction obtained proves that

$$\mu(u) \le c_0. \tag{3.12}$$

On the other hand, according to (1.14) and (3.9), we have

$$\begin{aligned} \|u^{(n-1)}\|_{C_\omega} &\le \max\left\{\left|\int_t^x u^{(n)}(s)ds\right| : 0 \le t \le x \le \omega\right\} \\ &\le \lambda\ell_0|u(0)| + (1-\lambda)\sum_{k=0}^{n-1}\ell_k\|u^{(k)}\|_{C_\omega} + c_0 \le \sum_{k=0}^{n-1}\ell_k\|u^{(k)}\|_{C_\omega} + c_0. \end{aligned} \tag{3.13}$$

Therefore, inequalities (2.22) are satisfied. However, by Lemma 2.4, inequalities (2.12) and (2.22) guarantee the validity of estimate (2.18). □

Proof of Corollary 1.3. For an arbitrary $u \in C_\omega^{n-1}$, we set

$$f(u)(t) = g\left(t, u(\tau_1(t)), \ldots, u^{(n-1)}(\tau_n(t))\right). \tag{3.14}$$

Then (1.4) takes the form (1.3). It is obvious that $f : C_\omega^{n-1} \to L_\omega$ is a continuous operator, satisfying condition (1.8), since $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ belongs to the Carathéodory class and conditions (1.6) and (1.7) are satisfied.

Let $u \in C_\omega^{n-1}$ and $\mu(u) = u(t_0)$. Then by Lemma 2.1 we have

$$0 \le |u(\tau_1(t))|^\lambda - \mu^\lambda(u) \le |u(\tau_1(t)) - u(t_0)|^\lambda \le \nu_0^\lambda \|u^{(n-1)}\|_{C_\omega}^\lambda \quad \text{for } t \in \mathbb{R}. \tag{3.15}$$

If along with this we take into account inequalities (1.17), then from (3.14) we get

$$\begin{gathered}\left(\int_0^{w} f(u)(s)ds\right)\operatorname{sgn}(\sigma u(0)) \geq h(\mu(u)) - \nu_0^{\lambda}\int_0^{w}[p_{11}(s)]_{-}ds \times \|u^{(n-1)}\|_{C_\omega}^{\lambda} \\ -\sum_{k=2}^{n}\int_0^{w} p_{1k}(s)ds\|u^{(k-1)}\|_{C_\omega} - \int_0^{w} q(s)ds \quad \text{for } \mu(u) > 0, \\ \left|\int_t^{x} f(u)(s)ds\right| \leq \ell h(\mu(u)) + \nu_0^{\lambda}\int_0^{w} p_{21}(s)ds\|u^{(n-1)}\|_{C_\omega}^{\lambda} \\ +\sum_{k=2}^{n}\int_0^{w} p_{2k}(s)ds\|u^{(k-1)}\|_{C_\omega} + \int_0^{w} q(s)ds \quad \text{for } 0 \leq t \leq x \leq \omega,\end{gathered} \tag{3.16}$$

where

$$h(x) = x^{\lambda}\int_0^{w} p_{11}(s)ds, \tag{3.17}$$

and, in view of condition (1.18), h is increasing and $h(x) \to +\infty$ as $x \to +\infty$.

If $\lambda < 1$, then in view of (1.19) there exists $\varepsilon > 0$ such that

$$\varepsilon\nu_0^{\lambda}\int_0^{w}(\ell[p_{11}(s)]_{-} + p_{21}(s))ds + \sum_{k=2}^{n}\nu_{k-1}\int_0^{w}(\ell p_{1k}(s) + p_{2k}(s))ds < 1. \tag{3.18}$$

Set

$$\begin{gathered}\varepsilon_\lambda = \varepsilon, \qquad c_0 = \int_0^{w} q(s)ds + \varepsilon^{-\lambda/(1-\lambda)}\nu_0^{\lambda}\left(\int_0^{w}([p_{11}(s)]_{-} + p_{21}(s))ds\right) \quad \text{for } \lambda < 1, \\ \varepsilon_\lambda = 1, \quad c_0 = \int_0^{w} q(s)ds \quad \text{for } \lambda = 1, \\ \ell_{1k} = \int_0^{w} p_{1k+1}(s)ds \quad (k = 1,\dots,n-2), \\ \ell_{1n-1} = \varepsilon_\lambda\nu_0^{\lambda}\int_0^{w}[p_{11}(s)]_{-}ds + \int_0^{w} p_{1n-1}(s)ds, \\ \ell_{2k} = \int_0^{w} p_{2k+1}(s)ds \quad (k = 1,\dots,n-2), \\ \ell_{2n-1} = \varepsilon_\lambda\nu_0^{\lambda}\int_0^{w} p_{21}(s)ds + \int_0^{w} p_{2n-1}(s)ds.\end{gathered} \tag{3.19}$$

