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Research Article

Multiple Nodal Solutions for Some Fourth-Order Boundary Value Problems via Admissible Invariant Sets

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Existence and multiplicity results for nodal solutions are obtained for the fourth-order boundary value problem (BVP) $u^{(4)}(t) = f(t,u(t))$, $0 < t < 1$, $u(0) = u(1) = u''(0) = u''(1) = 0$, where $f : [0,1]\times R \to R$ is continuous. The critical point theory and admissible invariant sets are employed to discuss this problem.

1. Introduction

In this paper, we consider the existence of nodal solutions to the semilinear fourth-order equation:

$$u^{(4)}(t) = f(t,u(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $f : [0,1] \times R \to R$ is continuous.

Owning to the importance of *higher-order* differential equations in physics, the existence and multiplicity of the solutions to such problems have been studied by many authors. They obtained the existence of solutions by the cone expansion or compression fixed point theorem [1–6]; sub-sup solution method [7–9]; critical point theory [10–13]; Morse theory [14, 15]; and eta [16, 17]. There are also papers which study nodal solutions for elliptic equations [18, 19]. In particular, in [20], Han and Li obtained multiple positive, negative, and sign-changing solutions by combining the critical point theory and the method of sub-sup solutions for the (BVP) (1.2). The main result is as follows:

(H_1) there exist a strict subsolution α and a strict supersolution β of (BVP) (1.2) with $\alpha < \beta$, $\alpha(0) = \alpha(1) = \alpha''(0) = \alpha''(1) = 0$, and $\beta(0) = \beta(1) = \beta''(0) = \beta''(1) = 0$;

(H_2) $f(t,u)$ is strictly increasing in u;

(H_3) $f(t,u)$ is locally Lipschitz continuous in u;

(H_4) there exist $\mu \in (0, 1/2)$ and $\Lambda > 0$ such that $0 < F(t,u) = \int_0^1 f(t,v)dv \le \mu u f(t,u)$ for all $|u| \ge \Lambda$ and $t \in [0,1]$.

Theorem 1.1 (see [20]). *Assume that (H_1)–(H_4) hold. Then, (BVP) (1.2) has at least four solutions.*

Motivated by their ideas, we cannot help wondering if there are no strict subsolution and supersolution of (BVP) (1.2), can we still get the nodal solutions just by critical point theory? In this paper, we will use the admissible invariant sets and critical point theory to settle this problem. But we should point out that in all theorems of our paper, the nonlinearity $f(t,u)$ is assumed to be odd in u, while no such symmetry is required in [20].

The paper is organized as follows: in Section 2, we give some preliminaries, including the critical point theorems which will be used in our main results and some concepts concerning the partially ordered Banach space. The main results and proofs are established in Section 3.

2. Preliminaries

Let E be a Hilbert space and $X \subset E$ a Banach space densely embedded in E. Assume that E has a closed convex cone P_E and that $P := P_E \cap X$ has interior points in X, that is, $P = \dot{P} \cup \partial P$ with $\dot{P}$ the interior and ∂P the boundary of P in X.

Let $J \in C^1(E, R)$ and $J'(u) = u - A(u)$ for $u \in E$. We use the following notation: $K = K(J) = \{u \in E : J'(u) = 0\}$, $J^b = \{u \in E : J(u) \le b\}$, $K_c = \{u \in E : J(u) = c,\ J'(u) = 0\}$, $K([a,b]) = \{u \in E : J(u) \in [a,b],\ J'(u) = 0\}$ for $a, b, c \in R$. Let $\|\cdot\|$ and $\|\cdot\|_X$ denote the norms in E and X, respectively.

Lemma 2.1 (see [21]). *Assume E is a Hilbert space, and M is a closed convex set of E, $J'(u) = u - A(u)$, and $A(M) \subset M$. Then, there exists a pseudogradient vector field $W := -\mathrm{id} + B$ for J, and $B(M) \subset M$. Furthermore, if J is even, $M = -M$, then W is odd.*

Consider the pseudogradient flow σ on E associated with the vector field $W = -\mathrm{id} + B$,

$$\begin{gathered} \frac{d}{dt}\sigma(t,u) = -W(\sigma(t,u)), \quad t \ge 0; \\ \sigma(0,u) = u. \end{gathered} \tag{2.1}$$

We see that σ *is odd in* u, *if* W *is odd in* u. Since $u + \lambda(-W(u)) = (1-\lambda)u + \lambda B(u) \in M$ for $u \in M \setminus K$ and $0 \le \lambda \le 1$, the Brezis-Martin theorem [22] implies that $\sigma(t, M) \subset M$ for $t \ge 0$.

