

## Research Article

# Solvability of the Dirichlet Problem for Elliptic Equations in Weighted Sobolev Spaces on Unbounded Domains

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This paper is concerned with the study of the Dirichlet problem for a class of second-order linear elliptic equations in weighted Sobolev spaces on unbounded domains of  $\mathbb{R}^n$ ,  $n \geq 3$ . We state a regularity result and we can deduce an existence and uniqueness theorem.

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## 1. Introduction

Consider the Dirichlet problem

$$\begin{aligned} u &\in W^{2,p}(\Omega) \cap \overset{\circ}{W}^{1,p}(\Omega), \\ Lu &= f, \quad f \in L^p(\Omega), \end{aligned} \tag{1.1}$$

where  $\Omega$  is a sufficiently regular open subset of  $\mathbb{R}^n$  ( $n \geq 3$ ),  $p \in ]1, +\infty[$ ,  $L$  is the uniformly elliptic second-order linear differential operator defined by

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a \tag{1.2}$$

with coefficients  $a_{ij} = a_{ji} \in L^\infty(\Omega)$ ,  $i, j = 1, \dots, n$ .

It is well known that if  $\Omega$  is bounded, the above problem has been largely studied by several authors under various hypotheses of discontinuity on the leading coefficients and considering the case  $p = 2$ . In particular, some  $W^{2,2}$ -bounds for the solutions of the

problem (1.1) and related existence and uniqueness theorems have been obtained. Among the other results on this subject, we quote here the classical result of [1], where the author assumed that the  $a_{ij}$ 's belong to  $W^{1,n}(\Omega)$ . This result was later generalized in different ways, supposing that the derivatives of the leading coefficients belong to some wider spaces. More recently, a relevant contribution to the theory has been given in [2–5], where the coefficients  $a_{ij}$  are assumed to be in the class VMO and  $p \in ]1, +\infty[$ ; observe here that VMO contains the class  $W^{1,n}(\Omega)$ .

If the set  $\Omega$  is unbounded, under assumptions similar to those required in [1], problem (1.1) has for instance been studied in [6] with  $p = 2$ , and in [7] with  $p \in ]1, +\infty[$ . Instead, in [8, 9], the leading coefficients satisfy restrictions similar to those in [2, 3].

In [10], we extended some results of [8, 9] to a weighted case. More precisely, we denoted by  $\rho$  a weight function belonging to a suitable class and such that

$$\inf_{\Omega} \rho > 0, \quad \lim_{|x| \rightarrow +\infty} \rho(x) = +\infty. \quad (1.3)$$

Then we considered the problem

$$\begin{aligned} u &\in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ Lu &= f, \quad f \in L_s^p(\Omega), \end{aligned} \quad (1.4)$$

where  $s \in \mathbb{R}$ ,  $W_s^{2,p}(\Omega)$ ,  $\overset{\circ}{W}_s^{1,p}(\Omega)$ , and  $L_s^p(\Omega)$  are some weighted Sobolev spaces and the weight functions are a suitable power of  $\rho$ . We obtained that the operator  $L$  has closed range and that for the problem (1.4) a uniqueness result holds.

In this paper, we study again the problem (1.4). We state a regularity result which allows us to obtain the solvability of the problem.

A similar weighted case was studied in [11] with the leading coefficients satisfying hypotheses of Miranda's type and when  $p = 2$ .

## 2. Weight functions and weighted spaces

Let  $G$  be any Lebesgue measurable subset of  $\mathbb{R}^n$  and let  $\Sigma(G)$  be the collection of all Lebesgue measurable subsets of  $G$ . Let  $F \in \Sigma(G)$ . Denote by  $|F|$  the Lebesgue measure of  $F$ , by  $\chi_F$  the characteristic function of  $F$ , and by  $\mathfrak{D}(F)$  the class of restrictions to  $F$  of functions  $\zeta \in C^\infty(\mathbb{R}^n)$  with  $\overline{F} \cap \text{supp } \zeta \subseteq F$ . Moreover, if  $X(F)$  is a space of functions defined on  $F$ , we denote by  $X_{\text{loc}}(F)$  the class of all functions  $g : F \rightarrow \mathbb{R}$  such that  $\zeta g \in X(F)$  for any  $\zeta \in \mathfrak{D}(F)$ . Finally, for any  $x \in \mathbb{R}^n$  and  $r \in \mathbb{R}_+$ , we put  $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ ,  $B_r = B(0, r)$ , and  $F(x, r) = F \cap B(x, r)$ .

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . We introduce a class of weight functions defined on  $\Omega$ . Denote by  $\mathcal{A}(\Omega)$  the set of all measurable functions  $\rho : \Omega \rightarrow \mathbb{R}_+$  such that

$$\gamma^{-1}\rho(y) \leq \rho(x) \leq \gamma\rho(y), \quad \forall y \in \Omega, \quad \forall x \in \Omega(y, \rho(y)), \quad (2.1)$$

where  $\gamma \in \mathbb{R}_+$  is independent of  $x$  and  $y$ .

