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Research Article

Critical Point Theory Applied to a Class of the Systems of the Superquadratic Wave Equations

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We show the existence of a nontrivial solution for a class of the systems of the superquadratic nonlinear wave equations with Dirichlet boundary conditions and periodic conditions with a superquadratic nonlinear terms at infinity which have continuous derivatives. We approach the variational method and use the critical point theory which is the Linking Theorem for the strongly indefinite corresponding functional.

1. Introduction

In this paper, we consider the existence of a nontrivial solution for the following class of the systems of the superquadratic wave equations with Dirichlet boundary condition and periodic condition

$$
\begin{gathered}
u_{tt}-u_{xx}=av+F_u(x,t,u,v), \quad \text{in}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times R,\\
v_{tt}-v_{xx}=bu+F_v(x,t,u,v), \quad \text{in}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times R,\\
u\left(\pm\frac{\pi}{2},t\right)=v\left(\pm\frac{\pi}{2},t\right)=0,\\
u(x,t+\pi)=u(x,t)=u(-x,t)=u(x,-t),\\
v(x,t+\pi)=v(x,t)=v(-x,t)=v(x,-t),
\end{gathered}
\tag{1.1}
$$

where $F : [-(\pi/2), \pi/2] \times R \times R \times R \rightarrow R$ is a superquadratic function at infinity which has continuous derivatives $F_r(x,t,r,s)$, $F_s(x,t,r,s)$ with respect to r, s, for almost any $(x,t) \in (-(\pi/2),\ \pi/2) \times R$. Moreover, we assume that F satisfies the following conditions:

(F1) $F(x,t,0,0) = F_x(x,t,0,0) = F_t(x,t,0,0) = 0$; $F_{xx}(x,t,0,0) = F_{tt}(x,t,0,0) = F_{xt}(x,t,0,0) = 0$, $F(x,t,r,s) > 0$ if $(r,s) \neq (0,0)$, $\inf_{(x,t)\in(-(\pi/2),\pi/2)\times R,\, |r|^2+|s|^2=R^2} F(x,t,r,s) > 0$,

(F2) $|F_r(x,t,r,s)| + |F_s(x,t,r,s)| \le c(|r|^\nu + |s|^\nu)\forall x, t, r, s$;

(F3) $rF_r(x,t,r,s) + sF_s(x,t,r,s) \ge \mu F(x,t,r,s)\forall x, t, r, s$;

(F4) $|F_r(x,t,r,s)| + |F_s(x,t,r,s)| \le d(F(x,t,r,s)^{\delta_1} + F(x,t,r,s)^{\delta_2})$;

where $c > 0$, $d > 0$, $R > 0$, $\mu > 2$, $\nu > 1$ and $1/2 < \delta_1 \le \delta_2 \le 1/r$, for some $1 < r < 2$.

As the physical model for these systems we can find crossing two beams with travelling waves, which are suspended by cable under a load. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. Choi and Jung investigate in [1–3] the existence and multiplicity of solutions of the single nonlinear wave equation with Dirichlet boundary condition.

Let us set

$$\mathcal{L}(u,v) = (Lu, Lv), \quad Lu = u_{tt} - u_{xx}. \tag{1.2}$$

Then, system (1.1) can be rewritten by

$$\begin{gathered} \mathcal{L}U = \nabla\left(\frac{1}{2}(AU,U) + F(x,t,u,v)\right), \\ U\left(\pm\frac{\pi}{2},t\right) = \begin{pmatrix}0\\0\end{pmatrix}, \\ U(x,t+\pi) = U(x,t) = U(-x,t) = U(x,-t), \end{gathered} \tag{1.3}$$

where ∇ is the gradient operator, $U = \binom{u}{v}$, $A = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} \in M_{2\times2}(R)$.

We note that $\sqrt{ab}$, $-\sqrt{ab}$ are two eigenvalues of the matrix $A = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix}$, and that

$$-\sqrt{ab}\|U\|_E^2 \le (AU,U)_{R^2} \le \sqrt{ab}\|U\|_E^2, \quad U = (u,v). \tag{1.4}$$

Let λ_{mn} be the eigenvalues of the eigenvalue problem $u_{tt} - u_{xx} = \lambda u$ in $(-(\pi/2),\pi/2) \times R$, $u(\pm(\pi/2),t) = 0$, $u(x,t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$.