Definition 2.2 (see [21, 23]). With the flow σ, a subset $M \subset E$ is called an invariant set if $\sigma(t, M) \subset M$ for $t \ge 0$.

Let us assume that

(Φ) $K(J) \subset X$, $J'(u) = u - A(u)$ for $u \in E$, $A : X \to X$ is continuous.

Under condition (Φ), we have $\sigma(t, x) \in X$ for $x \in X$ and σ is continuous in $(t, x) \in R \times X$.

Definition 2.3 (see [21]). Let $M \subset X$ be an invariant set under σ. M is said to be an admissible invariant set for J if (a) M is the closure of an open set in X, that is, $M = \dot{M} \cup \partial M$; (b) if $u_n = \sigma(t_n, v)$ for some $v \notin M$ and $u_n \to u$ in E as $t_n \to \infty$ for some $u \in K$, then $u_n \to u$ in X; (c) if $u_n \in K \cap M$ such that $u_n \to u$ in E, then $u_n \to u$ in X; (d) for any $u \in \partial M \setminus K$, $\sigma(t, u) \in \dot{M}$ for $t > 0$.

Lemma 2.4 (see [24]). *Let $J \in C^1(E, R)$ and (Φ) hold. Assume J is even, bounded from below, $J(0) = 0$ and satisfies (PS) condition. Assume that the positive cone P is an admissible invariant set for J and $K_c \cap \partial P = \varnothing$ for all $c < 0$. Suppose there is a linear subspace $F \subset X$ with $\dim F = n$, such that $\sup_{F \cap \partial B_\rho} J(u) < 0$ for some $\rho > 0$, where $B_\rho = \{u \in X : \|u\|_X \leq \rho\}$. Then, J has at least n pairs of critical points with negative critical values. More precisely,*

(i) *if $\inf_P J \leq \inf_E J$, J has at least one pair of critical points in $\dot{P} \cup (-\dot{P})$, and at least $n - 1$ pairs of critical points in $X \setminus (P \cup (-P))$;*

(ii) *if $\inf_E J < \inf_P J$, J has at least one pair of critical points in $\dot{P} \cup (-\dot{P})$, and at least n pairs of critical points in $X \setminus (P \cup (-P))$.*

Lemma 2.5 (see [21]). *Let $J \in C^1(E, R)$ and (Φ) hold. Assume J is even, $J(0) = 0$, and J satisfies (PS) condition. Assume that the positive cone P is an admissible invariant set for J and $K_c \cap \partial P = \varnothing$ for all $c < 0$. Suppose there exist linear subspaces $F \subset X$ and $H \subset E$ with $\dim F = n$, $\operatorname{codim} H = k \geq 1$ ($k = 0$, resp.), $n > k$, such that for some $\rho > 0$, $\sup_{F \cap \partial B_\rho(0)} J(u) < 0$ and $\inf_H J(u) > -\infty$. Then, J has at least $(n - k)$ ($(n - 1)$, resp.) pairs of critical points in $X \setminus (P \cup (-P))$ with negative critical values.*

Lemma 2.6 (see [21]). *Let $J \in C^1(E, R)$ and (Φ) hold. Assume J is even, $J(0) = 0$ and J satisfies (PS) condition. Assume that the positive cone P is an admissible invariant set for J and $K_c \cap \partial P = \varnothing$ for all $c > 0$. Suppose there exist linear subspaces $F \subset X$ and $H \subset E$ with $\dim F = n$, $\operatorname{codim} H = k \geq 1$, $n > k + 1$, such that for some $\rho > \gamma > 0$, $\sup_{F \cap \partial B_\rho(0)} J(u) \leq 0$ and $\inf_{H \cap \partial B_\gamma(0)} J(u) > 0$. Then for $k \geq 1$ ($k = 0$, resp.), J has at least $(n - k - 1)$ ($(n - 1)$, resp.) pairs of critical points in $X \setminus (P \cup (-P))$ with positive critical values.*