We note that the class of all functions  $\rho : \Omega \rightarrow \mathbb{R}_+$  which are Lipschitz continuous in  $\Omega$  with Lipschitz coefficient  $< 1$  is contained in  $\mathcal{A}(\Omega)$  (see [12]).

For  $\rho \in \mathcal{A}(\Omega)$ , we put

$$S_\rho = \left\{ z \in \partial\Omega : \lim_{x \rightarrow z} \rho(x) = 0 \right\}. \quad (2.2)$$

It is known that

$$\rho \in L_{loc}^\infty(\overline{\Omega}), \quad \rho^{-1} \in L_{loc}^\infty(\overline{\Omega} \setminus S_\rho), \quad (2.3)$$

(see [12, 13]).

We assign an unbounded open subset  $\Omega$  of  $\mathbb{R}^n$ .

From now on, let  $\rho_1$  be a function such that  $\rho_1 \in \mathcal{A}(\mathbb{R}^n)$  and

$$\inf_{\Omega} \rho_1 > 0, \quad \lim_{|x| \rightarrow +\infty} \rho_1(x) = +\infty. \quad (2.4)$$

For example,

$$\rho_1 : x \in \mathbb{R}^n \longrightarrow 1 + a|x|, \quad a \in ]0, 1[. \quad (2.5)$$

We put

$$\rho = \rho_{1|\Omega}, \quad (2.6)$$

and note that  $\rho^{-1} \in L^\infty(\Omega)$ .

For any  $a \in ]0, 1[$  and  $x \in \mathbb{R}^n$ , we set

$$I_a(x) = \Omega(x, a\rho_1(x)), \quad E_a(x) = \{y \in \mathbb{R}^n \mid x \in B(y, a\rho_1(y))\}, \quad (2.7)$$

and note that

$$c'\rho_1^n(x) \leq |E_a(x)| \leq c''\rho_1^n(x) \quad \forall x \in \mathbb{R}^n, \quad (2.8)$$

where  $c', c'' \in \mathbb{R}_+$  depend only on  $n, a, \rho$  (see [12]).

If  $v$  is a real function defined in  $\Omega$ , we denote by  $v_0$  the zero extension of  $v$  in  $\mathbb{R}^n$ .

We begin to prove the following.

**Lemma 2.1.** *If  $v, g$  are two nonnegative functions in  $L_{loc}^1(\overline{\Omega}), L_{loc}^1(\mathbb{R}^n)$ , respectively, then for any  $a \in ]0, 1[$ ,*

$$\int_{\Omega} v(x)\|g\|_{L^1(E_a(x))} dx = \int_{\mathbb{R}^n} g(x)\|v\|_{L^1(I_a(x))} dx, \quad (2.9)$$

and for  $v \in ]1, +\infty[$ , the following also hold:

$$\int_{\mathbb{R}^n} g(x)\|v\|_{L^1(I_a(x))}^v dx \leq \left( \int_{\Omega} v(x)\|g\|_{L^1(E_a(x))}^{1/v} dx \right)^v. \quad (2.10)$$

*Proof.* The equality (2.9) follows by

$$\begin{aligned} \int_{\Omega} v(x)\|g\|_{L^1(E_a(x))} dx &= \int_{\mathbb{R}^n} dx \int_{E_a(x)} v_0(x)g(y)dy \\ &= \int_{\mathbb{R}^n} dy \int_{B(y, a\rho_1(y))} v_0(x)g(y)dx \\ &= \int_{\mathbb{R}^n} g(x)\|v\|_{L^1(I_a(x))} dx. \end{aligned} \quad (2.11)$$

Prove now the inequality (2.10). We observe that:

$$\begin{aligned}
 \int_{\mathbb{R}^n} g(x) \|v\|_{L^1(I_a(x))}^p dx &= \int_{\mathbb{R}^n} g(x) dx \left( \int_{B(x, a\rho_1(x))} v_0(y) dy \right)^p \\
 &= \int_{\mathbb{R}^n} g(x)^{1/\nu} h(x) dx \int_{B(x, a\rho_1(x))} v_0(y) dy \\
 &= \int_{\mathbb{R}^n} dy \int_{E_a(y)} g(x)^{1/\nu} h(x) v_0(y) dx \\
 &= \int_{\Omega} v(y) dy \int_{E_a(y)} g(x)^{1/\nu} h(x) dx,
 \end{aligned} \tag{2.12}$$

where we have put

$$h(x) = g(x)^{1-1/\nu} \left( \int_{B(x, a\rho_1(x))} v_0(y) dy \right)^{\nu-1}. \tag{2.13}$$

On the other hand,

$$\begin{aligned}
 \int_{E_a(y)} g(x)^{1/\nu} h(x) dx &\leq \left( \int_{E_a(y)} g(x) dx \right)^{1/\nu} \left( \int_{E_a(y)} h(x)^{\nu/(\nu-1)} dx \right)^{(\nu-1)/\nu} \\
 &= \left( \int_{E_a(y)} g(x) dx \right)^{1/\nu} \left( \int_{E_a(y)} g(x) \|v\|_{L^1(I_a(x))}^p dx \right)^{(\nu-1)/\nu}.
 \end{aligned} \tag{2.14}$$

