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Research Article

Blowup for a Non-Newtonian Polytropic Filtration System Coupled via Nonlinear Boundary Flux

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We study the global existence and the global nonexistence of a non-Newtonian polytropic filtration system coupled via nonlinear boundary flux. We first establish a weak comparison principle, then discuss the large time behavior of solutions by using modified upper and lower solution methods and constructing various upper and lower solutions. Necessary and sufficient conditions on the global existence of all positive (weak) solutions are obtained.

1. Introduction

In this paper,we consider the following Neumann problem:

$$(u^{k_1})_t = \Delta_m u, \qquad (v^{k_2})_t = \Delta_n v, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\nabla_m u \cdot \nu = u^{\alpha} v^{p}, \qquad \nabla_n v \cdot \nu = u^{q} v^{\beta}, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \overline{\Omega}, \tag{1.3}$$

where $\Delta_k u = \operatorname{div}(|\nabla u|^{k-1}\nabla u) = \sum_{i=1}^{N}(|\nabla u|^{k-1}u_{x_i})_{x_i}$, $\nabla_k u = (|\nabla u|^{k-1}u_{x_1}, \dots, |\nabla u|^{k-1}u_{x_N})$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward normal vector on the boundary $\partial\Omega$, $k_1, k_2, m, n > 0$, $\alpha, \beta \geq 0$, $p, q > 0$, and $u_0(x), v_0(x) \in C^1(\overline{\Omega})$ are positive and satisfy the compatibility conditions.

Parabolic equations like (1.1) appear in population dynamics, chemical reactions, heat transfer, and so on. In particular, (1.1) may be used to describe the nonstationary flows in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, (1.1) are called the non-Newtonian polytropic filtration equations (see [1–6] and the references therein). For the Neuman problem (1.1)–(1.3), the local existence of solutions in time has been established, see survey in [4].

Recall the single quasilinear parabolic equation with nonlinear boundary condition

$$\begin{aligned}&\left(u^k\right)_t = \Delta u, \quad x \in \Omega,\ t > 0,\\&\frac{\partial u}{\partial \nu} = u^\alpha, \quad x \in \partial\Omega,\ t > 0,\\&u(x,0) = u_0(x), \quad x \in \overline{\Omega},\end{aligned} \tag{1.4}$$

with $k > 0$, $\alpha \geq 0$.It is known that the solutions of (1.4) exist globally if and only if $\alpha \leq k$ for $0 < k \leq 1$; they exist globally if and only if $\alpha \leq (k+1)/2$ when $k > 1$ (see [7–10]).

In [11, 12], M. Wang and S. Wang studied the following nonlinear diffusion system with nonlinear boundary conditions

$$\begin{aligned}&\left(u^{k_1}\right)_t = \Delta u, \quad \left(v^{k_2}\right)_t = \Delta v, \quad x \in \Omega,\ t > 0,\\&\frac{\partial u}{\partial \nu} = u^\alpha v^p, \quad \frac{\partial v}{\partial \nu} = u^q v^\beta, \quad x \in \partial\Omega,\ t > 0,\\&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega},\end{aligned} \tag{1.5}$$

with $k_1, k_2 > 0$, $\alpha, \beta \geq 0$, $p, q > 0$. In [11], they obtained the necessary and sufficient conditions to the global existence of solutions for $0 < k_1, k_2 \leq 1$. In [12], they considered the case of $k_1 > 1$ or $k_2 > 1$ and obtained the necessary and sufficient blowup conditions for the special case $\Omega = B_R(0)$ (the ball centered at the origin in $\mathbb{R}^N$ with radius R). However, for the general domain Ω, they only gave some sufficient conditions to the global existence and the blowup of solutions.

In [2], Wang considered the following system with nonlinear boundary conditions:

$$\begin{aligned}&\left(u^{k_1}\right)_t = \left(|u_x|^{m-1}u_x\right)_x, \quad \left(v^{k_2}\right)_t = \left(|v_x|^{n-1}v_x\right)_x, \quad x \in (0,1),\ t > 0,\\&u_x(0,t) = 0, \quad u_x(1,t) = \lambda u^\alpha v^p(1,t), \quad t > 0,\\&v_x(0,t) = 0, \quad v_x(1,t) = \lambda u^q v^\beta(1,t), \quad t > 0,\\&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in [0,1],\end{aligned} \tag{1.6}$$

with $\lambda > 0$. They obtained the necessary and sufficient conditions on the global existence of all positive (weak) solutions.

Sun and Wang in [13] studied the nonlinear equation with nonlinear boundary condition

$$\begin{aligned}&\left(u^k\right)_t = \Delta_m u, \quad x \in \Omega,\ t > 0,\\&\nabla_m u \cdot \nu = u^\alpha, \quad x \in \partial\Omega,\ t > 0,\\&u(x,0) = u_0(x), \quad x \in \overline{\Omega}.\end{aligned} \tag{1.7}$$

They proved that all positive (weak) solutions of (1.7) exist globally if and only if $\alpha \le k$ when $k \le m$; they exist globally if and only if $\alpha \le m(k+1)/(m+1)$ when $k > m$.

The main purpose of this paper is to study the influence of nonlinear power exponents on the existence and nonexistence of global solutions of (1.1)–(1.3). By using upper- and lower-solution methods, we obtain the necessary and sufficient conditions on the existence of global (weak) solutions to (1.1)–(1.3). Our main results are stated as follows.

Theorem 1.1. *If $k_1 \ge m$, $k_2 \ge n$, then all positive (weak) solutions of* (1.1)–(1.3) *exist globally if and only if $\alpha \le m(k_1+1)/(m+1)$, $\beta \le n(k_2+1)/(n+1)$ and $pq \le (m(k_1+1)/(m+1)-\alpha)(n(k_2+1)/(n+1)-\beta)$.*

Theorem 1.2. *If $k_1 < m$, $k_2 \ge n$, then all positive (weak) solutions of* (1.1)–(1.3) *exist globally if and only if $\alpha \le k_1$, $\beta \le n(k_2+1)/(n+1)$ and $pq \le (k_1-\alpha)(n(k_2+1)/(n+1)-\beta)$.*

Theorem 1.3. *If $k_1 \ge m$, $k_2 < n$, then all positive (weak) solutions of* (1.1)–(1.3) *exist globally if and only if $\alpha \le m(k_1+1)/(m+1)$, $\beta \le k_2$ and $pq \le (m(k_1+1)/(m+1)-\alpha)(k_2-\beta)$.*

Theorem 1.4. *If $k_1 < m$, $k_2 < n$, then all positive (weak) solutions of* (1.1)–(1.3) *exist globally if and only if $\alpha \le k_1$, $\beta \le k_2$ and $pq \le (k_1-\alpha)(k_2-\beta)$.*

Remark 1.5. If $m = n = 1$, $0 < k_1, k_2 \le 1$, the results in [11] are included in Theorem 1.4, and if $m = n = 1$, $k_1 > 1$ or $k_2 > 1$, Theorems 1.1–1.3 improve the results of [12].

Remark 1.6. If we extend the solution to (1.6) to the interval $[-1,1]$ by symmetry, we get a solution to the same problem (1.6) with the condition at $x = 0$, substituted by a condition at $x = -1$, $-u_x(-1,t) = \lambda u^\alpha v^p(-1,t)$, $-v_x(-1,t) = \lambda u^q v^\beta(-1,t)$, $t > 0$. Conversely, symmetric solutions to this latter problem are solutions to the original problem (1.6). The problem (1.1)–(1.3) is the more general N-dimensional version of the problem (1.6). Theorems 1.1–1.4 extend the results of the problem (1.6) into multidimensional case and it seems to be a natural extension of Wang [2].