Our main result is the following.

Theorem 1.1. *Let F satisfy the conditions (F1), (F2), (F3), and (F4). Assume that*

$$\lambda_{mn}^2 - ab \neq 0 \ \forall m,n \text{ with } (m,n) \neq (0,0), \tag{1.5}$$

$$a > 0, \quad b > 0, \tag{1.6}$$

$$\sqrt{ab} < 1. \tag{1.7}$$

Then, system (1.3) *has a nontrivial solution* (u,v).

In Section 2, we obtain some results on the nonlinear term F. In Section 3, we approach the variational method and recall the critical point theorem which is the linking theorem for the strongly indefinite functional. This plays a crucial role to find a nontrivial solution. In Section 4, we prove Theorem 1.1.

2. Some results on the nonlinear term F

The eigenvalue problem for $u(x,t)$,

$$\begin{gathered} u_{tt} - u_{xx} = \lambda u \quad \text{in} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\ u\left(\pm\frac{\pi}{2}, t\right) = 0, \qquad u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t) \end{gathered} \tag{2.1}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \quad (m, n = 0, 1, 2, \ldots), \tag{2.2}$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x \quad \text{for } n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x \quad \text{for } m > 0, n \geq 0. \end{aligned} \tag{2.3}$$

Let Q be the square $[-(\pi/2), \pi/2] \times [-(\pi/2), \pi/2]$ and H_0 the Hilbert space defined by

$$H_0 = \left\{ u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t \text{ and } \int_Q u = 0 \right\}. \tag{2.4}$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, by

$$u = \sum h_{mn} \phi_{mn}. \tag{2.5}$$

We define a subspace $\mathfrak{D}$ of H_0 as follows:

$$\mathfrak{D} = \left\{ u \in \sum h_{mn}\phi_{mn} : \sum_{mn} \lambda_{mn}^2 h_{mn}^2 < +\infty \right\}. \tag{2.6}$$

Then, this space is a Banach space with norm

$$\|u\| = \left[\sum \lambda_{mn}^2 h_{mn}^2 \right]^{1/2}. \tag{2.7}$$

Let us set $E = \mathfrak{D} \times \mathfrak{D}$. We endow the Hilbert space E with the norm

$$\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2. \tag{2.8}$$

We are looking for the weak solutions of (1.3) in $\mathfrak{D} \times \mathfrak{D}$, that is, (u,v) such that $u \in \mathfrak{D}$, $v \in \mathfrak{D}$, $Lu = av + F_u(x,t,u,v)$, $Lv = bu + F_v(x,t,u,v)$. Since $|\lambda_{mn}| \geq 1$ for all m, n, we have the following lemma.

Lemma 2.1. (i) $\|u\| \geq \|u\|_{L^2(Q)}$, *where* $\|u\|_{L^2(Q)}$ *denotes the* L^2 *norm of* u.
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(Q)} = 0$.
(iii) $u_{tt} - u_{xx} \in \mathfrak{D}$ *implies* $u \in \mathfrak{D}$.

Lemma 2.2. *Suppose that* c *is not an eigenvalue of* $L : \mathfrak{D} \to H_0$, $Lu = u_{tt} - u_{xx}$, *and let* $f \in H_0$. *Then, one has* $(L-c)^{-1} f \in \mathfrak{D}$.

Proof. Let λ_{mn} be an eigenvalue of L. We note that $\{\lambda_{mn} : |\lambda_{mn}| < |c|\}$ is finite. Let

$$f = \sum h_{mn}\phi_{mn}, \tag{2.9}$$

then

$$(L-c)^{-1} f = \sum \frac{1}{\lambda_{mn} - c} h_{mn}\phi_{mn}. \tag{2.10}$$

Hence, we have the inequality

$$\left\|(L-c)^{-1} f\right\|^2 = \sum \lambda_{mn}^2 \frac{1}{(\lambda_{mn} - c)^2} h_{mn}^2 \leq C \sum h_{mn}^2 \tag{2.11}$$

for some C, which means that

$$\left\|(L-c)^{-1} f\right\| \leq C_1 \|f\|_{L^2(Q)}, \quad C_1 = \sqrt{C}. \tag{2.12}$$

□

By (F1) and (F3), we obtain the lower bound for $F(x,t,u,v)$ in the term of $|u|^\mu + |v|^\mu$.