Then by the Young inequality, inequalities (3.16) result in inequalities (1.11) and (1.12). On the other hand, the numbers ℓ_{ik} $(i = 1,2;\ k = 1,\dots,n-1)$ satisfy inequality (1.15) since for $\lambda < 1$ (for $\lambda = 1$) the functions p_{ik} $(i = 1,2;\ k = 1,\dots,n)$ satisfy inequality (3.18) (inequality (1.20)).

Therefore, all the conditions of Theorem 1.1 are fulfilled which guarantee the existence of at least one ω-periodic solution of (1.4). □

Proof of Corollary 1.4. Without loss of generality, it can be assumed that

$$\int_0^{\omega} q(s)ds < c_0. \tag{3.20}$$

Then in view of (1.21), (1.22) from (3.14), inequalities (1.13) and (1.14) hold, where

$$\ell_k = \int_0^{\omega} p_{k+1}(s)ds \quad (k = 0, \dots, n-1). \tag{3.21}$$

On the other hand, according to (1.23), the numbers ℓ_k $(k = 0, \dots, n-1)$ satisfy (1.16). Therefore, all the conditions of Theorem 1.2 are fulfilled. □

Proof of Theorem 1.5. For $v(t) \equiv 0$, inequalities (1.24) and (1.25) yield inequalities (1.11) and (1.12), where

$$h(x) = \ell_{10} x, \qquad c = \int_0^{\omega} \left| f(0)(s) \right| ds. \tag{3.22}$$

Consequently, all the conditions of Theorem 1.1 are satisfied which guarantee the existence of at least one ω-periodic solution of (1.3).

It remains to prove that (1.3) has no more than one ω-periodic solution. Let u_1 and u_2 be arbitrary ω-periodic solutions of (1.3) and

$$u(t) = u_2(t) - u_1(t). \tag{3.23}$$

If we assume that $\mu(u) > 0$, then from (1.24) we find

$$\ell_{10}\mu(u) \le \sum_{k=1}^{n-1} \ell_{1k} \| u^{(k)} \|_{C_\omega}. \tag{3.24}$$

It is obvious that this inequality is valid also for $\mu(u) = 0$. Due to (3.24), from (1.25) it follows

$$\| u^{(n-1)} \|_{C_\omega} \le \max\left\{ \left| \int_t^x \left(f(u_1 + u)(s) - f(u_1)(s) \right) ds \right| : 0 \le t \le x \le \omega \right\} \le \sum_{k=1}^{n-1} \ell_k \| u^{(k)} \|_{C_\omega}, \tag{3.25}$$

where $\ell_k = \ell \ell_{1k} + \ell_{2k}$ $(k = 1, \dots, n-1)$. Moreover, in view of (1.15) the numbers ℓ_k $(k = 1, \dots, n-1)$ satisfy inequality (2.17). On the other hand, by Lemma 2.1 from (3.24) we have

$$\ell_{10}\mu(u) \le \left(\sum_{k=1}^{n-1} \ell_{1k} \nu_k \right) \| u^{(n-1)} \|_{C_\omega}. \tag{3.26}$$

Consequently, inequalities (2.16) are satisfied, where

$$\varphi(x) = \left(\sum_{k=1}^{n-1} \ell_{10}^{-1} \ell_{1k} \nu_k \right) x, \qquad c_0 = 0. \tag{3.27}$$

If now we apply Lemma 2.3, then it becomes evident that $u(t) \equiv 0$, that is, $u_1(t) \equiv u_2(t)$. □

Proof of Corollary 1.7. To prove the corollary, it is sufficient to state that the operator f, given by equality (3.14), satisfies the conditions of Theorem 1.5.