Remark 1.7. If $k_1 = k_2$, $m = n$, $\alpha = \beta$, $p = q = 0$, the conclusions of Theorems 1.1 and 1.4 are consistent with those of the single equation (1.7). This paper generalizes the results of the single equation (1.7) to the system (1.1)–(1.3).

The rest of this paper is organized as follows. Some preliminaries will be given in Section 2. Theorems 1.1–1.4 will be proved in Sections 3–5, respectively.

2. Preliminaries

As it is well known that degenerate and singular equations need not possess classical solutions, we give a precise definition of a weak solution to (1.1)–(1.3).

Definition 2.1. Let $T > 0$ and $Q_T = \Omega \times (0,T]$. A vector function $(u(x,t), v(x,t))$ is called a weak upper (or lower) solution to (1.1)–(1.3) in Q_T if

(i) $u(x,t), v(x,t) \in L^\infty(0,T;W^{1,\infty}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \cap C(\overline{Q_T})$;

(ii) $(u(x,0), v(x,0)) \ge (\le)(u_0(x), v_0(x))$;

(iii) for any positive two functions $\varphi_1, \varphi_2 \in L^1(0,T;W^{1,2}(\Omega)) \cap L^2(Q_T)$, one has

$$\begin{aligned}
&\iint_{Q_T} \left[(u^{k_1})_t \varphi_1 + \nabla_m u \cdot \nabla \varphi_1\right] dx\, dt \geq (\leq) \int_0^T \int_{\partial\Omega} u^\alpha v^p \varphi_1\, ds\, dt, \\
&\iint_{Q_T} \left[(v^{k_2})_t \varphi_2 + \nabla_n v \cdot \nabla \varphi_2\right] dx\, dt \geq (\leq) \int_0^T \int_{\partial\Omega} u^q v^\beta \varphi_2\, ds\, dt.
\end{aligned} \tag{2.1}$$

In particular, $(u(x,t), v(x,t))$ is called a weak solution of (1.1)–(1.3) if it is both a weak upper and a lower solution. For every $T < \infty$, if $(u(x,t), v(x,t))$ is a solution of (1.1)–(1.3) in Q_T, we say that $(u(x,t), v(x,t))$ is global.

Next we give some preliminary propositions and lemmas.

Proposition 2.2 (comparison principle). *Assume that* u_0, v_0 *are positive* $C^1(\overline{\Omega})$ *functions and* (u,v) *is any weak solution of* (1.1)–(1.3). *Also assume that* $(\underline{u}, \underline{v}) \geq (\delta_0, \delta_0) > 0$ *and* $(\overline{u}, \overline{v})$ *are a lower and an upper solution of* (1.1)–(1.3) *in* Q_T, *respectively, with nonlinear boundary flux* $(\underline{\lambda}\underline{u}^\alpha \underline{v}^p, \underline{\lambda}\underline{u}^q \underline{v}^\beta)$ *and* $(\overline{\lambda}\overline{u}^\alpha \overline{v}^p, \overline{\lambda}\overline{u}^q \overline{v}^\beta)$, *where* $0 < \underline{\lambda} < 1 < \overline{\lambda}$. *Then we have* $(\overline{u}, \overline{v}) \geq (u,v) \geq (\underline{u}, \underline{v})$ *in* Q_T.

Proof. For small $\sigma > 0$, letting $\varphi_\sigma(z) = \min\{1, \max\{z/\sigma, 0\}\}$, $z \in \mathbb{R}$, and setting $\varphi_1 = \varphi_\sigma(\underline{u} - u)$, according to the definition of upper and lower solutions, we have

$$\begin{aligned}
&\iint_{Q_\tau} \left[(\underline{u}^{k_1} - u^{k_1})_t \varphi_\sigma(\underline{u} - u) + \left(\nabla_m \underline{u} - \nabla_m u\right) \cdot \nabla \varphi_\sigma(\underline{u} - u)\right] dx\, dt \\
&\quad \leq \int_0^\tau \int_{\partial\Omega} \left(\underline{\lambda}\underline{u}^\alpha \underline{v}^p - u^\alpha v^p\right) \varphi_\sigma(\underline{u} - u) ds\, dt.
\end{aligned} \tag{2.2}$$

Define

$$\chi(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{2.3}$$

As in [14],by letting $\sigma \to 0$ we get

$$\iint_{Q_\tau} (\underline{u}^{k_1} - u^{k_1})_t \chi(\underline{u} - u) dx\, dt \leq \int_0^\tau \int_{\partial\Omega} \left(\underline{\lambda}\underline{u}^\alpha \underline{v}^p - u^\alpha v^p\right) \chi(\underline{u} - u) ds\, dt, \tag{2.4}$$

that is,

$$\int_\Omega (\underline{u}^{k_1} - u^{k_1})_+ |_{t=\tau}\, dx \leq \int_0^\tau \int_{\partial\Omega} \left(\underline{\lambda}\underline{u}^\alpha \underline{v}^p - u^\alpha v^p\right)_+ ds\, dt, \tag{2.5}$$

where $W_+ = \max\{W, 0\}$. Similarly, we have

$$\int_\Omega (\underline{v}^{k_2} - v^{k_2})_+ |_{t=\tau}\, dx \leq \int_0^\tau \int_{\partial\Omega} \left(\underline{\lambda}\underline{u}^q \underline{v}^\beta - u^q v^\beta\right)_+ ds\, dt. \tag{2.6}$$

Since $\underline{\lambda} < 1$, $\underline{u}, \underline{v} \geq \delta_0 > 0$ and $\underline{u}(x,0) \leq u_0(x)$, $\underline{v}(x,0) \leq u_0(x)$, it follows from the continuity of $\underline{u}$, $\underline{v}$, u and v that there exists a $\tau > 0$ sufficiently small such that $\underline{\lambda u}^{\alpha}\underline{v}^{p} \leq u^{\alpha}v^{p}$, $\underline{\lambda u}^{q}\underline{v}^{\beta} \leq u^{q}v^{\beta}$ for $(x,t) \in Q_{\tau}$. It follows from (2.5) and (2.6) that $(u,v) \geq (\underline{u},\underline{v})$ in Q_{τ}.

Denote $\tau^* = \sup\{\tau \in [0,T] : \underline{u}(x,t) \leq u(x,t), \underline{v}(x,t) \leq v(x,t)$ for all $(x,t) \in \overline{Q_{\tau}}\}$. We claim that $\tau^* = T$. Otherwise, from the continuity of $\underline{u}$, $\underline{v}$, u and v there exists $\epsilon > 0$ such that $\tau^* + \epsilon < T$ and $\underline{\lambda u}^{\alpha}\underline{v}^{p} \leq u^{\alpha}v^{p}$, $\underline{\lambda u}^{q}\underline{v}^{\beta} \leq u^{q}v^{\beta}$ for all $(x,t) \in \overline{Q_{\tau^*+\epsilon}}$. By (2.5) and (2.6) we obtain that $\underline{u}(x,t) \leq u(x,t)$, $\underline{v}(x,t) \leq v(x,t)$ in $Q_{\tau^*+\epsilon}$, which contradicts the definition of τ^*. Hence $(\underline{u},\underline{v}) \leq (u,v)$ for all $(x,t) \in Q_T$.

Obviously, $\delta = \min\{\min_{\overline{\Omega}} u_0(x), \min_{\overline{\Omega}} v_0(x)\} > 0$ is a lower solution of (1.1)–(1.3) in Q_T. Therefore, $u, v \geq \delta > 0$ in Q_T. Using this fact, as in the above proof we can prove that $(\overline{u},\overline{v}) \geq (u,v)$ in Q_T.