Lemma 2.3. *Assume that* F *satisfies the conditions (F1) and (F3). Then, there exist* a_0, $b_0 \in R$ *with* $a_0 > 0$ *such that*

$$F(x,t,r,s) \geq a_0\left(|r|^\mu + |s|^\mu\right) - b_0, \quad \forall x,t,r,s. \tag{2.13}$$

Proof. Let r, s be such that $r^2 + s^2 \geq R^2$. Let us set $\varphi(\xi) = F(x,t,\xi r,\xi s)$ for $\xi \geq 1$. Then,

$$\varphi(\xi)' = rF_r(x,t,\xi r,\xi s) + sF_s(x,t,\xi r,\xi s) \geq \frac{\mu}{\xi}\varphi(\xi). \tag{2.14}$$

Multiplying by $\xi^{-\mu}$, we get

$$(\xi^{-\mu}\varphi(\xi))' \geq 0, \tag{2.15}$$

hence $\varphi(\xi) \geq \varphi(1)\xi^{\mu}$ for $\xi \geq 1$. Thus, we have

$$\begin{aligned} F(x,t,r,s) &\geq F\left(x,t,\frac{Rr}{\sqrt{r^2+s^2}},\frac{Rs}{\sqrt{r^2+s^2}}\right)\left(\frac{\sqrt{r^2+s^2}}{R}\right)^{\mu} \\ &\geq c_0\left(\frac{\sqrt{r^2+s^2}}{R}\right)^{\mu} \geq a_0(|r|^{\mu}+|s|^{\mu})-b_0 \end{aligned} \tag{2.16}$$

for some a_0, b_0, where $c_0 = \inf\{F(x,t,r,s) \mid (x,t) \in Q, r^2+s^2 = R^2\}$. □

Lemma 2.4. *Assume that F satisfies the conditions $(F1)$, $(F2)$, and $(F3)$. Then,*

(i) $\int_Q F(x,t,0,0)dx\,dt = 0$, $\int_Q F(x,t,u,v)dx\,dt > 0$ *if* $(u,v) \neq (0,0)$, $\mathrm{grad}(\int_Q F(x,t,u,v))dx\,dt = o(\|(u,v)\|_E)$ *as* $(u,v) \to (0,0)$;

(ii) *there exist* $a_0 > 0$, $\mu > 2$ *and* $b_1 \in R$ *such that*

$$\int_Q F(x,t,u,v)dx\,dt \geq a_0\|(u,v)\|^{\mu}_{L^{\mu}} - b_1 \quad \forall (u,v) \in E, \tag{2.17}$$

(iii) $(u,v) \to \mathrm{grad}(\int_Q F(x,t,u,v))dx\,dt$ *is a compact map;*

(iv) *if* $\int_Q [uF_u(x,t,u,v)+vF_v(x,t,u,v)]dx\,dt - 2\int_Q F(x,t,u,v)dx\,dt = 0$, *then* $\mathrm{grad}(\int_Q F(x,t,u,v)dx\,dt) = 0$;

(v) *if* $\|(u_n,v_n)\|_E \to +\infty$ *and* $(\int_Q [u_nF_u(x,t,u_n,v_n)+v_nF_v(x,t,u_n,v_n)]dx\,dt - 2\int_Q F(x,t,u_n,v_n)dx\,dt)/\|(u,v)\|_E \to 0$, *then, there exists* $((u_{h_n},v_{h_n}))_n$ *and* $w \in E$ *such that*

$$\frac{\mathrm{grad}(\int_Q F(x,y,u_n,v_n)dx\,dt)}{\|(u_{h_n},v_{h_n})\|_E} \longrightarrow w, \qquad \frac{(u_{h_n},v_{h_n})}{\|(u_{h_n},v_{h_n})\|_E} \rightharpoonup (0,0). \tag{2.18}$$

Proof. (i) It follows from (F1) and (F2), since $1 < \nu$;

(ii) by Lemma 2.3, for $U = (u,v) \in E$,

$$\int_Q F(x,t,U)dx\,dt \geq a_0\|U\|^{\mu}_{L^{\mu}}dx\,dt - b_1, \tag{2.19}$$

where $b_1 \in R$, thus, (ii) holds;

(iii) it is easily obtained with standard arguments;

(iv) it is implied by (F3) and the fact that $F(x,t,u,v) > 0$ for $(u,v) \neq (0,0)$;

