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Research Article

Existence of Solutions of Periodic Boundary Value Problems for Impulsive Functional Duffing Equations at Nonresonance Case

Xingyuan Liu[1] and Yuji Liu[2]

[1] *Department of Mathematics, Shaoyang University, Shaoyang, Hunan 422000, China*
[2] *Department of Mathematics, Guangdong University of Business Studies, Guangzhou, Guangdong 510320, China*

Correspondence should be addressed to Xingyuan Liu, liuxingyuan999@sohu.com

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This paper deals with the existence of solutions of the periodic boundary value problem of the impulsive Duffing equations: $x''(t) + \alpha x'(t) + \beta x(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t)))$, a.e. $t \in [0, T]$, $\Delta x(t_k) = I_k(x(t_k), x'(t_k))$, $k = 1, \ldots, m$, $\Delta x'(t_k) = J_k(x(t_k), x'(t_k))$, $k = 1, \ldots, m$, $x^{(i)}(0) = x^{(i)}(T)$, $i = 0, 1$. Sufficient conditions are established for the existence of at least one solution of above-mentioned boundary value problem. Our method is based upon Schaeffer's fixed-point theorem. Examples are presented to illustrate the efficiency of the obtained results.

1. Introduction

In recent years, many authors studied the solvability of the periodic boundary value problems (PBVPs for short) for second-order ordinary or functional differential equations with or without impulse effects; see [1–24] and the references therein. For example, consider the following PBVP:

$$x''(t) = f(t, x(t)), \quad t \in (0, 2\pi),$$
$$x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \tag{1.1}$$

the well-known result is that if f satisfies the nonresonance condition

$$-(N+1)^2 + \epsilon \le f_u(t, u) \le -\epsilon - N^2, \tag{1.2}$$

where N is a nonnegative integer and ϵ is a positive constant, then PBVP(1.1) has a unique solution; see [1].

For PBVP of the Duffing equation

$$\begin{gathered} x''(t) + cx'(t) + g(t, x(t)) = e(t), \quad t \in (0, 2\pi), \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1.3}$$

in [16], the authors proved the following results.

Theorem 1.1. *Suppose g is a L^2-Caratheodory function, and there are $a \leq A$, $r < 0 < R$ such that*

$$g(t, x) \geq A, \quad x \geq R,\ t \in [0, 2\pi], \qquad g(t, x) \leq a, \quad x \leq r,\ t \in [0, 2\pi], \tag{1.4}$$

and further there is $r \in L(0, 2\pi)$ with $\|r\|_\infty < 1 + C^2$ such that

$$\overline{\lim_{|x| \to \infty}} \frac{g(t, x)}{x} < r(t), \quad t \in (0, 2\pi). \tag{1.5}$$

Then, PBVP(1.3) has at least one solution for each $e \in L^2(0, 2\pi)$ with $a \leq (1/2\pi)\int_0^{2\pi} e(s)ds \leq A$.

In [17], Nieto and Rodríguez-López studied the following PBVP:

$$\begin{gathered} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma(t), \quad t \in (0, T), \\ x(0) = x(T), \qquad x'(0) = x'(T). \end{gathered} \tag{1.6}$$

They gave Green's function to express the unique solution for the correspondence second-order functional differential equation with periodic boundary conditions and the functional dependence given by the piecewise constant function. Using upper and lower solution methods, they presented sufficient conditions to assure the existence of solutions of PBVP(1.6). The authors in [3, 21] also studied the solvability of PBVP(1.6) by the similar method.

In [2], Ding et al. studied the PBVP:

$$\begin{gathered} x''(t) + f(t, x(t), x(\theta(t))) = 0, \quad t \in (0, T), \\ x(0) = x(T), \qquad x'(0) = x'(T). \end{gathered} \tag{1.7}$$

Sufficient conditions for the existence of solutions of PBVP(1.7) are given by using upper and lower solution method.

The PBVPs,

$$\begin{gathered} x''(t) + \rho^2 x(t) = f(t, x(t)), \quad t \in (0, 2\pi), \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1.8}$$

$$\begin{gathered} -x''(t) + \rho^2 x(t) = f(t, x(t)), \quad t \in (0, 2\pi), \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1.9}$$

were studied in [8, 18–20]. In [19], based upon Krasnosel′skii fixed-point theorems, Jiang proved that PBVP(1.8) and PBVP(1.9) with singularity have at least one positive solution provided that $f(t, x)$ is superlinear or sublinear at $x = 0$ and $x = +\infty$. In [21], Wang, who utilized the Schauder fixed-point theorem, proved that PBVP(1.9) without singularity has at least one positive solution provided that $f(t,x)$ is sublinear at $x = \infty$. In [18], Zhang and Wang established multiplicity results to positive solutions of PBVP(1.8) and PBVP(1.9) with f being a Caratheodory function.