For convenience, we denote $0 < \underline{\lambda} < 1 < \overline{\lambda}$, which are fixed constants, and let $\delta = \min\{\min_{\overline{\Omega}} u_0(x), \min_{\overline{\Omega}} v_0(x)\} > 0$. □

Proposition 2.3. *Assume $k_1 \geq m$, $k_2 \geq n$ and that $\alpha > m(k_1+1)/(m+1)$ or $\beta > n(k_2+1)/(n+1)$ holds. Then the solutions of* (1.1)–(1.3) *blow up in finite time.*

Proof. Without loss of generality, assume $\alpha > m(k_1+1)/(m+1)$. Consider the single equation

$$\begin{aligned} \left(w^{k_1}\right)_t &= \nabla_m w, \quad x \in \Omega,\ t > 0, \\ \nabla_m w \cdot \nu &= \delta^p w^{\alpha}, \quad x \in \partial\Omega,\ t > 0, \\ w(x,0) &= u_0(x), \quad x \in \overline{\Omega}. \end{aligned} \tag{2.7}$$

We know from [13] that w blows up in finite time. Since $v \geq \delta$, by the comparison principle, (w,δ) is a lower solution of (1.1)–(1.3) and (u,v) blows up in finite time if $\alpha > m(k_1+1)/(m+1)$. □

The following propositions can be proved in the similar procedure.

Proposition 2.4. *Assume $k_1 < m$, $k_2 \geq n$ and that $\alpha > k_1$ or $\beta > n(k_2+1)/(n+1)$ holds. Then the solutions of* (1.1)–(1.3) *blow up in finite time.*

Proposition 2.5. *Assume $k_1 \geq m$, $k_2 < n$ and that $\alpha > m(k_1+1)/(m+1)$ or $\beta > k_2$ holds. Then the solutions of* (1.1)–(1.3) *blow up in finite time.*

Proposition 2.6. *Assume $k_1 < m$, $k_2 < n$ and that $\alpha > k_1$ or $\beta > k_2$ holds. Then the solutions of* (1.1)–(1.3) *blow up in finite time.*

Let $\varphi_k(x)$ $(k = m, n)$ be the first eigenfunction of

$$-\Delta_k \varphi = \lambda \varphi^k(x) \text{ in } \Omega, \qquad \varphi_k(x) = 0 \text{ on } \partial\Omega \tag{2.8}$$

with the first eigenvalue λ_k, normalized by $\|\varphi_k(x)\|_{\infty} = 1$, then $\lambda_k > 0$, $\varphi_k(x) > 0$ in Ω and $\varphi_k(x) \in W_0^{1,k+1} \cap C^1(\Omega)$ and $\partial\varphi_k(x)/\partial\nu < 0$ on $\partial\Omega$ (see [15–17]).

Thus there exist some positive constants A_k, B_k, C_k, D_k such that

$$A_k \le -\frac{\partial \varphi_k(x)}{\partial \nu} \le B_k, \quad |\nabla \varphi_k(x)| \ge C_k, \quad x \in \partial\Omega; \quad |\nabla \varphi_k(x)| \le D_k, \quad x \in \overline{\Omega}. \tag{2.9}$$

We have also $|\nabla \varphi_k(x)| \ge E_k$ provided $x \in \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \varepsilon_k\}$ with $E_k = C_k/2$ and some positive constant ε_k. For the fixed ε_k, there exists a positive constant F_k such that $\varphi_k(x) \ge F_k$ if $x \in \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon_k\}$.

At the end of this section, we describe two simple lemmas without proofs.

Lemma 2.7. *Suppose that positive constants A, B, C, D satisfy $AB < CD$, then for any two positive constants a, b, there exist two positive constants l_1, l_2 such that $al_1^C > l_2^A$ and $bl_2^D > l_1^B$.*

Lemma 2.8. *For any constant $j \ge 0$, there exist positive constants $f_i(j)$ $(i = 1, 2)$ which depend only on j and $\varphi(x)$, such that*

$$f_1(j)\left(\varphi(x) + s^j\right) \le \left(\varphi(x) + s\right)^j \le f_2(j)\left(\varphi(x) + s^j\right) \quad \forall s \ge 1, \tag{2.10}$$

where $\varphi(x)$ is a positive bounded function.

3. Proof of Theorem 1.1

Lemma 3.1. *Suppose $k_1 \ge m$, $k_2 \ge n$, $\alpha \le m(k_1 + 1)/(m + 1)$, $\beta \le n(k_2 + 1)/(n + 1)$, $pq \le (m(k_1 + 1)/(m + 1) - \alpha)(n(k_2 + 1)/(n + 1) - \beta)$. Then all positive solutions of* (1.1)–(1.3) *exist globally.*

Proof. Construct

$$\begin{aligned}
\overline{u}(x,t) &= e^{l_1 t}\left(M + \overline{\lambda}^{-1/m} e^{-L_1 \varphi_m e^{(k_1-m)l_1 t/(m+1)}} (2M)^{(p+\alpha)/m} L_1^{-1} \left(A_m c_m^{m-1}\right)^{-1/m}\right),\\
\overline{v}(x,t) &= e^{l_2 t}\left(M + \overline{\lambda}^{-1/n} e^{-L_2 \varphi_n e^{(k_2-n)l_2 t/(n+1)}} (2M)^{(q+\beta)/n} L_2^{-1} \left(A_n c_n^{n-1}\right)^{-1/n}\right),
\end{aligned} \tag{3.1}$$

where $c_m = C_m$ if $m \ge 1$, $c_m = D_m$ if $m < 1$, and $c_n = C_n$ if $n \ge 1$, $c_n = D_n$ if $n < 1$, φ_m, φ_n, A_m, A_n, C_m, C_n, D_m, D_n are defined in (2.8) and (2.9), l_1, l_2 are positive constants to be determined, $M = \max\{1, \|u_0\|_\infty, \|v_0\|_\infty\}$ and

$$\begin{aligned}
L_1 &= \overline{\lambda}^{-1/m} \max\left\{\frac{k_1 - m}{m+1} 2^{(p+\alpha+m)/m} M^{(p+\alpha-m)/m} \left(A_m c_m^{m-1}\right)^{-1/m}, 2^{(p+\alpha)/m} M^{(p+\alpha-m)/m} \left(A_m c_m^{m-1}\right)^{-1/m}\right\},\\
L_2 &= \overline{\lambda}^{-1/n} \max\left\{\frac{k_2 - n}{n+1} 2^{(q+\beta+n)/n} M^{(q+\beta-n)/n} \left(A_n c_n^{n-1}\right)^{-1/n}, 2^{(q+\beta)/n} M^{(q+\beta-n)/n} \left(A_n c_n^{n-1}\right)^{-1/n}\right\}.
\end{aligned} \tag{3.2}$$