(v) by Lemma 2.3 and (F3), for $U=(u,v)$,

$$\begin{aligned}&\int_Q [uF_u(x,t,u,v)+vF_v(x,t,u,v)]dx\,dt-2\int_Q F(x,t,u,v)dx\,dt\\&\quad\geq(\mu-2)\int_Q F(x,t,u,v)dx\,dt\geq(\mu-2)(a_0\|U\|_{L^\mu}^{\mu}-b_1).\end{aligned}\tag{2.20}$$

By (F2),

$$\left\|\operatorname{grad}\left(\int_Q F(x,t,u,v)dx\,dt\right)\right\|_E\leq C'\|F_U(x,t,U)\|_{L^r}\leq C''\||U|^{\nu}\|_{L^r}\tag{2.21}$$

for some $1<r<2$ and suitable constants C',C''. To get the conclusion it suffices to estimate $\||U|^{\nu}/\|U\|_E\|_{L^r}$ in terms of $\|U\|_{L^\mu}^{\mu}/\|U\|_E$. If $\mu\geq r\nu$, then this is a consequence of Hölder inequality. If $\mu<r\nu$, by the standard interpolation arguments, it follows that $\||U|^{\nu}/\|U\|_E\|_{L^r}\leq C(\|U\|_{L^\mu}^{\mu}/\|U\|_E)^{\nu/\mu}\|U\|_E^{l}$, where l is such that $l=-1+\nu/\mu$. Thus, we prove (v). □

Lemma 2.5. *Assume that F satisfies the conditions (F1), (F2), (F3), and (F4). Then, there exist φ, $\psi:[0,+\infty]\to R$ continuous and such that $\psi(s)/s\to 0$ as $s\to 0$, $\varphi(s)>0$ if $s>0$,*

(i) $\|\operatorname{grad}\int_Q F(x,t,u,v)dx\,dt\|_E^2\leq\psi(\int_Q F(x,t,u,v)dx\,dt)$, $\forall(u,v)\in E$,

(ii) $\int_Q[uF_u(x,t,u,v)+vF_v(x,t,u,v)]dx\,dt-2\int_Q F(x,t,u,v)dx\,dt\geq\varphi(u,v)$, $\forall(u,v)\in E$.

Proof. (i) By (F4), for all $U=(u,v)\in E$,

$$\begin{aligned}\left\|\operatorname{grad}\left(\int_Q F(x,t,U)dx\,dt\right)\right\|_E&\leq\|F_U(x,t,U)\|_{L^r}\\&\leq C_1\left\|F(x,t,U)^{\delta_1}+F(x,t,U)^{\delta_2}\right\|_{L^r}\\&\leq C_2\left(\left\|F(x,t,U)^{\delta_1}\right\|_{L^r}+\left\|F(x,t,U)^{\delta_2}\right\|_{L^r}\right)\\&\leq C_3\left(\left\|F(x,t,U)^{\delta_1}\right\|_{L^{1/\delta_1}}+\left\|F(x,t,U)^{\delta_2}\right\|_{L^{1/\delta_2}}\right)\\&\leq C_4\left(\|F(x,t,U)\|_{L^1}^{\delta_1}+\|F(x,t,U)\|_{L^1}^{\delta_2}\right)\\&=C_5\left(\left(\int_Q F(x,t,U)dx\,dt\right)^{\delta_1}+\left(\int_Q F(x,t,U)dx\,dt\right)^{\delta_2}\right),\end{aligned}\tag{2.22}$$

where $1<r<1/\delta_1,1/\delta_2<2$, C_1, C_2, C_3, C_4 and C_5 are constants. Since δ_1, $\delta_2>1/2$, we prove (i).

(ii) By (F3),

$$\begin{aligned}&\int_Q [uF_u(x,t,u,v)+vF_v(x,t,u,v)]dx\,dt-2\int_Q F(x,t,u,v)dx\,dt\\&\quad\geq(\mu-2)\int_Q F(x,t,U)dx\,dt\geq(\mu-2)\big(a_0\|U\|^{\mu}_{L^{\mu}}-b_1\big).\end{aligned}\tag{2.23}$$

Thus, we prove (ii). □

3. Variational approach and linking theorem

Now we are looking for the weak solutions of system (1.3). We shall approach the variational method and recall the linking theorem for the strongly indefinite functional. We observe that the weak solutions of (1.3) coincide with the critical points of the corresponding functional

$$\begin{gathered}I:E\longrightarrow R\in C^{1,1},\\ I(U)=\frac{1}{2}\int_Q \mathcal{L}U\cdot U dx\,dt-\frac{1}{2}\int_Q (AU,U)_{R^2}dx\,dt-\int_Q F(x,t,u,v)dx\,dt.\end{gathered}\tag{3.1}$$

Now, we recall the linking theorem for strongly indefinite functional (cf. [4]).