Lemma 2.7 (see [21, 25]). *Assume $J \in C^1(E, R)$ is even, $J(0) = 0$, satisfies (Φ) and $(PS)_c$ condition for $c > 0$. Assume that P is an admissible invariant set for J, $K_c \cap \partial P = \varnothing$ for all $c > 0$. $E = \overline{\bigoplus_{j=1}^{\infty} E_j}$, where E_j are finite-dimensional subspaces of X, and for each k, let $Y_k = \bigoplus_{j=1}^{k} E_j$ and $Z_k = \overline{\bigoplus_{j=k}^{\infty} E_j}$. Assume for each k there exist $\rho_k > \gamma_k > 0$ such that $\lim_{k \to \infty} a_k < \infty$, where $a_k = \max_{Y_k \cap \partial B_{\rho_k}(0)} J(u)$, $b_k = \inf_{Z_k \cap \partial B_{\gamma_k}(0)} J(u) \to \infty$ as $k \to \infty$. Then, J has a sequence of critical points $u_n \in X \setminus (P \cup (-P))$ such that $J(u_n) \to \infty$ as $n \to \infty$, provided $Z_k \cap \partial B_{\gamma_k}(0) \cap P = \varnothing$ for large k.*

Next, we need some basic concepts of ordered Banach spaces.

Definition 2.8. An ordered real Banach space is a pair (X, P), where X is a real Banach space and P a closed convex subset of X such that $(-P) \cap P = \{0\}$ and $R^+ \cdot P \subset P$. The partial order on X is given by the cone P. For $u, v \in X$, we write

$$\begin{gathered} u \leq v \Longleftrightarrow v - u \in P; \\ u < v \Longleftrightarrow u \leq v, \quad \text{but } u \neq v;. \\ u \ll v \Longleftrightarrow v - u \in \dot{P}. \end{gathered} \tag{2.2}$$

If P has nonempty interior, then it is called a solid cone. If every ordered interval is bounded, then P is called a normal cone. An operator $A : D(A) \to X$ is called order preserving (in the literature sometimes increasing) if

$$u \leq v \Longrightarrow Au \leq Av; \tag{2.3}$$

strictly order preserving if

$$u < v \Longrightarrow Au < Av; \tag{2.4}$$

and strongly order preserving if

$$u < v \Longrightarrow Au \ll Av. \tag{2.5}$$

3. Main results

In this section, we will employ the abstract results in Section 2 to establish some existence theorems on sign-changing solutions of (BVP) (1.2). Firstly, we give some lemmas to change (BVP) (1.2) to a variational problem. Let $C[0,1]$ be the usual real Banach space with the norm $\|u\|_C = \max_{t\in[0,1]}|u(t)|$ for all $u \in C[0,1]$. We can easily verify that

$$C_0[0,1] = \{u \in C[0,1] : u(0) = u(1) = 0\} \tag{3.1}$$

is also a Banach space with respect to $\|\cdot\|_C$. Let

$$P = \{u \in C_0[0,1] : u(t) \geq 0 \ \forall t \in [0,1]\}, \tag{3.2}$$

then P is a normal solid cone in $C_0[0,1]$ and

$$\dot{P} = \{u \in C_0[0,1] : u(t) > 0 \ \forall t \in (0,1)\}. \tag{3.3}$$

By $L^2[0,1]$, we denote the usual real Hilbert space with the inner product $(u,v) = \int_0^1 u(t)v(t)dt$ for all $u, v \in L^2[0,1]$.