We know that $-L_1\varphi_m e^{(k_1-m)l_1t/(m+1)}e^{-L_1\varphi_m e^{(k_1-m)l_1t/(m+1)}} \ge -e^{-1}$ since $-ye^{-y} \ge -e^{-1}$ for any $y > 0$. Thus for $(x,t) \in \Omega \times \mathbb{R}^+$, a simple computation shows

$$\begin{aligned}
\left(\overline{u}^{k_1}\right)_t &= k_1 l_1 e^{k_1 l_1 t}\left(M + \overline{\lambda}^{-1/m} e^{-L_1\varphi_m e^{(k_1-m)l_1t/(m+1)}}(2M)^{(p+\alpha)/m} L_1^{-1}\left(A_m c_m^{m-1}\right)^{-1/m}\right)^{k_1} \\
&\quad + k_1 e^{k_1 l_1 t}\left(M + \overline{\lambda}^{-1/m} e^{-L_1\varphi_m e^{(k_1-m)l_1t/(m+1)}}(2M)^{(p+\alpha)/m} L_1^{-1}\left(A_m c_m^{m-1}\right)^{-1/m}\right)^{k_1-1} \\
&\quad \times \overline{\lambda}^{-1/m}(2M)^{(p+\alpha)/m} L_1^{-1}\left(A_m c_m^{m-1}\right)^{-1/m}\frac{(k_1-m)l_1}{m+1}\left(-L_1\varphi_m\right)e^{(k_1-m)l_1t/(m+1)}e^{-L_1\varphi_m e^{(k_1-m)l_1t/(m+1)}} \\
&\ge \frac{1}{2}k_1 l_1 e^{k_1 l_1 t}.
\end{aligned} \tag{3.3}$$

In addition, we have

$$\begin{aligned}
\Delta_m \overline{u} &\le \overline{\lambda}\lambda_m (2M)^{p+\alpha}\left(A_m c_m^{m-1}\right)^{-1}\varphi_m^m e^{ml_1t}e^{m(k_1-m)l_1t/m}e^{-L_1 m\varphi_m e^{(k_1-m)l_1t/(m+1)}} \\
&\quad + \overline{\lambda}L_1 m(2M)^{p+\alpha}\left(A_m c_m^{m-1}\right)^{-1}e^{k_1l_1t}e^{-L_1 m\varphi_m e^{(k_1-m)l_1t/(m+1)}}\left|\nabla\varphi_m\right|^{m+1} \\
&\le \overline{\lambda}\left(\lambda_m + L_1 m D_m^{m+1}\right)(2M)^{p+\alpha}\left(A_m c_m^{m-1}\right)^{-1}e^{k_1l_1t}.
\end{aligned} \tag{3.4}$$

Similarly, we can get

$$\left(\overline{v}^{k_2}\right)_t \ge \frac{1}{2}k_2 l_2 e^{k_2 l_2 t}, \qquad \Delta_n \overline{v} \le \overline{\lambda}\left(\lambda_n + L_2 n D_n^{n+1}\right)(2M)^{q+\beta}\left(A_n c_n^{n-1}\right)^{-1}e^{k_2l_2t}. \tag{3.5}$$

Noting $\varphi_m = \varphi_n = 0$ on $\partial\Omega$, we have on the boundary that

$$\begin{aligned}
\nabla_m \overline{u}\cdot\nu &\ge \overline{\lambda}(2M)^{p+\alpha}e^{m(k_1+1)l_1t/(m+1)}, \qquad \nabla_n \overline{v}\cdot\nu \ge \overline{\lambda}(2M)^{q+\beta}e^{n(k_2+1)l_2t/(n+1)}; \\
\overline{u}^{\alpha}\overline{v}^{p} &\le (2M)^{p+\alpha}e^{(\alpha l_1+pl_2)t}, \qquad \overline{u}^{q}\overline{v}^{\beta} \le (2M)^{q+\beta}e^{(ql_1+\beta l_2)t}.
\end{aligned} \tag{3.6}$$

Since $pq \le (m(k_1+1)/(m+1)-\alpha)(n(k_2+1)/(n+1)-\beta)$, there exist constants l_1, l_2 large such that

$$\begin{aligned}
&\frac{m(k_1+1)l_1}{m+1} \ge \alpha l_1 + pl_2, \qquad \frac{n(k_2+1)l_2}{n+1} \ge ql_1 + \beta l_2; \\
&l_1 \ge 2\overline{\lambda}\left(\lambda_m + L_1 m D_m^{m+1}\right)(2M)^{p+\alpha}\left(k_1 A_m c_m^{m-1}\right)^{-1}, \\
&l_2 \ge 2\overline{\lambda}\left(\lambda_n + L_2 n D_n^{n+1}\right)(2M)^{q+\beta}\left(k_2 A_n c_n^{n-1}\right)^{-1}.
\end{aligned} \tag{3.7}$$

By (3.3)–(3.7), we know that $(\overline{u},\overline{v})$ is a global upper solution of (1.1)–(1.3). The global existence of solutions to (1.1)–(1.3) follows from the comparison principle. □

Lemma 3.2. *Suppose $k_1 \ge m$, $k_2 \ge n$, $\alpha \le m(k_1+1)/(m+1)$, $\beta \le n(k_2+1)/(n+1)$, $pq > (m(k_1+1)/(m+1)-\alpha)(n(k_2+1)/(n+1)-\beta)$. Then all positive solutions of* (1.1)–(1.3) *blow up in finite time.*

Proof.

Case 1. $k_1 > m$, $k_2 > n$. Let $d_m = C_m$ if $m < 1$, $d_m = D_m$ if $m \geq 1$, and $d_n = C_n$ if $n < 1$, $d_n = D_n$ if $n \geq 1$. In light of $pq > (m(k_1+1)/(m+1) - \alpha)(n(k_2+1)/(n+1) - \beta)$, we choose l_1, l_2 such that

$$\frac{m}{m+1} \leq pl_2 - \left(\frac{m(k_1+1)}{m+1} - \alpha\right)l_1, \qquad \frac{n}{n+1} \leq ql_1 - \left(\frac{n(k_2+1)}{n+1} - \beta\right)l_2. \tag{3.8}$$

For the above l_1, l_2, we set $\underline{u} = (1/(b-ct)^{l_1})e^{-a\varphi_m(x)/(b-ct)^{r_1}}$, $\underline{v} = (1/(b-ct)^{l_2})e^{-a\varphi_n(x)/(b-ct)^{r_2}}$, where $r_1 = ((k_1-m)l_1+1)/(m+1)$, $r_2 = ((k_2-n)l_2+1)/(n+1)$, $b = \max\{1, \delta^{-1/l_1}, \delta^{-1/l_2}\}$, and

$$a = \min\left\{1, \underline{\lambda}^{1/m}\left(B_m d_m^{m-1}\right)^{-1/m} b^{-(\alpha l_1 + pl_2)/m}, \underline{\lambda}^{1/n}\left(B_n d_n^{n-1}\right)^{-1/n} b^{-(ql_1+\beta l_2)/n}\right\}, \tag{3.9}$$

$$c = \min\left\{\frac{ma^{m+1}E_m^{m+1}}{k_1 l_1}, \frac{na^{n+1}E_n^{n+1}}{k_2 l_2}, \frac{\lambda_m(k_1-m)a^{m+1}F_m^{m+1}}{k_1 l_1}, \frac{\lambda_n(k_2-n)a^{n+1}F_n^{n+1}}{k_2 l_2}\right\}. \tag{3.10}$$

By a direct computation, for $x \in \Omega$, $0 < t < c/b$, we obtain that

$$\left(\underline{u}^{k_1}\right)_t \leq k_1 l_1 c e^{-ak_1\varphi_m(x)/(b-ct)^{r_1}} (b-ct)^{-(k_1 l_1+1)}, \tag{3.11}$$

$$\Delta_m \underline{u} = \frac{\lambda_m a^m \varphi_m^m e^{-am\varphi_m(x)/(b-ct)^{r_1}}}{(b-ct)^{m(l_1+r_1)}} + \frac{ma^{m+1}e^{-am\varphi_m(x)/(b-ct)^{r_1}}\left|\nabla\varphi_m\right|^{m+1}}{(b-ct)^{m(l_1+r_1)+r_1}}. \tag{3.12}$$

If $x \in \Omega_1 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon_m\}$, we have $\varphi_m \geq F_m$, and thus

$$\Delta_m \underline{u} \geq \frac{\lambda_m a^m F_m^m e^{-am\varphi_m(x)/(b-ct)^{r_1}}}{(b-ct)^{m(l_1+r_1)}}. \tag{3.13}$$

On the other hand, since $-ye^{-y} \geq -e^{-1}$ for any $y > 0$, we have

$$\left(\underline{u}^{k_1}\right)_t \leq k_1 l_1 c e^{-ak_1\varphi_m(x)/(b-ct)^{r_1}} (b-ct)^{-(k_1 l_1+1)} = \frac{k_1 l_1 c e^{-am\varphi_m(x)/(b-ct)^{r_1}}}{a(k_1-m)F_m e(b-ct)^{m(l_1+r_1)}}. \tag{3.14}$$

We have by (3.10), (3.13), and (3.14) that $(\underline{u}^{k_1})_t \leq \Delta_m \underline{u}$ for $(x,t) \in \Omega_1 \times (0, b/c)$.