Lemma 3.1 (linking theorem). *Let E be a real Hilbert space with $E=E_1\oplus E_2$ and $E_2=E_1^{\perp}$. one supposes that*

(I1) $I\in C^1(E,R)$, *satisfies $(P.S.)^*$ condition;*

(I2) $I(u)=1/2(Lu,u)+bu$, *where $Lu=L_1P_1u+L_2P_2u$ and $L_i:E_i\longrightarrow E_i$ is bounded and self-adjoint, $i=1,2$;*

(I3) *b' is compact;*

(I4) *there exists a subspace $\widetilde{E}\subset E$ and sets $S\subset E$, $T\subset\widetilde{E}$ and constants $\alpha>w$ such that:*

(i) *$S\subset E_1$ and $I|_S\geq\alpha$;*

(ii) *T is bounded and $I|_{\partial T}\leq w$;*

(iii) *S and ∂T link.*

Then, I possesses a critical value $c\geq\alpha$.

Let E^-, E^0, E^+ be the subspace of E on which the functional $U\mapsto(1/2)\int_Q\mathcal{L}U\cdot U$ is positive definite, null, negative definite, and E^-, E^0 and E^+ are mutually orthogonal. Let P^+ be the projection for E onto E^+, P^0 the one from E onto E^0, and P^- the one from E onto E^-. Let $(E_n)_n$ be a sequence of closed subspaces of E with the conditions

$$\begin{gathered}E_n=E_n^-\oplus E^0\oplus E_n^+,\quad\text{where }E_n^+\subset E^+,E_n^-\subset E^-\forall n,\\ (E_n^+\text{ and }E_n^-\text{ are subspaces of }E),\dim E_n<+\infty,E_n\subset E_{n+1},\cup_{n\in N}E_n\text{ is dense in }E.\end{gathered}\tag{3.2}$$

Let P_{E_n} be the orthogonal projections from E onto E_n.

Let us prove that the functional I satisfies the linking geometry.

Lemma 3.2. *Assume that the conditions* (1.5), (1.6), *and* (1.7) *hold. Then, for any F with (F1), (F2), (F3), and (F4),*

(i) *there exist a small number $\rho > 0$ and a small ball $B_\rho \subset E^+$ with radius ρ such that if $U \in \partial B_\rho$, then*

$$\alpha = \inf\ I(U) > 0, \tag{3.3}$$

(ii) *there is an $e \in E^+$, $R > \rho$ and a large ball D_R with radius $R > 0$ such that if*

$$W = \left(\overline{D}_R \cap \left(E^0 \oplus E^-\right)\right) \oplus \{re \mid 0 < r < R\}, \tag{3.4}$$

then

$$\sup_{U \in \partial W} I(U) \le 0. \tag{3.5}$$

Proof. (i) By (1.7) and (i) of Lemma 2.4, we can find a small number ρ such that, for $U \in E^+$,

$$\begin{aligned} I(U) &= \frac{1}{2}\int_Q \mathcal{L}U\cdot U - \frac{1}{2}\int_Q (AU,U)_{R^2} - \int_Q F(x,t,u,v)dx\,dt \\ &\ge \frac{1}{2}\left(1 - \frac{\sqrt{ab}}{\lambda_{00}}\right)\|U\|_E^2 - 0(\|U\|_E). \end{aligned} \tag{3.6}$$

Since $\sqrt{ab} < 1 = \lambda_{00}$, there exist a small number $\rho > 0$ and a small ball $B_\rho \subset E^+$ with radius ρ such that if $U \in \partial B_\rho$, then $\inf\ I(U) > 0$. Thus, the assertion (1) holds;