Let u and $v \in C_\omega^{n-1}$. Then by virtue of conditions (1.29), and Lemma 2.1, from (3.14), we obtain inequalities (1.24) and (1.25), where

$$\begin{aligned} \ell_{1k} &= \int_0^w p_{1k+1}(s)ds \quad (k = 0,\dots,n-2), \quad \ell_{1n-1} = \nu_0 \int_0^w [p_{11}(s)]_- ds + \int_0^w p_{1n-1}(s)ds, \\ \ell_{2k} &= \int_0^w p_{2k+1}(s)ds \quad (k = 0,\dots,n-2), \quad \ell_{2n-1} = \nu_0 \int_0^w [p_{21}(s)]_- ds + \int_0^w p_{2n-1}(s)ds. \end{aligned} \tag{3.28}$$

On the other hand, in view of (1.18) and (1.20), $\ell_{10} > 0$ and condition (1.15) holds, where $\ell = \ell_{20}/\ell_{10}$. □

Proof of Theorem 1.6. For $v(t) \equiv 0$, (1.26)–(1.28) yield conditions (1.13), (1.14), and (1.16), where

$$c_0 = \int_0^w |f(0)(s)|ds, \quad \ell_k = 0 \quad (k = 1,\dots,n-1). \tag{3.29}$$

Consequently, all the conditions of Theorem 1.2 are satisfied which guarantee the existence of at least one ω-periodic solution of (1.3).

Suppose now that u_1 and u_2 are arbitrary ω-periodic solutions of (1.3) and $u(t) = u_2(t) - u_1(t)$. If we assume that $\mu(u) > 0$, then in view of (1.26) we obtain the contradiction

$$0 = \left(\int_0^w u^{(n)}(s)ds\right)\operatorname{sgn}(\sigma u(0)) = \left(\int_0^w (f(u_1+u)(s) - f(u_1)(s))ds\right)\operatorname{sgn}(\sigma u(0)) > 0. \tag{3.30}$$

Thus, it is proved that $\mu(u) = 0$.

On the other hand, (1.27) implies

$$\|u^{(n-1)}\|_{C_\omega} \le \ell_0 \|u\|_{C_\omega}. \tag{3.31}$$

Therefore, inequalities (2.22) are satisfied, where $c_0 = 0$, $\ell_k = 0$ $(k = 1,\dots,n-1)$, and $\ell_0\nu_0 < 1$. Hence by Lemma 2.4 it follows that $u(t) \equiv 0$, that is, $u_1(t) \equiv u_2(t)$. □

Proof of Corollary 1.8. For any $u \in C_\omega^{n-1}$, we set

$$f(u)(t) = g(u)(t). \tag{3.32}$$

Then conditions (1.33) and (1.34) imply conditions (1.26)–(1.28), where $\ell_0 = \int_0^\omega p(s)ds$. Consequently, the operator f satisfies all the conditions of Theorem 1.6. □

4. Examples

From the main results of the present paper new (and optimal in some sense) sufficient conditions for the existence of periodic solutions of linear and sublinear differential equations with deviated arguments and differential equations with bounded right-hand sides follow. To illustrate the above mentioned, let us consider the differential equations

$$u^{(n)}(t) = \sum_{k=1}^{n} g_k(t)\left|u^{(k-1)}(\tau_k(t))\right|^{\lambda_k}\operatorname{sgn}\left(u^{(k-1)}(\tau_k(t))\right) + g_0(t), \tag{4.1}$$

$$u^{(n)}(t) = \sum_{k=1}^{n} g_k(t)\left|u^{(k-1)}(\tau_k(t))\right|^{\lambda_{0k}}\left(1 + \left|u(\tau_1(t))\right|\right)^{-\lambda_k}u(\tau_1(t)) + g_0(t), \tag{4.2}$$

$$u^{(n)}(t) = \sum_{k=1}^{n} g_k(t)u^{(k-1)}(\tau_k(t)) + g_0(t), \tag{4.3}$$

$$u^{(n)}(t) = \sum_{k=1}^{m} g_{0k}(t)\frac{\left|u(\tau(t))\right|^{\lambda_k}}{1 + \left|u(\tau(t))\right|^{\lambda_k}}\operatorname{sgn} u(\tau(t)) + g_0(t), \tag{4.4}$$

where m is a natural number, $\lambda_k > 0$ $(k = 1, \ldots, m)$, $\lambda_{0k} \geq 0$ $(k = 1, \ldots, n)$ are constants, $g_k \in L_\omega$ $(k = 0, \ldots, n)$, $g_{0k} \in L_\omega$ $(k = 1, \ldots, m)$, while $\tau_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \ldots, n)$ and $\tau : \mathbb{R} \to \mathbb{R}$ are measurable on every finite interval functions satisfying, respectively, conditions (1.7) and (1.32).

Corollaries 1.3 and 1.4 imply the following corollaries.