Therefore, from (2.12) and (2.14), we deduce that

$$\int_{\mathbb{R}^n} g(x) \|v\|_{L^1(I_a(x))}^p dx \leq \left( \int_{\mathbb{R}^n} g(x) \|v\|_{L^1(I_a(x))}^p dx \right)^{(\nu-1)/\nu} \int_{\Omega} v(y) \|g\|_{L^1(E_a(y))}^{1/\nu} dy. \tag{2.15}$$

By (2.15), it obviously follows

$$\left( \int_{\mathbb{R}^n} g(x) \|v\|_{L^1(I_a(x))}^p dx \right)^{1/\nu} \leq \int_{\Omega} v(x) \|g\|_{L^1(E_a(x))}^{1/\nu} dx, \tag{2.16}$$

and (2.16) yields the inequality (2.10).  $\square$

If  $k \in \mathbb{N}_0$ ,  $1 \leq p < +\infty$  and  $s \in \mathbb{R}$ , consider the space  $W_s^{k,p}(\Omega)$  of distributions  $u$  on  $\Omega$  such that  $\rho^s \partial^\alpha u \in L^p(\Omega)$  for  $|\alpha| \leq k$ , equipped with the norm

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\rho^s \partial^\alpha u\|_{L^p(\Omega)}. \tag{2.17}$$

Moreover, denote by  $\overset{\circ}{W}_s^{k,p}(\Omega)$  the closure of  $C^\infty(\Omega)$  in  $W_s^{k,p}(\Omega)$  and put  $W_s^{0,p}(\Omega) = L^p_s(\Omega)$ . A more detailed account of properties of the above-defined spaces can be found, for instance, in [14].

From Lemma 2.1 we can deduce another lemma which we will need in the proof of our regularity result.

**Lemma 2.2.** *Let  $p \in [1, +\infty[$ ,  $s \in \mathbb{R}$ , and  $a \in ]0, 1]$ . Then  $u \in L_s^p(\Omega)$  if and only if  $u \in L_{loc}^p(\overline{\Omega})$  and the function  $x \in \mathbb{R}^n \rightarrow \rho_1^{s-n/p}(x)\|u\|_{L^p(I_a(x))}$  belongs to  $L^p(\mathbb{R}^n)$ . In addition, there exist  $c_1, c_2 \in \mathbb{R}_+$  such that*

$$c_1 \|u\|_{L_s^p(\Omega)}^p \leq \int_{\mathbb{R}^n} \rho_1^{sp-n}(x) \|u\|_{L^p(I_a(x))}^p dx \leq c_2 \|u\|_{L_s^p(\Omega)}^p \quad \forall u \in L_s^p(\Omega), \quad (2.18)$$

where  $c_1$  and  $c_2$  depend only on  $n, p, s, a$ , and  $\rho$ . Moreover, if  $p_0 \in [1, p[$  and  $u \in L_s^{p_0}(\Omega)$ , then the function  $x \in \mathbb{R}^n \rightarrow \rho_1^{s-n/p_0}(x)\|u\|_{L^{p_0}(I_a(x))}$  belongs to  $L^p(\mathbb{R}^n)$  and the following estimate holds:

$$\int_{\mathbb{R}^n} \rho_1^{sp-n}(x) \|u\|_{L^{p_0}(I_a(x))}^p dx \leq c_3 \|u\|_{L_s^{p_0}(\Omega)}^p \quad (2.19)$$

with  $c_3 \in \mathbb{R}_+$  dependent only on  $n, p, p_0, s, a$ , and  $\rho$ .

*Proof.* The first part of the lemma follows from (2.9) for  $g = \rho_1^{sp-n}$  and  $v = |u|^p$ , if one uses (2.1) and (2.8). The second part of the lemma follows in a similar way from the inequality (2.10), if one puts  $g = \rho_1^{sp-n}$ ,  $v = |u|^{p_0}$ , and  $\nu = p/p_0$ .  $\square$

### 3. An embedding lemma

We now recall the definitions of the function spaces in which the coefficients of the operator will be chosen. If  $\Omega$  has the property

$$|\Omega(x, r)| \geq Ar^n \quad \forall x \in \Omega, \forall r \in ]0, 1], \quad (3.1)$$

where  $A$  is a positive constant independent of  $x$  and  $r$ , it is possible to consider the space  $\text{BMO}(\Omega, \tau)$  ( $\tau \in \mathbb{R}_+$ ) of functions  $g \in L_{loc}^1(\overline{\Omega})$  such that

$$[g]_{\text{BMO}(\Omega, \tau)} = \sup_{\substack{x \in \Omega \\ r \in ]0, \tau]}} \int_{\Omega(x, r)} \left| g - \int_{\Omega(x, r)} g \right| < +\infty, \quad (3.2)$$

where

$$\int_{\Omega(x, r)} g = |\Omega(x, r)|^{-1} \int_{\Omega(x, r)} g. \quad (3.3)$$