It is well known that the solution of (BVP) (1.2) in $C^4[0,1]$ is equivalent to the solution of the following integral equation in $C[0,1]$:

$$u(t) = \int_0^1 G(t,s)\int_0^1 G(s,\tau)f(\tau,u(\tau))d\tau\, ds, \quad t \in [0,1], \tag{3.4}$$

where $G(t,s)$ is the Green's function of the linear boundary value problem $-u''(t) = 0$ for all $t \in [0,1]$ subject to $u(0) = u(1) = 0$, that is,

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1; \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{3.5}$$

Define operators $T, A_f : C[0,1] \to C[0,1]$ by

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)u(s)ds, \\ A_f u(t) &= f(t,u(t)). \end{aligned} \tag{3.6}$$

Since $T : C[0,1] \to C_0[0,1]$, (3.4) is equivalent to the following operator equation in $C_0[0,1]$:

$$u = T^2 A_f u. \tag{3.7}$$

Remark 3.1. It is easy to see that

(i) $G : [0,1] \times [0,1] \to [0,1]$ is nonnegative continuous;
(ii) $\max_{(t,s)\in[0,1]\times[0,1]} G(t,s) = 1/4$;
(iii) $A_f : C[0,1] \to C[0,1]$ is bounded and continuous.

Lemma 3.2 (see [20]). $T : L^2[0,1] \to C_0[0,1]$ *is a linear completely continuous operator and also a linear completely continuous operator from* $L^2[0,1]$ *to* $L^2[0,1]$*. In addition,* $T : C_0[0,1] \to C_0[0,1]$ *is strongly order-preserving.*

From the definition of T, we can obtain that $Tu \ne 0$ for all $u \in L^2[0,1]$ with $u \ne 0$. Therefore, $Tu_1 \ne Tu_2$ for all $u_1, u_2 \in L^2[0,1]$ with $u_1 \ne u_2$. It is well known that all eigenvalues of T are

$$\{\lambda_k\}_{k\in N} = \frac{1}{k^2\pi^2}, \tag{3.8}$$

which have the corresponding orthonormal eigenfunctions

$$\{e_k\}_{k\in N} = \left\{\sqrt{2}\sin k\pi t\right\}_{k\in N}, \tag{3.9}$$

and $\lambda_1 > \lambda_2 > \cdots > \lambda_k > \cdots > 0 \ \ \forall k \in N$.

Lemma 3.3 (see [10]). (i) *The operator equation*

$$u = T^2 A_f u \tag{3.10}$$

has a solution in $C[0,1]$ *if and only if the operator equation*

$$v = TA_f Tv \tag{3.11}$$

has a solution in $L^2[0,1]$.

(ii) *The uniqueness of the solution for these two above equations is also equivalent.*

Remark 3.4. From the proof of Lemma 3.3 [10], it is very clear if $u \in L^2[0,1]$ is a solution for (3.11), then $Tu \in C_0[0,1]$ is a solution for (3.7). Furthermore, if $u \in C_0[0,1]$ is a solution for (3.11), then $Tu \in C_0[0,1]$ is a solution for (3.7) with the same sign, which follows from Lemma 3.2.

Lemma 3.5 (see [10]). *Let* $\Psi(u) = \int_0^1 \int_0^{u(t)} f(t,v)dv\, dt$, $u \in C[0,1]$. *Then,*

(i) Ψ *is Fréchet differentiable on* $C[0,1]$ *and* $(\Psi'(u))(w) = (A_f, w)$ *for all* $u, w \in C[0,1]$;

(ii) $\Psi \circ T$ *is Fréchet differentiable on* $L^2[0,1]$ *and* $(\Psi \circ T)'(v) = TA_f Tv$ *for all* $v \in L^2[0,1]$.

Choose $E = L^2[0,1]$ and $X = C_0[0,1]$ to be our Hilbert space and Banach space, respectively. Define a functional $J : E \to R$:

$$J(u) = \frac{1}{2}\|u\|^2 - \Psi(Tu), \quad u \in E. \tag{3.12}$$

Then, according to Lemma 3.5, we have

$$J'(u) = u - TA_f Tu \quad \forall u \in E. \tag{3.13}$$

Hence, Lemma 3.3 implies that the operator equation $u = T^2 A_f u$ has a solution in X if and only if the functional J has a critical point in E. Thus, (BVP) (1.2) has been transformed into a variational problem.

We refer the following assumption:

(f_1) $f : [0,1] \times R \to R$ is continuous and increasing in u.