If $x \in \Omega_2 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \varepsilon_m\}$, then $|\nabla\varphi_m| \geq E_m$, and hence

$$\Delta_m \underline{u} \geq \frac{ma^{m+1}E_m^{m+1}e^{-ak_1\varphi_m(x)/(b-ct)^{r_1}}}{(b-ct)^{m(l_1+r_1)+r_1}} = \frac{ma^{m+1}E_m^{m+1}e^{-ak_1\varphi_m(x)/(b-ct)^{r_1}}}{(b-ct)^{k_1 l_1+1}}. \tag{3.15}$$

We follow from (3.10), (3.11), and (3.15) that $(\underline{u}^{k_1})_t \leq \Delta_m \underline{u}$ for $(x,t) \in \Omega_2 \times (0, b/c)$.

Similarly, we can get $(\underline{v}^{k_2})_t \leq \Delta_n \underline{v}$ for $(x,t) \in \Omega \times (0, b/c)$ also.
We have on the boundary that

$$\nabla_m \underline{u} \cdot \nu = \frac{a^m |\nabla \varphi_m|^{m-1} e^{-am\varphi_m(x)/(b-ct)^{r_1}} (-\partial \varphi_m/\partial \nu)}{(b-ct)^{m(l_1+r_1)}} \leq \frac{a^m B_m d_m^{m-1}}{(b-ct)^{m(l_1+r_1)}},$$
$$\nabla_n \underline{v} \cdot \nu = \frac{a^n |\nabla \varphi_n|^{n-1} e^{-an\varphi_n(x)/(b-ct)^{r_2}} (-\partial \varphi_n/\partial \nu)}{(b-ct)^{n(l_2+r_2)}} \leq \frac{a^n B_n d_n^{n-1}}{(b-ct)^{n(l_2+r_2)}}; \tag{3.16}$$

$$\underline{u}^\alpha \underline{v}^p = \frac{1}{(b-ct)^{\alpha l_1 + p l_2}}, \qquad \underline{u}^q \underline{v}^\beta = \frac{1}{(b-ct)^{q l_1 + \beta l_2}}. \tag{3.17}$$

Moreover, by (3.8) we have that

$$m(l_1+r_1) \leq \alpha l_1 + p l_2, \qquad n(l_2+r_2) \leq q l_1 + \beta l_2. \tag{3.18}$$

Equations (3.9), (3.16)–(3.18) imply that $\nabla_m \underline{u} \cdot \nu \leq \lambda \underline{u}^\alpha \underline{v}^p$, $\nabla_n \underline{v} \cdot \nu \leq \lambda \underline{u}^q \underline{v}^\beta$ on $\partial\Omega$. Therefore $(\underline{u}, \underline{v})$ is a lower solution of (1.1)–(1.3).

Case 2. $k_1 > m$, $k_2 = n$. Set $\underline{u}$ as above with $\underline{v} = (1/(b-ct)^{l_2}) e^{-a\varphi_n(x)/(b-ct)^{1/n}}$.

Case 3. $k_1 = m$, $k_2 > n$. Set $\underline{v}$ as above with $\underline{u} = (1/(b-ct)^{l_1}) e^{-a\varphi_m(x)/(b-ct)^{1/m}}$.

Case 4. $k_1 = m$, $k_2 = n$. Set $\underline{u} = (1/(b-ct)^{l_1}) e^{-a\varphi_m(x)/(b-ct)^{1/m}}$, $\underline{v} = (1/(b-ct)^{l_2}) e^{-a\varphi_n(x)/(b-ct)^{1/n}}$.

By similar arguments, we conform that $(\underline{u}, \underline{v})$ is a lower solution of (1.1)–(1.3), which blows up in finite time. We know by the comparison principle that the solution (u, v) blows up in finite time. □

We get the proof of Theorem 1.1 by combining Proposition 2.3 and Lemmas 3.1 and 3.2.

4. Proof of Theorems 1.2 and 1.3

Lemma 4.1. *Suppose $k_1 < m$, $k_2 \geq n$, $\alpha \leq k_1$, $\beta \leq n(k_2+1)/(n+1)$ with $pq \leq (k_1 - \alpha)(n(k_2+1)/(n+1) - \beta)$. Then all positive solutions of (1.1)–(1.3) exist globally.*

Proof. Take

$$\overline{u}(x,t) = R_1 e^{l_1 t} \log\left((1-\varphi_m(x)) e^{(k_1-m)l_1 t/m} + R_2\right),$$
$$\overline{v}(x,t) = e^{l_2 t}\left(M + \overline{\lambda}^{1/n} e^{-L\varphi_n(x) e^{(k_2-n)l_2 t/(n+1)}} (2M)^{(k_2+1)/(n+1)} L^{-1} \left(A_n c_n^{n-1}\right)^{-1/n}\right) \tag{4.1}$$

for $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$, where $c_n = C_n$ if $n \geq 1$, $c_n = D_n$ if $n < 1$, R_2 satisfying $R_2 \log R_2 \geq 2(m-k_1)/m$, and constants R_1, M, L, l_1, l_2 are to be determined. By performing direct calculations, we have, for $(x,t) \in \Omega \times \mathbb{R}^+$,

$$\begin{aligned}
&\left(\overline{u}^{k_1}\right)_t \geq \frac{k_1 l_1}{2} R^{k_1} e^{k_1 l_1 t}\left(\log\left((1-\varphi_m(x))e^{(k_1-m)l_1 t/m}+R_2\right)\right)^{k_1} \geq \frac{k_1 l_1}{2} R^{k_1} e^{k_1 l_1 t}\left(\log R_2\right)^{k_1},\\
&\Delta_m \overline{u} = \sum_{i=1}^{N}\left(\frac{R_1^m e^{k_1 l_1 t}\left(-\left|\nabla\varphi_m(x)\right|^{m-1}\varphi_{m_{x_i}}\right)}{\left((1-\varphi_m(x))e^{(k_1-m)l_1 t/m}+R_2\right)^m}\right)_{x_i} \leq \frac{\lambda_m R_1^m e^{k_1 l_1 t}}{R_2^m}.
\end{aligned} \tag{4.2}$$

By setting $c_m = C_m$ if $m \geq 1$, $c_m = D_m$ if $m < 1$, we have on the boundary that

$$\begin{aligned}
&\nabla_m \overline{u}\cdot\nu \geq \frac{R_1^m e^{k_1 l_1 t} c_m^{m-1} A_m}{(1+R_2)^m}, \qquad \nabla_n \overline{v}\cdot\nu \geq \overline{\lambda}(2M)^{n(k_2+1)/(n+1)} e^{n(k_2+1)l_2 t/(n+1)};\\
&\overline{u}^{\alpha}\overline{v}^{p} \leq \left(R_1 \log(1+R_2)\right)^{\alpha}(2M)^p e^{\alpha l_1 t + p l_2 t}, \qquad \overline{u}^{q}\overline{v}^{\beta} \leq \left(R_1 \log(1+R_2)\right)^{q}(2M)^{\beta} e^{q l_1 t + \beta l_2 t}.
\end{aligned} \tag{4.3}$$