If  $g \in \text{BMO}(\Omega) = \text{BMO}(\Omega, \tau_A)$ , where

$$\tau_A = \sup \left\{ \tau \in \mathbb{R}_+ : \sup_{\substack{x \in \Omega \\ r \in ]0, \tau]}} \frac{r^n}{|\Omega(x, r)|} \leq \frac{1}{A} \right\}, \quad (3.4)$$

we will say that  $g \in \text{VMO}(\Omega)$  if  $[g]_{\text{BMO}(\Omega, \tau)} \rightarrow 0$  for  $\tau \rightarrow 0^+$ . A function

$$\eta[g] : ]0, 1] \rightarrow \mathbb{R}_+ \quad (3.5)$$

is called a *modulus of continuity* of  $g$  in  $\text{VMO}(\Omega)$  if

$$[g]_{\text{BMO}(\Omega, \tau)} \leq \eta[g](\tau) \quad \forall \tau \in ]0, 1], \quad \lim_{\tau \rightarrow 0^+} \eta[g](\tau) = 0. \quad (3.6)$$

For  $t \in [1, +\infty[$  and  $\lambda \in [0, n[$ , we denote by  $M^{t, \lambda}(\Omega)$  the set of all functions  $g$  in  $L_{loc}^t(\overline{\Omega})$  such that

$$\|g\|_{M^{t, \lambda}(\Omega)} = \sup_{\substack{r \in ]0, 1] \\ x \in \Omega}} r^{-\lambda/t} \|g\|_{L^t(\Omega(x, r))} < +\infty \quad (3.7)$$

endowed with the norm defined by (3.7). Then we define  $\widetilde{M}^{t,\lambda}(\Omega)$  as the closure of  $L^\infty(\Omega)$  in  $M^{t,\lambda}(\Omega)$  and  $M_o^{t,\lambda}(\Omega)$  as the closure of  $C_o^\infty(\Omega)$  in  $M^{t,\lambda}(\Omega)$ . In particular, we put  $M^t(\Omega) = M^{t,0}(\Omega)$ ,  $\widetilde{M}^t(\Omega) = \widetilde{M}^{t,0}(\Omega)$ , and  $M_o^t(\Omega) = M_o^{t,0}(\Omega)$ . In order to define the modulus of continuity of a function  $g$  in  $\widetilde{M}^{t,\lambda}(\Omega)$ , recall first that for a function  $g \in M^{t,\lambda}(\Omega)$  the following characterization holds:

$$g \in \widetilde{M}^{t,\lambda}(\Omega) \iff \lim_{\tau \rightarrow 0^+} p_g(\tau) = 0, \quad (3.8)$$

where

$$p_g(\tau) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x,1)| \leq \tau}} \|\chi_E g\|_{M^{t,\lambda}(\Omega)}, \quad \tau \in \mathbb{R}_+. \quad (3.9)$$

Thus the *modulus of continuity* of  $g \in \widetilde{M}^{t,\lambda}(\Omega)$  is a function

$$\tilde{\sigma}[g] : ]0, 1[ \longrightarrow \mathbb{R}_+, \quad (3.10)$$

such that

$$p_g(\tau) \leq \tilde{\sigma}[g](\tau) \quad \forall \tau \in ]0, 1[, \quad \lim_{\tau \rightarrow 0^+} \tilde{\sigma}[g](\tau) = 0. \quad (3.11)$$

A more detailed account of properties of the above defined function spaces can be found in [6, 15, 16].

We consider the following condition:

(h<sub>0</sub>)  $\Omega$  has the cone property,  $p \in ]1, +\infty[$ ,  $s \in \mathbb{R}$ ,  $k, h, t$  are numbers such that

$$k \in \mathbb{N}, \quad h \in \{0, 1, \dots, k-1\}, \quad t \geq p, \quad t > p \quad \text{if } p = \frac{n}{k-h}, \quad g \in M^t(\Omega). \quad (3.12)$$

From [17, Theorem 3.1] we have the following.

**Lemma 3.1.** *If the assumption (h<sub>0</sub>) holds, then for any  $u \in W_s^{k,p}(\Omega)$ , it results that  $g \partial^h u \in L_s^p(\Omega)$  and*

$$\|g \partial^h u\|_{L_s^p(\Omega)} \leq c \|g\|_{M^t(\Omega)} \|u\|_{W_s^{k,p}(\Omega)} \quad (3.13)$$

with  $c$  dependent only on  $\Omega$ ,  $n$ ,  $k$ ,  $h$ ,  $p$ , and  $t$ .