(ii) let us choose an element $e \in E^+$. Let $U \neq (0,0) \in E^0 \oplus E^- \oplus \{re \mid r > 0\}$. We note that

$$\begin{aligned} &\text{if } U \in E^+, \text{ then} \int_Q (\mathcal{L}U\cdot U - (AU,U)_{R^2})dx\,dt \ge \tau_1\|U\|_E^2, \\ &\text{if } U \in E^-, \text{ then} \int_Q (\mathcal{L}U\cdot U - (AU,U)_{R^2})dx\,dt \le -\tau_2\|U\|_E^2 \end{aligned} \tag{3.7}$$

for some $\tau_1 > 0$, $\tau_2 > 0$. Let us choose a sequence $(U_n)_n$, $U_n = (u_n, v_n) \neq (0,0) \in E^0 \oplus E^- \oplus \{re \mid r > 0\}$ such that $\|U_n\|_E \to \infty$. Let us set $\breve{U}_n = U_n/\|U_n\|_E$. By Lemma 2.3, we have that

$$\begin{aligned} \frac{I(U_n)}{\|U_n\|_E^2} &\le \|\mathcal{L} - A\|\|P^+\breve{U}_n\|_E^2 - a_0\|\breve{U}_n\|_{L^\mu}^\mu\|U_n\|_E^{\mu-2} + \frac{b_0}{\|U_n\|^2} - \tau_2\|P^-\breve{U}_n\|_E^2 \\ &= \|\mathcal{L} - A\|\frac{r^2\|e\|_E^2}{\|U_n\|_E^2} - a_0\|\breve{U}_n\|_{L^\mu}^\mu\|U_n\|_E^{\mu-2} + \frac{b_0}{\|U_n\|^2} - \tau_2\|P^-\breve{U}_n\|_E^2. \end{aligned} \tag{3.8}$$

Since $\|U_n\|_E \to \infty$, two possible cases arise. For the case $\|\breve{U}_n\|_{L^\mu} \to 0$ it follows that $\breve{U}_n \rightharpoonup 0$, hence $P^+\breve{U}_n \to 0$ and $P^0\breve{U}_n \to 0$. Thus $\|P^-\breve{U}_n\|_E \to 1$. Hence

$$\limsup_{n\to\infty} \frac{I(u_n)}{\|U_n\|_E^2} \le -\tau_2. \tag{3.9}$$

For the case $\|\breve{U}_n\|_{L^\mu} \ge \varepsilon > 0$ (3.6) implies

$$\lim_{n\to\infty} \frac{I(u_n)}{\|U_n\|_E^2} = -\infty. \tag{3.10}$$

Thus, we can choose a large number $R > 0$ and a large ball $D_R \subset E^0 \oplus E^-$ with radius $R > 0$ such that if $W = \overline{D}_R \cap (E^0 \oplus E^-)) \oplus \{re \mid 0 < r < R\}$, then $\sup_{U\in\partial W} I(U) \le 0$. So the assertion (ii) holds. □

We shall prove that the functional I satisfies the $(P.S.)_c^*$ condition with respect to $(E_n)_n$ for any $c \in R$.

Lemma 3.3. *Assume that the conditions* (1.5), (1.6), *and* (1.7) *hold. Then, for any F with (F1), (F2), (F3), and (F4), the functional I satisfies the* $(P.S.)_c^*$ *condition with respect to* $(E_n)_n$ *for any real number c.*

Proof. Let $c \in R$ and (h_n) be a sequence in N such that $h_n \to +\infty$,$(U_n)_n$ be a sequence such that

$$U_n = (u_n, v_n) \in E_{h_n}, \forall n, I(U_n) \longrightarrow c, P_{E_{h_n}} \nabla I(U_n) \longrightarrow 0. \tag{3.11}$$

We claim that $(U_n)_n$ is bounded. By contradiction we suppose that $\|U_n\|_E \to +\infty$ and set $\widehat{U}_n = U_n/\|U_n\|_E$. Then

$$\begin{aligned}\langle P_{E_{h_n}} \nabla I(U_n), \widehat{U}_n\rangle &= \langle \nabla I(U_n), \widehat{U}_n\rangle \\ &= 2\frac{I(U_n)}{\|U_n\|_E} - \frac{\int_Q \nabla F(x,t,U_n)\cdot U_n dx\,dt - 2\int_Q F(x,t,U_n) dx\,dt}{\|U_n\|_E} \longrightarrow 0,\end{aligned} \tag{3.12}$$

hence

$$\frac{\int_Q \nabla F(x,t,U_n)\cdot U_n dx\,dt - 2\int_Q F(x,t,U_n) dx\,dt}{\|U_n\|_E} \longrightarrow 0. \tag{3.13}$$