Lemma 3.6. *Under* (f_1), (Φ) *is satisfied, and* $A := TA_f T : C_0[0,1] \to C_0[0,1]$ *is strongly order-preserving.*

Proof. The proof is similar to [20], and we omit it here. □

Lemma 3.7. *Under* (f_1), $M = P$ *is an admissible invariant set for* J.

Proof. We know that $A : C_0[0,1] \to C_0[0,1]$ is strongly order-preserving, so does B given in Lemma 2.1. The Brezis-Martin theory implies that P and $-P$ are invariant sets under the negative pesudogradient flow of J. Requirement (a) is satisfied automatically. For (d), we note that for all $v \in P \setminus \{0\}$, we have $B(v) \in \dot{P}$, similar to the proof in [23], $\sigma(t, \partial P) \in \dot{P}$. To prove (b), let $u_n = \sigma(t_n, v)$ for some $v \in X \setminus (P \cup (-P))$, so $u_n \in X = C_0[0,1]$, let $t_n \to \infty$ be a sequence such that $u_n \to u$ in $E = L^2[0,1]$ for some $u \in K(J) \subset X = C_0[0,1]$, then $u_n \to u$ in $X = C_0[0,1]$. For (c), if $u_n \in K(J) \cap (P \cup (-P)) \subset X$, then $J'(u_n) = 0$, if $u_n \to u$ in $E = L^2[0,1]$, for $J \in C^1(E, R)$, then $J'(u) = 0$ and $u \in K(J) \subset X = C_0[0,1]$, so $u_n \to u$ in $X = C_0[0,1]$, and the proof is completed. □

Lemma 3.8 (see [15]). *Any bounded sequence* $\{u_n\} \subset L^2[0,1]$ *such that* $J'(u) \to 0$ *as* $n \to \infty$ *has a convergent subsequence.*

Next, we make more assumptions:

(f_2) $\lim_{|u| \to \infty} f(t,u)/u < \pi^4$, uniformly for $t \in [0,1]$;

(f_3) $\lim_{|u| \to 0} f(t,u)/u > k^4\pi^4$, uniformly for $t \in [0,1]$ and some $k \geq 2$;

(f_4) $f(t,u)$ is odd in u.

Theorem 3.9 (sublinear nonlinearity). *Under* (f_1)–(f_4), *(BVP)* (1.2) *has at least one pair of one-sign solutions* $u_1 > 0$, $-u_1 < 0$, *and at least* $k-1$ *pairs of nodal solutions* u_i *for* $i = 2, \dots, k$.

Proof. It is easy to see that $J \in C^1(E,R)$ and (Φ) holds. P is an admissible invariant set for J, and $K_c(J) \cap \partial P = \varnothing$ for $c \neq 0$. Also, J is even, $J(0) = 0$. By (f_2), there exist $\delta > 0$, $\Lambda > 0$ such that $F(t,u) \leq (1/2)(\pi^4 - \delta)u^2 + \Lambda$ for all $u \in R$, then

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu(t))dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\pi^4 - \delta)\int_0^1 (Tu)^2 dt - \Lambda \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\pi^4 - \delta}{2\pi^4}\|u\|^2 - \Lambda \\ &= \frac{\delta}{2\pi^4}\|u\|^2 - \Lambda \geq -\Lambda. \end{aligned} \tag{3.14}$$

So J is *coercive*, bounded from below, and satisfies (PS) condition.

Take $F = \bigoplus_{i=1}^k \{e_i\}$; from ($f_3$), there exist $\eta > 0$, $\delta_1 > 0$ such that $|s| < \eta$, $F(t,s) \geq (1/2)(k^4\pi^4 + \delta_1)s^2$, choose $\rho = 4\eta$, then $|u| \leq \rho \Rightarrow |Tu| = |\int_0^1 G(t,s)u(s)ds| \leq (1/4)|u| \leq \eta$, and

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu)dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{2}(k^4\pi^4 + \delta_1)\int_0^1 (Tu)^2 dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{k^4\pi^4 + \delta_1}{2k^4\pi^4}\|u\|^2 = -\frac{\delta_1}{2k^4\pi^4}\|u\|^2 < 0, \end{aligned} \tag{3.15}$$

so $\sup_{F \cap \partial B_\rho} J(u) < 0$ for $\rho > 0$ small. Result follows from Lemma 2.4. □

Next, we consider an asymptotically linear problem:

(f_5) $\lim_{|u| \to \infty} f(t,u)/u \in (k^4\pi^4, (k+1)^4\pi^4)$, uniformly for $t \in [0,1]$;

(f_6) $\lim_{|u| \to 0} f(t,u)/u \in (l^4\pi^4, (l+1)^4\pi^4)$, uniformly for $t \in [0,1]$.