#### 4. A regularity result

Assume that  $\Omega$  is an unbounded open subset of  $\mathbb{R}^n$ ,  $n \geq 3$ , with the uniform  $C^{1,1}$ -regularity property, and let  $\rho$  be the function defined by (2.6). Moreover, let  $p \in ]1, +\infty[$  and  $s \in \mathbb{R}$ . Consider in  $\Omega$  the differential operator

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \quad (4.1)$$

with the following conditions on the coefficients:

(h<sub>1</sub>)

$$\begin{aligned}
a_{ij} &= a_{ji} \in L^\infty(\Omega) \cap \text{VMO}_{\text{loc}}(\overline{\Omega}), \quad i, j = 1, \dots, n, \\
\exists \nu > 0 : \sum_{i,j=1}^n a_{ij} \xi_i \xi_j &\geq \nu |\xi|^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^n,
\end{aligned} \tag{4.2}$$

there exist functions  $e_{ij}$ ,  $i, j = 1, \dots, n$ ,  $g$  and  $\mu \in \mathbb{R}_+$  such that(h<sub>2</sub>)

$$\begin{aligned}
e_{ij} &= e_{ji} \in L^\infty(\Omega) \cap \text{VMO}(\Omega), \quad i, j = 1, \dots, n, \\
\sum_{i,j=1}^n e_{ij} \xi_i \xi_j &\geq \mu |\xi|^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^n, \\
g &\in L^\infty(\Omega), \quad \lim_{r \rightarrow +\infty} \sum_{i,j=1}^n \|e_{ij} - g a_{ij}\|_{L^\infty(\Omega \setminus B_r)} = 0,
\end{aligned} \tag{4.3}$$

(h<sub>3</sub>)

$$a_i \in \widetilde{M}^{t_1}(\Omega), \quad i = 1, \dots, n, \quad a \in \widetilde{M}^{t_2}(\Omega), \tag{4.4}$$

where

$$\begin{aligned}
t_1 &> n \quad \text{if } p \leq n, & t_1 &= p \quad \text{if } p > n, \\
t_2 &> \frac{n}{2} \quad \text{if } p \leq \frac{n}{2}, & t_2 &= p \quad \text{if } p > \frac{n}{2}.
\end{aligned} \tag{4.5}$$

Observe that under the assumptions (h<sub>1</sub>)–(h<sub>3</sub>), it follows that the operator  $L : W_s^{2,p}(\Omega) \rightarrow L_s^p(\Omega)$  is bounded from Lemma 3.1.

**Theorem 4.1.** *Suppose that the assumptions (h<sub>1</sub>), (h<sub>2</sub>), and (h<sub>3</sub>) hold, and let  $u$  be a solution of the problem*

$$\begin{aligned}
u &\in W_{\text{loc}}^{2,q}(\overline{\Omega}) \cap \overset{\circ}{W}_{\text{loc}}^{1,q}(\overline{\Omega}) \cap L_s^{p_0}(\Omega), \\
Lu &\in L_s^p(\Omega),
\end{aligned} \tag{4.6}$$

where  $q \in ]1, p]$  and  $p_0 \in [1, p]$ . Then  $u$  belongs to  $W_s^{2,p}(\Omega)$ .

*Proof.* By [8, Lemma 4.1], we have

$$u \in W_{\text{loc}}^{2,p}(\overline{\Omega}) \cap \overset{\circ}{W}_{\text{loc}}^{1,p}(\overline{\Omega}). \tag{4.7}$$

We choose  $r, r' \in \mathbb{R}_+$ , with  $r < r' < 1$ , and a function  $\phi \in C^\infty(\mathbb{R}^n)$  such that

$$\begin{aligned}
\phi|_{B_r} &= 1, \quad \text{supp } \phi \subset B_{r'}, \\
\sup_{\mathbb{R}^n} |\partial^\alpha \phi| &\leq c_\alpha (r' - r)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}_0^n,
\end{aligned} \tag{4.8}$$

where  $c_\alpha \in \mathbb{R}_+$  depends only on  $\alpha$ .

We fix  $y \in \mathbb{R}^n$  and put

$$\psi = \psi_y : x \in \mathbb{R}^n \longrightarrow \phi\left(\frac{x-y}{\rho_1(y)}\right). \quad (4.9)$$

Clearly, we have

$$\begin{aligned} \psi|_{B(y, r\rho_1(y))} &= 1, \quad \text{supp } \psi \subset B(y, r'\rho_1(y)), \\ \sup_{\mathbb{R}^n} |\partial^\alpha \psi| &\leq c_\alpha \rho_1^{-|\alpha|}(y) (r' - r)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}_0^n. \end{aligned} \quad (4.10)$$

Since  $\psi u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}^{1,p}(\Omega)$ , from [8, Theorem 3.1] it follows that

$$\|\psi u\|_{W^{2,p}(\Omega)} \leq c_1 (\|L(\psi u)\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega)}) \quad (4.11)$$

with  $c_1$  depending on  $n, p, \Omega, \nu, \mu, t_1, t_2, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}], \eta[e_{ij}], \tilde{\sigma}[a_i], \tilde{\sigma}[a]$ , where  $r_0 \in \mathbb{R}_+$  depends on  $n, p, \Omega, \mu, \|e_{ij}\|_{L^\infty(\Omega)}, \eta[e_{ij}]$  and  $\zeta_{2r_0}$  is a function in  $C^\infty(\mathbb{R}^n)$  such that

$$0 \leq \zeta_{2r_0} \leq 1, \quad \zeta_{2r_0}|_{B_{2r_0}} = 1, \quad \text{supp } \zeta_{2r_0} \subset B_{4r_0}. \quad (4.12)$$