In paper [24], Liu and Ge studied the following equation:

$$x''(t) - p(t)x'(t) + q(t)x(t) = \lambda f\big(t, x(t-\tau(t))\big) + r(t); \tag{1.10}$$

they established sufficient conditions for the existence of positive periodic solutions of (1.10).

In [11], Peng studied the existence of periodic solutions of the functional Duffing equation:

$$x''(t) + cx'(t) + g_1\big(t, x(t-\tau_1(t))\big) + g_2\big(t, x(t-\tau_2(t))\big) = p(t). \tag{1.11}$$

The studies on the existence of solutions of the periodic boundary value problems of the second-order impulsive differential equations,

$$\begin{gathered} x''(t) + f\big(t, x(t)\big) = 0, \quad t \in (0,T), \\ x(0) = x(T), \qquad x'(0) = x'(T), \\ \Delta x^{(i)}(t_k) = I_{i,k}\big(x(t_k)\big), \quad k = 1,\dots,p,\ i = 0,1, \end{gathered} \tag{1.12}$$

can be found in [3, 4, 12–14, 20, 22, 23] and the references therein. The methods used there are of lower and upper solutions methods, the monotone iterative technique.

In all above-mentioned papers, the results are based upon the following assumptions.

(**H**) f satisfies either the Lipschitzian condition, left Lipschitzian condition, right Lipschitzian condition, or Nagumo conditions.

To the best of our knowledge, the existence of solutions of periodic boundary value problems for impulsive Duffing functional differential equations has not been well studied till now.

In this paper, we investigate the following periodic boundary value problem for the impulsive functional Duffing equation:

$$\begin{gathered} x''(t) + \alpha x'(t) + \beta x(t) = f\big(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_n(t))\big), \quad \text{a.e. } t \in [0,T], \\ \Delta x(t_k) = I_k\big(x(t_k), x'(t_k)\big), \quad k = 1,\dots,m, \\ \Delta x'(t_k) = J_k\big(x(t_k), x'(t_k)\big), \quad k = 1,\dots,m, \\ x^{(i)}(0) = x^{(i)}(T), \quad i = 0,1, \end{gathered} \tag{1.13}$$

where m is a positive integer, $\alpha, \beta \in R$, $f : [0,T] \times R^{n+1} \to R$ is a impulsive Caratheodory function, $\alpha_i \in C^1([0,T],[0,T])$, whose inverse function is denoted by β_i with $\beta_i \in C^1[0,T]$, $I_k, J_k : R^2 \to R$ are continuous.

Our purpose here is to provide sufficient conditions for the existence of solutions of (1.13) at nonresonance case. This will be done by applying the well-known Schaeffer's fixed-point theorem, and we do not rely on the existence of upper and lower solutions and the assumption (**H**) mentioned above.

The main results and examples in this paper are established in Section 2. The proofs of the main results are presented in Section 3.

2. Main results and examples

In this section, we establish the main results and examples to illustrate the main theorems. To define solutions of PBVP (1.13), we introduce the following Banach spaces and definitions.

Suppose $u : J = [0,T] \to R$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. For $k = 0,\dots,m$, define the function $u_k : (t_k, t_{k+1}) \to R$ by $u_k(t) = u(t)$. Choose

$$X = \left\{ u : J \to R \;\middle|\; u_k \in C^0(t_k, t_{k+1}),\ k = 0,\dots,m,\ \text{there exist the limits}\ \begin{array}{l} \lim_{t\to t_k^-} u(t) = u(t_k), \\ \lim_{t\to t_k^+} u(t), \\ \lim_{t\to 0^+} u(t) = u(0), \\ \lim_{t\to T^-} u(t) = u(T) \end{array} \right\} \tag{2.1}$$

with the norm

$$\|u\|_X = \|u\| = \sup_{t\in[0,T]} \left|u(t)\right| \tag{2.2}$$

for $u \in X$. Choose

$$Y = X \times R^m \times R^m \tag{2.3}$$

with the norm

$$\|y\|_Y = \|y\| = \max\left\{ \sup_{t\in[0,T]} \left|u(t)\right|,\ \max_{1\le k\le m}\left\{\left|x_k\right|\right\},\ \max_{1\le k\le m}\left|y_k\right| \right\} \tag{2.4}$$

for $y = \{u, x_1, \dots, x_m, y_1, \dots, y_m\} \in Y$. Then, X and Y are real Banach spaces.