Since $pq < (m-\alpha)(n(k_2+1)/(n+1)-\beta)$, by Lemma 2.7 there exist two positive constants R_1, M such that $R_1 \log R_2 \geq \max\{1, \|u_0\|_\infty\}$, $M \geq \{1, \|u_0\|_\infty\}$, and

$$\begin{aligned}
&R_1^{m-\alpha} \geq \overline{\lambda}(2M)^p\left(\log(1+R_2)\right)^{\alpha}(1+R_2)^m\left(c_m^{m-1}A_m\right)^{-1},\\
&(2M)^{n(k_2+1)/(n+1)-\beta} \geq R_1^q\left(\log(1+R_2)\right)^q.
\end{aligned} \tag{4.4}$$

Set $L = \overline{\lambda}^{-1/n}\max\{((k_2-n)/(n+1))2^{(k_2+n+2)/(n+1)}M^{(k_2-n)/(n+1)}(A_n c_n^{n-1})^{-1/n}, 2^{(k_2+1)/(n+1)}M^{(k_2-n)/(n+1)}$ $\times(A_n c_n^{n-1})^{-1/n}\}$. By arguments in Lemma 3.1,for $(x,t) \in \Omega \times \mathbb{R}^+$, we have

$$\left(\overline{v}^{k_2}\right)_t \geq \frac{1}{2}k_2 l_2 e^{k_2 l_2 t}, \qquad \Delta_n \overline{v} \leq \overline{\lambda}\left(\lambda_n + LnD_n^{n+1}\right)(2M)^{n(k_2+1)/(n+1)}\left(A_n c_n^{n-1}\right)^{-1} e^{k_2 l_2 t}. \tag{4.5}$$

On the other hand, since $pq \leq (k_1-\alpha)(n(k_2+1)/(n+1)-\beta)$, there exist two positive constants l_1, l_2 such that

$$(k_1-\alpha)l_1 \geq p l_2, \qquad \left(\frac{n(k_2+1)}{n+1}-\beta\right)l_2 \geq q l_1; \tag{4.6}$$

$$l_1 \geq \frac{2\lambda_m R_1^{m-k_1}}{k_1\left(\log R_2\right)^{k_1} R_2^m}, \qquad l_2 \geq 2\overline{\lambda}\left(\lambda_n + LnD_n^{n+1}\right)(2M)^{n(k_2+1)/(n+1)}\left(k_2 A_n c_n^{n-1}\right)^{-1}. \tag{4.7}$$

By (4.2)–(4.7), it follows that $(\overline{u};\overline{v})$ is an upper solution of (1.1)–(1.3). Thus the solutions of (1.1)–(1.3) are global. □

Lemma 4.2. *Suppose* $k_1 < m$, $k_2 \geq n$, $\alpha \leq k_1$, $\beta \leq n(k_2+1)/(n+1)$ *with* $pq > (k_1-\alpha)(n(k_2+1)/(n+1)-\beta)$. *Then all positive solutions of* (1.1)–(1.3) *blow up in finite time.*

Proof. We first prove that there exist $l_1 \geq 1$, $l_2 \geq 1$ such that

$$\frac{mk_1 l_1 + m}{m-k_1} \leq \frac{m\alpha l_1 + \alpha}{m-k_1} + p l_2, \qquad \frac{n(k_2+1)l_2 + n}{n+1} \leq \frac{mq l_1 + q}{m-k_1} + \beta l_2. \tag{4.8}$$

When $\alpha < k_1$, $\beta < n(k_2+1)/(n+1)$, $pq > (k_1-\alpha)(n(k_2+1)/(n+1)-\beta)$ yields $m(k_1-\alpha)/((m-k_1)p) < mq/((m-k_1)(n(k_2+1)/(n+1)-\beta))$. Hence there exist $\mu > 0$ such that $m(k_1-\alpha)/((m-k_1)p) < \mu < mq/((m-k_1)(n(k_2+1)/(n+1)-\beta))$. Set $l_1 = \max\{1, 1/\mu, ((m-\alpha)/(m-k_1))/(\mu p - m(k_1-\alpha)/(m-k_1)), (n/(n+1)-q/(m-k_1))/[mq/(m-k_1)-(n(k_2+1)/(n+1)-\beta)\mu]\}$, and $l_2 = \mu l_1$.

When $\alpha = k_1$, $\beta \le n(k_1+1)/(n+1)$, take $l_2 = \max\{1, (m-\alpha)/(m-k_1)p\}$, $l_1 = \max\{1, (n/(n+1)-q/(m-k_1)+(n(k_2+1)/(n+1)-\beta)l_2)((m-k_1)/mq)\}$.

When $\alpha \le k_1$, $\beta = n(k_2+1)/(n+1)$, take $l_1 = \max\{1, (n/(n+1)-q/(m-k_1))((m-k_1)/mq)\}$, $l_2 = \max\{1, ((m-\alpha)/(m-k_1)+m(k_1-\alpha)l_1/(m-k_1))(1/p)\}$.

Let $d_n = C_n$ if $n<1$, $d_n = D_n$ if $n \ge 1$, and $d = \max\{|x| \mid x \in \overline{\Omega}\}$, $h(x) = \sum_{i=1}^N x_i + Nd + 1$, $y = ah^{1+1/m}(x) + (b-ct)^{-l_1}$.

Define $\underline{u}(x,t) = y^\theta$, $\underline{v} = (1/(b-ct)^{l_2})\exp\{-a\varphi_n(x)/(b-ct)^r\}$, where $\theta = (m+1/l_1)/(m-k_1)$, $r = ((k_2-n)l_2+1)/(n+1)$, $b = \max\{1, ((1/2)\delta^{1/\theta})^{-1/l_1}, \delta^{-1/l_2}\}$, and

$$a = \min\Bigg\{b^{-l_1}(2Nd+1)^{-(1+1/m)}, \underline{\lambda}^{1/n}\left(B_n d_n^{n-1}\right)^{-1/n} b^{-(q\theta l_1+\beta l_2)/n}, \underline{\lambda}^{1/m}\left(\theta^m\left(1+\frac{1}{m}\right)^m N^{m/2}(2Nd+1)2^{m(\theta-1)}\right)^{-1/m} b^{(-\alpha\theta l_1+pl_2)/m}\Bigg\}, \tag{4.9}$$

$$c = \min\left\{\frac{na^{n+1}E_n^{n+1}}{k_2 l_2}, \frac{\lambda_n(k_2-n)a^{n+1}F_n^{n+1}}{k_2 l_2}, \frac{a^m\theta^{m-1}(1+1/m)^m N^{(m+1)/2}}{k_1 l_1}\right\}. \tag{4.10}$$

By a direct computation, for $(x,t) \in \Omega \times (0, b/c)$, we have

$$\Delta_m \underline{u} \ge \left(a\theta\left(1+\frac{1}{m}\right)\right)^m N^{(m+1)/2} y^{k_1\theta-1} y^{m(\theta-1)-k_1\theta+1} \ge \left(\underline{u}^{k_1}\right)_t. \tag{4.11}$$

By similar arguments in Lemma 3.2, we have $(\underline{v}^{k_2})_t \le \Delta_n \underline{v}$ for $(x,t) \in \Omega \times (0, b/c)$.