Since

$$\begin{aligned} L(\psi u) &= -\sum_{i,j=1}^n a_{ij}(\psi u)_{x_i x_j} + \sum_{i=1}^n a_i(\psi u)_{x_i} + a\psi u \\ &= \psi Lu - 2\sum_{i,j=1}^n a_{ij}(\psi_{x_i} u)_{x_j} + \sum_{i,j=1}^n a_{ij}\psi_{x_i x_j} u + \sum_{i=1}^n a_i\psi_{x_i} u \end{aligned} \quad (4.13)$$

from (4.11) and (4.13), we have

$$\begin{aligned} \|\psi u\|_{W^{2,p}(\Omega)} &\leq c_2 \left( \|\psi Lu\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|(\psi_{x_i} u)_{x_j}\|_{L^p(\Omega)} \right. \\ &\quad \left. + \sum_{i,j=1}^n \|\psi_{x_i x_j} u\|_{L^p(\Omega)} + \sum_{i=1}^n \|a_i \psi_{x_i} u\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega)} \right) \end{aligned} \quad (4.14)$$

with  $c_2$  depending on the same parameters of  $c_1$ .

From Lemma 3.1 with  $s = 0$ , we have that

$$\|a_i \psi_{x_i} \mathbf{u}\|_{L^p(\Omega)} \leq c_3 \|a_i\|_{M^1(\Omega)} (\|\psi_{x_i} \mathbf{u}\|_{L^p(\Omega)} + \|(\psi_{x_i} \mathbf{u})_x\|_{L^p(\Omega)}) \quad (4.15)$$

with  $c_3$  dependent on  $\Omega$ ,  $n$ ,  $p$ , and  $t_1$ .

Using [18, Corollary 4.5], we can obtain the following interpolation estimates:

$$\begin{aligned} \|\psi_{x_i} \mathbf{u}\|_{L^p(\Omega)} + \|(\psi_{x_i} \mathbf{u})_{x_j}\|_{L^p(\Omega)} &\leq c_4 (\|(\psi_{x_i} \mathbf{u})_{xx}\|_{L^p(\Omega)}^a \|\psi_{x_i} \mathbf{u}\|_{L^{p_0}(\Omega)}^{1-a} + \|\psi_{x_i} \mathbf{u}\|_{L^{p_0}(\Omega)}), \\ \|\psi_{x_i x_j} \mathbf{u}\|_{L^p(\Omega)} &\leq c_5 (\|(\psi_{x_i x_j} \mathbf{u})_{xx}\|_{L^p(\Omega)}^a \|\psi_{x_i x_j} \mathbf{u}\|_{L^{p_0}(\Omega)}^{1-a} + \|\psi_{x_i x_j} \mathbf{u}\|_{L^{p_0}(\Omega)}), \\ \|\psi \mathbf{u}\|_{L^p(\Omega)} &\leq c_6 (\|(\psi \mathbf{u})_{xx}\|_{L^p(\Omega)}^a \|\psi \mathbf{u}\|_{L^{p_0}(\Omega)}^{1-a} + \|\psi \mathbf{u}\|_{L^{p_0}(\Omega)}), \end{aligned} \quad (4.16)$$

where  $a \in ]0, 1[$  depends on  $n$ ,  $p$ ,  $p_0$  and the constants  $c_4$ ,  $c_5$ , and  $c_6$  depend on  $\Omega$ ,  $n$ ,  $p$ ,  $p_0$ .

Thus by (4.14)–(4.16), with easy computations, we deduce the following bound:

$$\begin{aligned} \|\mathbf{u}\|_{W^{2,p}(I_r(y))} &\leq \|\psi \mathbf{u}\|_{W^{2,p}(\Omega)} \\ &\leq c_7 (r' - r)^{-2(1+a)} (\|L\mathbf{u}\|_{L^p(I_{r'}(y))} + \|\mathbf{u}\|_{W^{2,p}(I_{r'}(y))}^a \|\mathbf{u}\|_{L^{p_0}(I_{r'}(y))}^{1-a} + \|\mathbf{u}\|_{L^{p_0}(I_{r'}(y))}), \end{aligned} \quad (4.17)$$

where  $c_7 \in \mathbb{R}_+$  depends on  $n$ ,  $p$ ,  $p_0$ ,  $\Omega$ ,  $\rho$ ,  $\nu$ ,  $\mu$ ,  $t_1$ ,  $t_2$ ,  $\|a_{ij}\|_{L^\infty(\Omega)}$ ,  $\|e_{ij}\|_{L^\infty(\Omega)}$ ,  $\|g\|_{L^\infty(\Omega)}$ ,  $\eta[\zeta_{2r_0} a_{ij}]$ ,  $\eta[e_{ij}]$ ,  $\|a_i\|_{M^1(\Omega)}$ ,  $\tilde{\sigma}[a_i]$ ,  $\tilde{\sigma}[a]$ .