A function $F : [0,1] \times R^{n+1} \to R$ is called an impulsive Caratheodory function if

(i) $F(\bullet, u_0, u_1, \dots, u_n) \in X$ for each $u = (u_0, \dots, u_n) \in R^{n+1}$;

(ii) $F(t, \bullet, \dots, \bullet)$ is continuous for $t \neq t_k$ $(k = 1,\dots,m)$.

By a solution of PBVP (1.13), we mean a function $x : [0,T] \to R$ satisfying the following conditions:

(i) $x \in X$ is differentiable in (t_k, t_{k+1}) $(k = 0,1,\dots,m)$, there exist the limits $\lim_{t\to t_k^+} x'(t)$, $\lim_{t\to t_k^-} x'(t) = x'(t_k)$ $(k = 0,1,\dots,m)$, $\lim_{t\to 0^+} x'(t) = x'(0)$ and $\lim_{t\to T^-} x'(t) = x'(T)$;

(ii) $x' \in X$ is differentiable in (t_k, t_{k+1}) $(k = 0,1,\dots,m)$, there exist the limits $\lim_{t\to t_k^+} x''(t)$, $\lim_{t\to t_k^-} x''(t) = x''(t_k)$ $(k = 0,1,\dots,m)$, $\lim_{t\to 0^+} x''(t) = x''(0)$, and $\lim_{t\to T^-} x''(t) = x''(T)$;

(iii) $x'' \in X$;

(iv) the equations in (1.13) are satisfied.

Consider the following homogenous PBVP:

$$\begin{gathered} x''(t) + \alpha x'(t) + \beta x(t) = 0, \quad t \in [0,T], \\ \Delta x(t_k) = 0, \quad k = 1,\dots,m, \\ \Delta x'(t_k) = 0, \quad k = 1,\dots,m, \\ x(0) = x(T), \qquad x'(0) = x'(T). \end{gathered} \tag{2.5}$$

If $x \in X$ is a solution of problem (2.5), then

$$\begin{gathered} x''(t)x(t) + \alpha x'(t)x(t) + \beta[x(t)]^2 = 0, \\ x''(t)x'(t) + \alpha[x'(t)]^2 + \beta x(t)x'(t) = 0. \end{gathered} \tag{2.6}$$

Integrating them from 0 to T, one sees that

$$-\int_0^T [x'(t)]^2 dt + \beta\int_0^T [x(t)]^2 dt = 0, \qquad \alpha\int_0^T [x'(t)]^2 dt = 0. \tag{2.7}$$

If $\beta > 0$ and $\alpha \neq 0$ or $\beta < 0$, we get $x(t) \equiv 0$, then problem (2.5) has unique solution $x(t) = 0$ at the cases either $\beta > 0$ and $\alpha \neq 0$ or $\beta < 0$. We call PBVP (1.13) at nonresonance case. It suffices to consider the following cases:

$$\begin{aligned} &\text{Case 1. } \alpha \geq 0, \beta < 0, \qquad &&\text{Case 2. } \alpha < 0, \beta < 0, \\ &\text{Case 3. } \alpha > 0, \beta > 0, \qquad &&\text{Case 4. } \alpha < 0, \beta > 0. \end{aligned} \tag{2.8}$$

We set the following assumptions which should be used in the main results.

(A_1) $xI_k(x,y) \geq 0$ for all $x,y \in R$.

(A_1') $[I_k(x,y)]^2 + xI_k(x,y) \leq 0$ for all $x,y \in R$.

(A_2) $xJ_k(x,y) + yI_k(x,y) + \lambda I_k(x,y)J_k(x,y) \geq 0$ for all $x,y \in R$, $\lambda \in [0,1]$.

(A_2') $xJ_k(x,y) + yI_k(x,y) + \lambda I_k(x,y)J_k(x,y) \leq 0$ for all $x,y \in R$, $\lambda \in [0,1]$.

(A_3) There exist constants $\theta_k \geq 0$ such that $|I_k(x,y)| \leq \theta_k|x|$ for all $x,y \in R$ with $\sum_{k=1}^m \theta_k < 1$.

(C) There exist impulsive Caratheodory functions $h : [0,T] \times R^n \to R$, $g_i : [0,T] \times R \to R$, and function $r \in X$ and such that

(i) $f(t,x_0,\dots,x_n) = h(t,x_0,\dots,x_n) + \sum_{i=0}^n g_i(t,x_i) + r(t)$ holds for all $(t,x_0,\dots,x_n) \in [0,T] \times R^{n+1}$.