Theorem 3.10 (asymptotically linear case). *Under* (f_1), (f_4), (f_5), *and* (f_6), *(BVP)* (1.2) *has at least* n *pairs of nodal solutions provided* $k > l+2$ *or* $l > k+1$. *Here,* $n = k - l - 2$, *if* $k > l+2$; *and* $n = l - k - 1$, *if* $l > k+1$.

Proof. Take $k^4\pi^4 < b_1 \le b_2 < (k+1)^4\pi^4$ and $\Lambda > 0$ such that for $|u| \ge \Lambda$, $b_1 \le f(t,u)/u \le b_2$. Now let $\{u_n\}$ be a (PS) sequence for $J(u)$. Writing $u_n = v_n + w_n$ with $v_n \in E_k = \bigoplus_{i=1}^{k}\{e_i\}$, $w_n \in E_k^{\perp}$, and taking inner product of $J'(u_n)$ and $v_n - w_n$, we see that

$$\begin{aligned}
o(1)\cdot\|u_n\| &= \langle J'(u_n), v_n - w_n\rangle \\
&= \langle u_n, v_n - w_n\rangle - \int_0^1 f(t,Tu_n)(Tv_n - Tw_n)dt \\
&= \langle v_n + w_n, v_n - w_n\rangle - \int_{|Tu_n|\ge\Lambda} \frac{f(t,Tu_n)}{Tu_n}\left((Tv_n)^2 - (Tw_n)^2\right)dt \\
&\quad - \int_{|Tu_n|<\Lambda} f(t,Tu_n)(Tv_n - Tw_n)dt \\
&\le \|v_n\|^2 - \|w_n\|^2 - b_1\int_{|Tu_n|\ge\Lambda}(Tv_n)^2dt + b_2\int_{|Tu_n|\ge\Lambda}(Tw_n)^2dt \\
&\quad - \int_{|Tu_n|<\Lambda} f(t,Tu_n)(Tv_n - Tw_n)dt \\
&= \|v_n\|^2 - \|w_n\|^2 - b_1\int_0^1 (Tv_n)^2dt + b_1\int_{|Tu_n|<\Lambda}(Tv_n)^2 + b_2\int_0^1(Tw_n)^2dt \\
&\quad - b_2\int_{|Tu_n|<\Lambda}(Tw_n)^2dt - \int_{|Tu_n|<\Lambda} f(t,Tu_n)(Tv_n - Tw_n)dt \\
&\le \|v_n\|^2 - \|w_n\|^2 - \frac{b_1}{k^4\pi^4}\|v_n\|^2 + \frac{b_2}{(k+1)^4\pi^4}\|w_n\|^2 + b_1\int_{|Tu_n|<\Lambda}(Tv_n)^2dt \\
&\quad - b_2\int_{|Tu_n|<\Lambda}(Tw_n)^2dt - \int_{|Tu_n|<\Lambda} f(t,Tu_n)(Tv_n - Tw_n)dt \\
&\le \left(1 - \frac{b_1}{k^4\pi^4}\right)\|v_n\|^2 + \left(\frac{b_2}{(k+1)^4\pi^4} - 1\right)\|w_n\|^2 + \frac{b_1b_2}{b_2 - b_1}\int_{|Tu_n|<\Lambda}|Tu_n|^2dt \\
&\quad + \left(\int_{|Tu_n|\le\Lambda}|f(t,Tu_n)|^2dt\right)^{1/2}\left(\int_{|Tu_n|<\Lambda}(Tv_n - Tw_n)^2dt\right)^{1/2} \\
&\le \left(1 - \frac{b_1}{k^4\pi^4}\right)\|v_n\|^2 + \left(\frac{b_2}{(k+1)^4\pi^4} - 1\right)\|w_n\|^2 + \frac{b_1b_2}{b_2 - b_1}\Lambda^2 + C\left(\int_0^1|Tu_n|^2dt\right)^{1/2} \\
&\le \left(1 - \frac{b_1}{k^4\pi^4}\right)\|v_n\|^2 + \left(\frac{b_2}{(k+1)^4\pi^4} - 1\right)\|w_n\|^2 + \frac{b_1b_2}{b_2 - b_1}\Lambda^2 + C\|u_n\| \\
&\le -a\|u_n\|^2 + C\|u_n\| + C_1.
\end{aligned} \tag{3.16}$$