By a well-known lemma of monotonicity of Miranda (see [19, Lemma 3.1]), it follows from (4.17) that

$$\|\mathbf{u}\|_{W^{2,p}(I_{1/2}(y))} \leq c_8 \left( \|L\mathbf{u}\|_{L^p(I_1(y))} + \|\mathbf{u}\|_{L^{p_0}(I_1(y))} + \|\mathbf{u}\|_{L^{p_0}(I_1(y))}^{1-a} \|\mathbf{u}\|_{W^{2,p}(I_{1/2}(y))}^a \right), \quad (4.18)$$

and then, using Young's inequality, we deduce from (4.18) that

$$\|\mathbf{u}\|_{W^{2,p}(I_{1/2}(y))} \leq c_9 (\|L\mathbf{u}\|_{L^p(I_1(y))} + \|\mathbf{u}\|_{L^{p_0}(I_1(y))}) \quad (4.19)$$

with  $c_8, c_9 \in \mathbb{R}_+$  dependent on the same parameters of  $c_7$ .

From (4.19) it follows that

$$\int_{\mathbb{R}^n} \rho_1^{ps-n}(\mathbf{y}) \|\mathbf{u}\|_{W^{2,p}(I_{1/2}(\mathbf{y}))}^p d\mathbf{y} \leq c_{10} \left( \int_{\mathbb{R}^n} \rho_1^{ps-n}(\mathbf{y}) \|L\mathbf{u}\|_{L^p(I_1(\mathbf{y}))}^p d\mathbf{y} + \int_{\mathbb{R}^n} \rho_1^{ps-n}(\mathbf{y}) \|\mathbf{u}\|_{L^{p_0}(I_1(\mathbf{y}))}^p d\mathbf{y} \right), \quad (4.20)$$

where  $c_{10} \in \mathbb{R}_+$  depends on the same parameters of  $c_9$ .

By (4.20) and by Lemma 2.2, we have that

$$\|\mathbf{u}\|_{W_s^{2,p}(\Omega)} \leq c_{11} (\|L\mathbf{u}\|_{L_s^p(\Omega)} + \|\mathbf{u}\|_{L_s^{p_0}(\Omega)}) \quad (4.21)$$

with  $c_{11} \in \mathbb{R}_+$  dependent on the same parameters of  $c_{10}$  and on  $s$ .

Therefore, from (4.21), we have the result.  $\square$

### 5. Existence and uniqueness results

In this section, we will prove our existence and uniqueness theorem. To this aim, we need two preliminary lemmas.

Observe that it is possible to find a function  $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$  which is equivalent to  $\rho$  and such that

$$|\partial^\alpha \sigma(x)| \leq c_\alpha \sigma^{1-|\alpha|}(x) \quad \forall x \in \Omega, \quad \forall \alpha \in \mathbb{N}_0^n, \quad (5.1)$$

where  $c_\alpha$  is independent of  $x$  (see [12]).

**Lemma 5.1.** *The Dirichlet problem*

$$\begin{aligned} u &\in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}_s^{1,2}(\Omega), \\ -\Delta u + bu &= f, \quad f \in L_s^2(\Omega), \end{aligned} \quad (5.2)$$

where

$$b = 1 + \left| -s(s+1) \sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} + s \sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma} \right| \quad (5.3)$$

is uniquely solvable. Moreover, if  $f \in C_\infty^\infty(\Omega)$ , then the solution  $u$  belongs to  $L_s^q(\Omega)$  for all  $q$  in  $[1, +\infty]$ .

*Proof.* Note that  $u$  is a solution of problem (5.2) if and only if  $w = \sigma^s u$  is a solution of the problem

$$\begin{aligned} w &\in W^{2,2}(\Omega) \cap \overset{\circ}{W}^{1,2}(\Omega), \\ -\Delta(\sigma^{-s} w) + b\sigma^{-s} w &= f, \quad f \in L_s^2(\Omega). \end{aligned} \quad (5.4)$$

But for any  $i \in \{1, \dots, n\}$

$$\frac{\partial^2}{\partial x_i^2}(\sigma^{-s} w) = \sigma^{-s} w_{x_i x_i} - 2s\sigma^{-s-1} \sigma_{x_i} w_{x_i} + s(s+1)\sigma^{-s-2} \sigma_{x_i}^2 w - s\sigma^{-s-1} \sigma_{x_i x_i} w, \quad (5.5)$$

then (5.4) is equivalent to the problem

$$\begin{aligned} w &\in W^{2,2}(\Omega) \cap \overset{\circ}{W}^{1,2}(\Omega), \\ -\Delta w + \sum_{i=1}^n \alpha_i w_{x_i} + \alpha w &= g, \quad g \in L^2(\Omega), \end{aligned} \quad (5.6)$$

where we have put

$$\alpha_i = 2s \frac{\sigma_{x_i}}{\sigma}, \quad i = 1, \dots, n, \quad \alpha = b - s(s+1) \sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} + s \sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma}, \quad g = \sigma^s f. \quad (5.7)$$

Using [7, Theorem 5.2], [6, Equation (1.6)], and (5.1), we obtain that (5.6) is uniquely solvable and then problem (5.2) is uniquely solvable too.