Corollary 4.1. *Let*

$$\int_0^w g_1(s)ds \neq 0, \tag{4.5}$$

and either

$$0 < \lambda_k < 1 \quad (k = 1, \ldots, n) \tag{4.6}$$

or

$$0 < \lambda_1 < 1, \quad 0 < \lambda_k \leq 1 \quad (k = 2, \ldots, n), \quad \sum_{k=2}^{n} \nu_{k-1}\int_0^w \left|g_{2k}(s)\right|ds < (1 + \ell)^{-1}, \tag{4.7}$$

where

$$\ell = \frac{\int_0^1 \left|g_1(s)\right|ds}{\left|\int_0^1 g_1(s)ds\right|}. \tag{4.8}$$

Then (4.1) *has at least one ω-periodic solution.*

Corollary 4.2. *Let*

$$\int_0^w g_0(s)ds = 0, \tag{4.9}$$

and there exists a constant $\sigma \in \{-1, 1\}$ such that

$$\sigma g_k(t) \geq 0 \quad \text{for } t \in \mathbb{R} \ (k = 1, \ldots, n). \tag{4.10}$$

Let, moreover, either

$$0 \leq \lambda_{01} < \lambda_1, \quad 0 \leq \lambda_{0k} < 1, \quad \lambda_k \geq 1 \quad (k = 2, \ldots, n), \tag{4.11}$$

or

$$0 \le \lambda_{01} \le \lambda_1, \quad 0 \le \lambda_{0k} \le 1, \quad \lambda_k \ge 1 \quad (k = 2, \dots, n), \tag{4.12}$$

and

$$\sum_{k=1}^{n} \nu_{k-1} \int_0^w |g_k(s)| ds < 1. \tag{4.13}$$

Then (4.2) *has at least one* ω*-periodic solution.*

Remark 4.3. If

$$g_k(t) \equiv 0 \quad (k = 1, \dots, n), \qquad \int_0^w g_0(s) ds \not\equiv 0, \tag{4.14}$$

then (4.1) has no ω-periodic solution. Consequently, conditions (1.18) and (4.5) in Corollaries 1.3 and 4.1 are essential and they cannot be omitted.

Remark 4.4. If

$$\lambda_{0k} = 0, \quad \lambda_k \ge 1 \quad (k = 1, \dots, n), \quad \sum_{k=1}^{n} \int_0^w |g_k(s)| ds \le \varepsilon, \tag{4.15}$$

$$\left| \int_0^w g_0(s) ds \right| = \varepsilon, \tag{4.16}$$

where $\varepsilon > 0$, then (4.2) has no ω-periodic solution. Consequently, conditions (1.21) and (4.9) in Corollaries 1.4 and 4.2 cannot be replaced by condition (4.16) no matter how small ε would be.

Corollary 1.7 yields the following.

Corollary 4.5. *Let there exist a constant* $\sigma \in \{-1, 1\}$ *such that*

$$\begin{gathered} \sigma \int_0^w g_1(s) ds > 0, \\ \nu_0 \int_0^w \left(\ell [\sigma g_1(s)]_- + |g_1(s)| \right) ds + (1 + \ell) \sum_{k=2}^{n} \nu_{k-1} \int_0^w |g_k(s)| ds < 1, \end{gathered} \tag{4.17}$$

where ℓ *is a number given by equality* (4.8)*. Then* (4.3) *has one and only one* ω*-periodic solution.*

Suppose

$$\eta(\lambda) = \frac{1}{4\lambda} (\lambda - 1)^{(\lambda-1)/\lambda} (\lambda + 1)^{(\lambda+1)/\lambda} \quad \text{for } \lambda > 1, \qquad \eta(\lambda) = 1 \quad \text{for } \lambda = 1. \tag{4.18}$$

Theorem 1.6 results in the following.

Corollary 4.6. *Let there exist a constant* $\sigma \in \{-1, 1\}$ *such that*

$$\sigma g_{0k}(t) \ge 0 \quad \textit{for } t \in \mathbb{R} \ (k = 1, \dots, m). \tag{4.19}$$

If, moreover,

$$\sum_{k=1}^{m}\int_{0}^{w}|g_{0k}(s)|ds > 0, \tag{4.20}$$

$$\sum_{k=1}^{m}\eta(\lambda_k)\int_{0}^{w}|g_{0k}(s)|ds < \frac{1}{\nu_0}, \tag{4.21}$$

and equality (4.9) *holds, then* (4.4) *has one and only one* ω*-periodic solution.*

Remark 4.7. If $g_{0k}(t) \equiv 0$ $(k = 1, \dots, m)$ and equality (4.9) is fulfilled, then (4.4) has an infinite set of ω-periodic solutions. Consequently, the strict inequality (1.26) (the strict inequality (4.20)) in Theorem 1.6 (in Corollary 4.6) cannot be replaced by nonstrict one.

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