So $\{u_n\}$ is bounded, where $a = \min\{b_1/k^4\pi^4 - 1, 1 - b_2/(k+1)^4\pi^4\} > 0$. Then, $J(u)$ satisfies the (PS) condition.

If $k > l+2$, let $F = \bigoplus_{i=1}^{k}\{e_i\}$, and $H = \bigoplus_{i=l+2}^{\infty}\{e_i\}$, then $\dim F = k$, and $\operatorname{codim} H = l+1$.

From (f_6), we know that there exist $\delta > 0$, and $\eta > 0$ such that

$$\frac{1}{2}(l^4\pi^4 + \delta)u^2 \leq F(t,u) \leq \frac{1}{2}((l+1)^4\pi^4 - \delta)u^2 \quad \text{for } |u| \leq \eta. \tag{3.17}$$

Then, for $\|u\| \leq (1/4)\eta$, $|Tu| \leq \eta$, we can obtain, when $u \in H$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^1 F(t,Tu)dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}((l+1)^4\pi^4 - \delta)\int_0^1 (Tu)^2 dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{(l+1)^4\pi^4 - \delta}{2(l+2)^4\pi^4}\|u\|^2 \\ &= \left(\frac{1}{2} - \frac{(l+1)^4\pi^4 - \delta}{2(l+2)^4\pi^4}\right)\|u\|^2 > 0. \end{aligned} \tag{3.18}$$

So, choose $\gamma = (1/4)\eta$, then $\inf_{H\cap\partial B_\gamma(0)} J(u) > 0$.

From (f_5), we can get there exist $\theta > 0$, $\Lambda > 0$ such that

$$\frac{1}{2}(k^4\pi^4 + \theta)u^2 - \Lambda \leq F(t,u) \leq \frac{1}{2}((k+1)^4\pi^4 - \theta)u^2 + \Lambda \quad \forall u \in R. \tag{3.19}$$

Then, when $u \in F$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu)dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{2}(k^4\pi^4 + \theta)\int_0^1 (Tu)^2 dt + \Lambda \\ &\leq \frac{1}{2}\|u\|^2 - \frac{k^4\pi^4 + \theta}{2k^4\pi^4}\|u\|^2 + \Lambda \\ &= -\frac{\theta}{2k^4\pi^4}\|u\|^2 + \Lambda. \end{aligned} \tag{3.20}$$

Choose ρ large enough such that $\rho > \gamma > 0$, and $\sup_{F\cap\partial B_\rho(0)} J(u) \leq 0$, result follows from Lemma 2.6.

If $l > k+1$, let $F = \bigoplus_{i=1}^{l}\{e_i\}$, $H = \bigoplus_{i=k+2}^{\infty}\{e_i\}$, then $\dim F = l$, $\operatorname{codim} H = k+1$. From (3.17), when $u \in F$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu)dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{2}(l^4\pi^4 + \delta)\int_0^1 (Tu)^2 dt \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\|u\|^2 - \frac{l^4\pi^4+\delta}{2l^4\pi^4}\|u\|^2 \\
&= -\frac{\delta}{2l^4\pi^4}\|u\|^2 < 0.
\end{aligned} \tag{3.21}$$