Moreover, for $(x,t) \in \partial\Omega \times (0, b/c)$, we have

$$\begin{aligned}
\nabla_m \underline{u}\cdot\nu &\le \left(a\theta\left(1+\frac{1}{m}\right)\right)^m N^{m/2}(2Nd+1)2^{m(\theta-1)}(b-ct)^{-m(\theta-1)l_1}, \\
\nabla_n \underline{v}\cdot\nu &\le \frac{a^n B_n d_n^{n-1}}{(b-ct)^{n(l_2+r)}}; \\
\underline{u}^\alpha \underline{v}^p &= \left(ah(x)^{1+1/m} + (b-ct)^{-l_1}\right)^{\theta\alpha}(b-ct)^{-pl_2} \ge (b-ct)^{-(\alpha\theta l_1+pl_2)}, \\
\underline{u}^q \underline{v}^\beta &= \left(ah(x)^{1+1/m} + (b-ct)^{-l_1}\right)^{\theta q}(b-ct)^{-\beta l_2} \ge (b-ct)^{-(q\theta l_1+\beta l_2)}.
\end{aligned} \tag{4.12}$$

By (4.8), we have

$$m(\theta-1)l_1 \le \alpha\theta l_1 + pl_2, \qquad n\left(l_2+r\right) \le q\theta l_1 + \beta l_2. \tag{4.13}$$

By (4.9), (4.12), and (4.13), we have that $(\underline{u}, \underline{v})$ is a lower solution of (1.1)–(1.3), which with the comparison principle implies that the solutions of (1.1)–(1.3) blow up in finite time. □

It has been shown from Proposition 2.4 and Lemmas 4.1 and 4.2 that Theorem 1.2 is true. In a similar way to the proof of Theorem 1.2, we have Theorem 1.3.

5. Proof of Theorem 1.4

Lemma 5.1. *Suppose $k_1 < m$, $k_2 < n$, $\alpha \le k_1$, $\beta \le k_2$ with $pq \le (k_1-\alpha)(k_2-\beta)$. Then all positive solutions of* (1.1)–(1.3) *exist globally.*

Proof. Take $\overline{u} = a(1-\varphi_m(x)+e^{lt})^{m/(m-k_1)}$, $\overline{v} = b(1-\varphi_n(x)+e^{\theta lt})^{n/(n-k_2)}$, where $\theta = m(n-k_2)(k_1-\alpha)/n(m-k_1)p$ and a, b, l are the undetermined positive constants.

Calculating directly for $(x,t)\in\Omega\times\mathbb{R}^+$, we have by Lemma 2.8 that

$$
\begin{aligned}
(\overline{u}^{k_1})_t &\ge \frac{a^{k_1}mk_1 l}{2(m-k_1)}\left(1-\varphi_m(x)+e^{lt}\right)^{mk_1/(m-k_1)}\\
&\ge \frac{a^{k_1}mk_1 l}{2(m-k_1)}f_1\left(\frac{mk_1}{m-k_1}\right)\left(1-\varphi_m(x)+e^{mk_1lt/(m-k_1)}\right),\\
\Delta_m\overline{u} &\le \lambda_m\varphi_m^m\left(\frac{am}{m-k_1}\right)^m f_2\left(\frac{mk_1}{m-k_1}\right)\left(1-\varphi_m(x)+e^{mk_1lt/(m-k_1)}\right)\\
&\quad+\left(\frac{am}{m-k_1}\right)^m\frac{mk_1}{m-k_1}D_m^{m+1}f_2\left(\frac{mk_1}{m-k_1}\right)\left(1-\varphi_m(x)+e^{mk_1lt/(m-k_1)}\right)\\
&\le\left(\lambda_m+\frac{mk_1}{m-k_1}D_m^{m+1}\right)\left(\frac{am}{m-k_1}\right)^m f_2\left(\frac{mk_1}{m-k_1}\right)\left(1-\varphi_m(x)+e^{mk_1lt/(m-k_1)}\right);\\
(\overline{v}^{k_2})_t &\ge \frac{b^{k_2}nk_2\theta l}{2(n-k_2)}\left(1-\varphi_n(x)+e^{\theta lt}\right)^{nk_2/(n-k_2)}\\
&\ge \frac{b^{k_2}nk_2\theta l}{2(n-k_2)}f_1\left(\frac{nk_2}{n-k_2}\right)\left(1-\varphi_n(x)+e^{nk_2\theta lt/(n-k_2)}\right),\\
\Delta_n\overline{v} &\le\left(\lambda_n+\frac{nk_2}{n-k_2}D_n^{n+1}\right)\left(\frac{bn}{n-k_2}\right)^n f_2\left(\frac{nk_2}{n-k_2}\right)\left(1-\varphi_n(x)+e^{nk_2\theta lt/(n-k_2)}\right).
\end{aligned}
\tag{5.1}
$$

Let $c_m = C_m$ if $m\ge 1$, $c_m = D_m$ if $m<1$, and $c_n = C_n$ if $n \ge 1$, $c_n = D_n$ if $n<1$. We have on the boundary that

$$
\begin{aligned}
\nabla_m\overline{u}\cdot\nu &\ge \left(\frac{am}{m-k_1}\right)^m c_m^{m-1}A_m\left(1-\varphi_m(x)+e^{lt}\right)^{mk_1/(m-k_1)}\\
&\ge a^{m-\alpha}\left(\frac{m}{m-k_1}\right)^m c_m^{m-1}A_m f_1\left(\frac{m(k_1-\alpha)}{m-k_1}\right)\left(1+e^{m(k_1-\alpha)lt/(m-k_1)}\right)\overline{u}^{\alpha},\\
\nabla_n\overline{v}\cdot\nu &\ge b^{n-\beta}\left(\frac{n}{n-k_2}\right)^n c_n^{n-1}A_n f_1\left(\frac{n(k_2-\beta)}{n-k_2}\right)\left(1+e^{n(k_2-\beta)\theta lt/(n-k_2)}\right)\overline{v}^{\beta}\\
&\ge b^{n-\beta}\left(\frac{n}{n-k_2}\right)^n c_n^{n-1}A_n f_1\left(\frac{n(k_2-\beta)}{n-k_2}\right)\left(1+e^{mqlt/(m-k_1)}\right)\overline{v}^{\beta};\\
\overline{u}^{\alpha}\overline{v}^{p} &= b^p\left(1+e^{\theta lt}\right)^{np/(n-k_2)}\overline{u}^{\alpha}\le b^p f_2\left(\frac{np}{n-k_2}\right)\left(1+e^{m(k_1-\alpha)lt/(m-k_1)}\right)\overline{u}^{\alpha},\\
\overline{u}^{q}\overline{v}^{\beta} &= a^q\left(1+e^{lt}\right)^{mq/(m-k_1)}\overline{v}^{\beta}\le a^q f_2\left(\frac{mq}{m-k_1}\right)\left(1+e^{mqlt/(m-k_1)}\right)\overline{v}^{\beta}.
\end{aligned}
\tag{5.2}
$$

Since $pq \le (k_1-\alpha)(k_2-\beta) < (m-\alpha)(n-\beta)$, we know by Lemma 2.8 that there exist constants $a \ge \|u_0(x)\|_\infty$, $b \ge \|v_0(x)\|_\infty$ such that

$$\begin{aligned} a^{m-\alpha}\left(\frac{m}{m-k_1}\right)^m c_m^{m-1} A_m f_1\left(\frac{m(k_1-\alpha)}{m-k_1}\right) &\ge \overline{\lambda} b^p f_2\left(\frac{np}{n-k_2}\right), \\ b^{n-\beta}\left(\frac{n}{n-k_2}\right)^n c_n^{n-1} A_n f_1\left(\frac{n(k_2-\beta)}{n-k_2}\right) &\ge \overline{\lambda} a^q f_2\left(\frac{mq}{m-k_1}\right). \end{aligned} \tag{5.3}$$