Moreover, if  $f \in C_\infty^\infty(\Omega)$ , then also  $g \in C_\infty^\infty(\Omega)$ . Therefore, using the theorem in [20], we have that the solution  $w$  of (5.6) belongs to  $L^q(\Omega)$  for all  $q \in [1, +\infty]$ , and so the solution  $u$  of (5.2) lies in  $L_s^q(\Omega)$  for all  $q \in [1, +\infty]$ .  $\square$

**Lemma 5.2.** *The Dirichlet problem*

$$\begin{aligned} u &\in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ -\Delta u + bu &= f, \quad f \in L_s^p(\Omega), \end{aligned} \tag{5.8}$$

is uniquely solvable, where  $b$  is defined by (5.3).

*Proof.* Let  $f$  be a function in  $C^\infty(\Omega)$ . Then, by Lemma 5.1, there exists a unique  $u \in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}_s^{1,2}(\Omega) \cap L_s^q(\Omega)$  (for all  $q \in [1, +\infty]$ ) such that  $-\Delta u + bu = f$ .

Firstly, suppose that  $p \geq 2$ . It follows from Theorem 4.1 that  $u$  belongs to  $W_s^{2,p}(\Omega)$ . Moreover, by [10, Lemma 2.2],  $u$  lies in  $\overset{\circ}{W}_s^{1,p}(\Omega)$ .

Suppose now  $p < 2$ . Then  $u \in W_{\text{loc}}^{2,p}(\overline{\Omega}) \cap \overset{\circ}{W}_{\text{loc}}^{1,p}(\overline{\Omega}) \cap L_s^q(\Omega)$  (for all  $q \in [1, +\infty]$ ) and then, using again Theorem 4.1,  $u$  belongs to  $W_s^{2,p}(\Omega)$ . Moreover, by [10, Lemma 2.2],  $u$  lies in  $\overset{\circ}{W}_s^{1,p}(\Omega)$ .

Therefore, in both cases,  $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega)$  and it is a solution of the equation  $-\Delta u + bu = f$ , so that  $C_c^\infty(\Omega) \subseteq R(-\Delta + b)$ . Since  $C_c^\infty(\Omega)$  is dense in  $L_s^p(\Omega)$  (see [14, Proposition 1.1]) and  $R(-\Delta + b)$  is a closed subspace of  $L_s^p(\Omega)$  by [10, Theorem 4.1], we obtain that  $R(-\Delta + b) = L_s^p(\Omega)$ . The uniqueness of the solution follows from [10, Theorem 5.2].  $\square$

Finally, adding the following assumption on the coefficients of  $L$ :

(h<sub>4</sub>)

$$\begin{aligned} (e_{ij})_{x_h} &\in M_o^{t,n-t}(\Omega), \quad \text{with } t \in ]2, n], \quad i, j, h = 1, \dots, n, \\ a_i &\in M_o^{t_1}(\Omega), \quad i = 1, \dots, n, \\ a &= a' + a'', \quad a' \in M_o^{t_2}(\Omega), \quad a'' \in L^\infty(\Omega), \quad a''_0 = \text{ess inf}_\Omega a'' > 0, \\ g &\in \text{Lip}(\overline{\Omega}), \quad g_0 = \text{ess inf}_\Omega g > 0, \end{aligned} \tag{5.9}$$

we are now in position to state the following uniqueness and existence result.

**Theorem 5.3.** *Suppose that conditions (h<sub>1</sub>)–(h<sub>4</sub>) hold. In addition, assume that  $a \geq a_0 > 0$  a.e. in  $\Omega$ . Then the problem*

$$\begin{aligned} u &\in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ Lu &= f, \quad f \in L_s^p(\Omega), \end{aligned} \tag{5.10}$$

is uniquely solvable.

*Proof.* For each  $\tau \in [0, 1]$ , put

$$L_\tau = \tau(gL) + (1 - \tau)(-\Delta + b). \tag{5.11}$$

The function

$$\tau \in [0, 1] \longmapsto L_\tau \in B(W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), L_s^p(\Omega)) \tag{5.12}$$

is clearly continuous; moreover, it is easy to show that the coefficients of each operator  $L_\tau$  satisfy the hypotheses of [10, Theorem 5.2] (see also [16, Lemma 3.2]), and hence  $N(L_\tau) = \{0\}$ . On the other hand, it follows from [10, Theorem 4.1] that  $R(L_\tau)$  is closed for any  $\tau \in [0, 1]$ , so that [9, Lemma 4.1] can be used to obtain the existence of  $C_0 \in \mathbb{R}_+$  such that

$$\|u\|_{W_s^{2,p}(\Omega)} \leq C_0 \|L_\tau u\|_{L_s^p(\Omega)} \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \quad \forall \tau \in [0, 1]. \quad (5.13)$$

By Lemma 5.2, the problem

$$\begin{aligned} u &\in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ -\Delta u + bu &= h, \quad h \in L_s^p(\Omega), \end{aligned} \quad (5.14)$$

is uniquely solvable.

Therefore, this latter result and estimate (5.13) allow to use the method of continuity along a parameter in order to prove that the problem

$$\begin{aligned} u &\in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_s^{1,p}(\Omega), \\ (gL)u &= gf, \quad f \in L_s^p(\Omega) \end{aligned} \quad (5.15)$$

is likewise uniquely solvable. The proof is complete.  $\square$

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