By (v) of Lemma 2.4,

$$\frac{\operatorname{grad}\int_Q F(x,t,U_n) dx\,dt}{\|U_n\|_E} \text{ converges} \tag{3.14}$$

and $\widehat{U}_n \rightharpoonup 0$. We get

$$\frac{P_{E_{h_n}} \nabla I(U_n)}{\|U_n\|_E} = P_{E_{h_n}} \mathcal{L}\widehat{U}_n - A\widehat{U}_n - \frac{P_{E_{h_n}} \operatorname{grad}\left(\int_Q F(x,t,U_n)dx\,dt\right)}{\|U_n\|_E} \longrightarrow 0, \tag{3.15}$$

so $(P_{E_{h_n}} \mathcal{L}\widehat{U}_n - A\widehat{U}_n)_n$ converges. Since $(\widehat{U}_n)_n$ is bounded and $\mathcal{L} - A$ is a compact mapping, up to subsequence, $(\widehat{U}_n)_n$ has a limit. Since $\widehat{U}_n \rightharpoonup (0,0)$, we get $\widehat{U}_n \to (0,0)$, which is a contradiction to the fact that $\|\widehat{U}_n\|_E = 1$. Thus $(U_n)_n$ is bounded. We can now suppose that $U_n \rightharpoonup U$ for some $U \in E$. Since the mapping $U \mapsto \operatorname{grad}(\int_Q F(x,t,U)dx\,dt)$ is a compact mapping, $\operatorname{grad}(\int_Q F(x,t,U_n)dx\,dt) \to \operatorname{grad}(\int_Q F(x,t,u,v)dx\,dt)$. Thus, $(P_{E_{h_n}}(\mathcal{L}U_n - AU_n))_n$ converges. Since $\mathcal{L} - A$ is a compact operator and $(U_n)_n$ is bounded, we deduce that, up to a subsequence, $(U_n)_n$ converges to some U strongly with $\nabla I(U) = \lim \nabla I(U_n) = 0$. Thus, we prove the lemma. □

4. Proof of Theorem 1.1

Assume that the conditions (1.5), (1.6), and (1.7) hold and F satisfies (F1), (F2), (F3), and (F4). We note that $I(0,0) = 0$. By (iii) of Lemma 2.4, $U \mapsto \operatorname{grad}(\int_Q F(x,t,u,v)dx\,dt)$ is a compact mapping. By Lemma 3.2, there exists a small number $\rho > 0$ and a small ball $B_\rho \subset E^+$ with radius ρ such that if $U \in \partial B_\rho$, then $\alpha = \inf I(U) > 0$, and there is an $e \in E^+$, $R > \rho > 0$ and a large ball D_R with radius $R > 0$ such that if

$$W = (\overline{D}_R \cap (E^0 \oplus E^-)) \oplus \{re \mid 0 < r < R\}, \tag{4.1}$$

then

$$\sup_{U \in \partial W} I(U) \leq 0. \tag{4.2}$$

Let us set $\beta = \sup_W I$. We note that $\beta < +\infty$. Let $(E_n)_n$ be a sequence of subspaces of E satisfying (3.2). Clearly $E^0 \subset E_n$ for all n, and ∂B_ρ and ∂W link. We have, for all $n \in N$,

$$\sup_{\partial W \cap E_n} I < \inf_{\partial B_\rho \cap E_n} I. \tag{4.3}$$

Moreover, by Lemma 3.3, $I_n = I|_{E_n}$ satisfies the $(P.S.)^*_c$ condition for any $c \in R$. Thus by Lemma 3.1 (linking theorem), there exists a critical point U_n for I_n with

$$\alpha \leq \inf_{\partial B_\rho \cap E_n} I \leq I(U_n) \leq \sup_{W \cap E_n} I \leq \beta. \tag{4.4}$$

Since I_n satisfies the $(P.S.)^*_c$ condition, we obtain that, up to a subsequence, $U_n \to U$, with U a critical point for I such that $\alpha \leq I(U) \leq \beta$. Hence, $U \neq (0,0)$. Thus, system (1.5) has a nontrivial solution. Thus Theorem 1.1 is proved.

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