When $u \in H$, we know from (3.19),

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu)dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\left((k+1)^4\pi^4 - \theta\right)\int_0^1 (Tu)^2 dt - \Lambda \\
&\ge \frac{1}{2}\|u\|^2 - \frac{(k+1)^4\pi^4-\theta}{2(k+2)^4\pi^4}\|u\|^2 - \Lambda \\
&= \left(\frac{1}{2} - \frac{(k+1)^4\pi^4-\theta}{2(k+2)^4\pi^4}\right)\|u\|^2 - \Lambda \ge -\Lambda,
\end{aligned} \tag{3.22}$$

which means $\inf_H J(u) > -\infty$, then result follows from Lemma 2.5. □

Next, we consider a superlinear problem. Assume that
(f_7) there is $\mu > 2$ such that $0 < \mu F(t,u) \le f(t,u)u$ for $|u|$ large;
(f_8) there are $p > \mu$, $C > 0$ such that $F(t,u) \le C|u|^p$ for $|u|$ large.

Theorem 3.11 (superlinear nonlinearity). *Under (f_1), (f_4), (f_7), and (f_8), (BVP)* (1.2) *has infinitely many nodal solutions.*

Proof. From condition (f_7) by the standard argument, J satisfies $(\mathrm{PS})_c$ condition for every $c \in R$. Let $Z_k = \bigoplus_{i=k}^{\infty}\{e_i\}$. From (f_8), we obtain $|F(t,u)| \le C|u|^p + C_1$ for all $u \in R$. Define $\beta_k = \sup_{u\in Z_k,\|Tu\|=1}|Tu|$, it is very clear $\beta_k < \infty$ and $0 < \beta_{k+1} \le \beta_k$, so $\beta_k \to \beta \ge 0$ and $\beta \ne \pm\infty$. So if $u \in Z_k$,

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu)dt \\
&\ge \frac{1}{2}\|u\|^2 - \int_0^1 \left(C|Tu|^p + C_1\right)dt \\
&\ge \frac{1}{2}\|u\|^2 - C\beta_k^p\|Tu\|^p - C_1 \\
&\ge \frac{1}{2}\|u\|^2 - C\frac{\beta_k^p}{(k^2\pi^2)^p}\|u\|^p - C_1.
\end{aligned} \tag{3.23}$$

Choosing $r_k = (4C\beta_k^p)^{1/(2-p)}(k^2\pi^2)^{p/(p-2)}$, we obtain, if $u \in Z_k$ and $\|u\| = r_k$,

$$J(u) \ge \frac{(k^2\pi^2)^{2p/(p-2)}}{4(4C\beta_k^p)^{2/(p-2)}} - C_1 \longrightarrow \infty, \quad \text{if } k \longrightarrow \infty. \tag{3.24}$$

Let $Y_k = \bigoplus_{i=1}^{k}\{e_i\}$. From (f_7), after integrating, we obtain the existence of $C_2 > 0$ such that $F(t,u) \geq C_2|u|^\mu$ for $|u| \geq R$. Hence, we have $F(t,u) \geq C_2|u|^\mu - C_3$ for $u \in R$ and $C_3 > 0$ is constant. Therefore, when $u \in Y_k$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,Tu)dt \\ &\leq \frac{1}{2}\|u\|^2 - \int_0^1 \left(C_2|Tu|^\mu - C_3\right)dt \\ &\leq \frac{1}{2}\|u\|^2 - C\|Tu\|^\mu + C_3 \\ &\leq \frac{1}{2}\|u\|^2 - \frac{C}{(k^2\pi^2)^\mu}\|u\|^\mu + C_3. \end{aligned} \tag{3.25}$$

Noting $\mu > 2$, choose $\rho_k > r_k > 0$ large enough, such that $J(u)|_{u\in Y_k\cap\partial B_{\rho_k}} < 0$, and

$$\lim_{k\to\infty}\max_{u\in Y_k\cap\partial B_{\rho_k}} J(u) < \infty. \tag{3.26}$$

Result follows from Lemma 2.7. □

Remark 3.12. If there exist no strict supsolution and supersolution required in [20], just only using the functional J to get the critical point [10, 11], then we just know that (BVP) (1.2) has solutions, even we can know the sign of the critical point of the functional J because Tu is not strongly order-preserving in $L^2[0,1]$. In our paper, using admissible invariant sets in $C_0[0,1]$, we can settle the problem.

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