For the above constants a, b, we choose a constant l so large that

$$\begin{aligned} \left(\lambda_m + \frac{mk_1}{m-k_1} D_m^{m+1}\right)\left(\frac{am}{m-k_1}\right)^m f_2\left(\frac{mk_1}{m-k_1}\right) &\le \frac{a^{k_1} mk_1 l}{2(m-k_1)} f_1\left(\frac{mk_1}{m-k_1}\right), \\ \left(\lambda_n + \frac{nk_2}{n-k_2} D_n^{n+1}\right)\left(\frac{bn}{n-k_2}\right)^n f_2\left(\frac{nk_2}{n-k_2}\right) &\le \frac{b^{k_2} nk_2\theta l}{2(n-k_2)} f_1\left(\frac{nk_2}{n-k_2}\right). \end{aligned} \tag{5.4}$$

By (5.1)–(5.4), we know that $(\overline{u},\overline{v})$ is an upper solution of (1.1)–(1.3), Thus the solutions of (1.1)–(1.3) are global. □

Lemma 5.2. *Suppose $k_1 < m$, $k_2 < n$, $\alpha \le k_1$, $\beta \le k_2$ with $pq > (k_1-\alpha)(k_2-\beta)$. Then all positive solutions of* (1.1)–(1.3) *blow up in finite time.*

Proof. We first prove that there exist $l_1 \ge 1$, $l_2 \ge 1$ such that

$$\frac{mk_1 l_1 + m}{m-k_1} \le \frac{m\alpha l_1 + \alpha}{m-k_1} + \frac{npl_2 + p}{n-k_2}, \qquad \frac{nk_2 l_2 + n}{n-k_2} \le \frac{mql_1 + q}{m-k_1} + \frac{n\beta l_2 + \beta}{n-k_2}. \tag{5.5}$$

In fact, when $\alpha < k_1$, $\beta < k_2$, $pq > (k_1-\alpha)(k_2-\beta)$ yields $(m(k_1-\alpha)/(m-k_1))((n-k_2)/np) < (mq/(m-k_1))((n-k_2)/n(k_2-\beta))$. Hence there exists $\mu > 0$ such that $(m(k_1-\alpha)/(m-k_1))((n-k_2)/np) < \mu < (mq/(m-k_1))((n-k_2)/n(k_2-\beta))$. Set $l_1 = \max\{1, 1/\mu, ((m-\alpha)/(m-k_1) - p/(n-k_2))/((np/(n-k_2))\mu - m(k_1-\alpha)/(m-k_1)), ((n-\beta)/(n-k_2) - q/(m-k_1))/(mq/(m-k_1) - (n(k_2-\beta)/(n-k_2))\mu)\}$, and $l_2 = \mu l_1$.

When $k_1 \le \alpha$ and $\beta = k_2$, take $l_1 = \max\{1, ((n-\beta)/(n-k_2) - q/(m-k_1))((m-k_1)/mq)\}$, $l_2 = \max\{1, ((m-\alpha)/(m-k_1) - p/(n-k_2) + m(k_1-\alpha)l_1/(m-k_1))((n-k_2)/np)\}$.

When $k_1 = \alpha$ and $\beta \le k_2$, let $l_2 = \max\{1, ((m-\alpha)/(m-k_1) - p/(n-k_2))((n-k_2)/np)\}$, $l_1 = \max\{1, ((n-\beta)/(n-k_2) - q/(m-k_1) + n(k_2-\beta)l_2/(n-k_2))((m-k_1)/mq)\}$.

Take $y = ah^{1+1/m}(x) + (b-ct)^{-l_1}$, $z = ah^{1+1/n}(x) + (b-ct)^{-l_2}$, and $\underline{u} = y^\theta$, $\underline{v} = z^\sigma$, where $\theta = (m+1/l_1)/(m-k_1)$, $\sigma = (n+1/l_2)/(n-k_2)$, $b = \max\{1, ((1/2)\delta^{1/\theta})^{-1/l_1}, ((1/2)\delta^{1/\sigma})^{-1/l_2}\}$, and

$$\begin{aligned} a = \min\Bigg\{ & b^{-l_1}(2Nd+1)^{-(1+m)/m}, \left(\underline{\lambda}^{-1}\left[\frac{(1+m)\theta N^{1/2}2^{\theta-1}}{m}\right]^m (2Nd+1)\right)^{-1/m} b^{-(\alpha\theta l_1 + p\sigma l_2)/m}, \\ & b^{-l_2}(2Nd+1)^{-(1+n)/n}, \left(\underline{\lambda}^{-1}\left[\frac{(1+n)\sigma N^{1/2}2^{\sigma-1}}{n}\right]^n (2Nd+1)\right)^{-1/n} b^{-(q\theta l_1 + \beta\sigma l_2)/n} \Bigg\}, \end{aligned} \tag{5.6}$$

$$c = \min\left\{ \frac{a^m \theta^{m-1}(1+1/m)^m N^{(m+1)/2}}{k_1 l_1}, \frac{a^n \sigma^{n-1}(1+1/n)^n N^{(n+1)/2}}{k_2 l_2} \right\}. \tag{5.7}$$

By a direct computation for $(x,t) \in \Omega \times (0, b/c)$, we have

$$\Delta_m \underline{u} \geq \left(a\theta\left(1+\frac{1}{m}\right)\right)^m N^{(m+1)/2} y^{k_1\theta-1} y^{m(\theta-1)-k_1\theta+1} \geq \left(\underline{u}^{k_1}\right)_t, \tag{5.8}$$

$$\Delta_n \underline{v} \geq \left(a\sigma\left(1+\frac{1}{n}\right)\right)^n N^{(n+1)/2} y^{k_2\sigma-1} y^{n(\sigma-1)-k_2\sigma+1} \geq \left(\underline{v}^{k_2}\right)_t. \tag{5.9}$$

For $(x,t) \in \partial\Omega \times (0, b/c)$, we have

$$\begin{aligned} \nabla_m \underline{u} \cdot \nu &\leq \left(a\theta\left(1+\frac{1}{m}\right)\right)^m N^{m/2}(2Nd+1)2^{m(\theta-1)}(b-ct)^{-m(\theta-1)l_1}, \\ \nabla_n \underline{v} \cdot \nu &\leq \left(a\sigma\left(1+\frac{1}{n}\right)\right)^n N^{n/2}(2Nd+1)2^{n(\sigma-1)}(b-ct)^{-n(\sigma-1)l_2}; \\ \underline{u}^\alpha \underline{v}^p &= y^{\alpha\theta} z^{p\sigma} \geq (b-ct)^{-(\alpha\theta l_1 + p\sigma l_2)}, \\ \underline{u}^q \underline{v}^\beta &= y^{q\theta} z^{\beta\sigma} \geq (b-ct)^{-(q\theta l_1 + \beta\sigma l_2)}. \end{aligned} \tag{5.10}$$

Moreover, (5.5) implies

$$m(\theta-1)l_1 \leq \alpha\theta l_1 + p\sigma l_2, \qquad n(\sigma-1)l_2 \leq q\theta l_1 + \beta\sigma l_2. \tag{5.11}$$

It follows from (5.6), (5.8)–(5.11) that $(\underline{u}, \underline{v})$ is a lower solution of (1.1)–(1.3). Because $(\underline{u}, \underline{v})$ blows up in finite time, and so does (u, v). □

By Proposition 2.6 and Lemmas 5.1 and 5.2, we see that Theorem 1.4 holds.

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