(ii) There exist constants $q \geq 0$ and $\theta > 0$ such that

$$h(t,x_0,\dots,x_n)x_0 \geq \theta|x_0|^{q+1} \tag{2.9}$$

holds for all $(t,x_0,\dots,x_n) \in [0,T] \times R^{n+1}$.

(iii) $\lim_{|x|\to+\infty} \sup_{t\in[0,T]} (|g_i(t,x)|/|x|^q) = r_i \in [0,+\infty)$ for $i = 0,\dots,n$.

(C') There exist impulsive Caratheodory functions $h : [0,T] \times R^n \to R$, $g_i : [0,T] \times R \to R$, and function $r \in X$ and such that (C)(i) and (C)(iii) hold.

(ii) There exist constants $q \geq 0$ and $\theta > 0$ such that

$$h(t, x_0, \ldots, x_n)x_0 \leq -\theta|x_0|^{q+1} \tag{2.10}$$

holds for all $(t, x_0, \ldots, x_n) \in [0, T] \times R^{n+1}$.

Theorem 2.1. *Suppose $\alpha \geq 0$ and $\beta < 0$, (A_1), (A_2), (A_3), and (C) hold. Then, PBVP (1.13) has at least one solution if*

$$\theta > r_0 + \sum_{k=1}^{n} r_k \|\beta'_k\|^{q/(q+1)} \tag{2.11}$$

holds.

Theorem 2.2. *Suppose $\alpha < 0$ and $\beta < 0$, (A'_1), (A_2), (A_3), and (C) hold. Then, PBVP (1.13) has at least one solution if (2.11) holds.*

Theorem 2.3. *Suppose $\alpha > 0$ and $\beta > 0$, (A_1), (A_2), (A_3), and (C) hold. Then, PBVP (1.13) has at least one solution if (2.11) holds.*

Theorem 2.4. *Suppose $\alpha < 0$ and $\beta > 0$, (A_1), (A_2), (A_3), and (C) hold. Then, PBVP (1.13) has at least one solution if (2.11) holds.*

To illustrate our main results, we present two boundary value problems that our results can readily apply, whereas the known results in the current literature do not cover.

Example 2.5. Consider the following PBVP:

$$\begin{gathered}
x''(t) + \alpha x'(t) + \beta x(t) = \sum_{k=0}^{2q+1} \epsilon_k x^k(t) + r(t), \quad t \in [0, T],\ t \neq t_k,\ k = 1, \ldots, m, \\
\Delta x(t_k) = 0, \quad k = 1, \ldots, m, \\
\Delta x'(t_k) = b_k \left[x(t_k)\right]^3, \quad k = 1, \ldots, m, \\
x(0) = x(T), \qquad x'(0) = x'(T),
\end{gathered} \tag{2.12}$$

where q is a positive integer, $\epsilon_k \geq 0$, $T > 0$, r is a continuous function. Corresponding to PBVP (1.13), one finds that

$$\begin{gathered}
f(t, x_0, x_1, \ldots, x_n) = \sum_{k=0}^{2q+1} \epsilon_k x_0^k + r(t), \\
I_k(x, y) = 0, \quad J_k(x, y) = b_k x^3,
\end{gathered} \tag{2.13}$$

It is easy to see that

(i) (A_1) holds,

(ii) since $a_k \geq 0$, one gets that $yI_k(x, y) + xJ_k(x, y) + \lambda I_k(x, y)J_k(x, y) = b_k x^4 \geq 0$, then (A_2) holds,

(iii) (A_3) holds,

(iv) $h(t,x)=\epsilon_{2q+1}x^{2q+1}$ with $\alpha_{2q+1}>0$ and $g_k(t,x)=\alpha_k x^k$ for $k=0,1,\ldots,2q$ implies that (C) holds.

It follows from Theorem 2.1 that PBVP (2.12) has at least one solution.

Example 2.6. Consider the following PBVP:

$$\begin{gathered} x''(t)+\alpha x'(t)+\beta x(t)=\sum_{k=0}^{2q+1}\alpha_k x^k(t)+\sum_{k=1}^{n}\beta_k x^{2q+1}\left(\frac{1}{n+1}t\right)+r(t), \\ t\in[0,T],\ t\neq t_k,\ k=1,\ldots,m, \\ \Delta x(t_k)=0,\quad k=1,\ldots,m, \\ \Delta x'(t_k)=b_k\left[x(t_k)\right]^5,\quad k=1,\ldots,m, \\ x(0)=x(T),\qquad x'(0)=x'(T), \end{gathered} \tag{2.14}$$

where $\alpha\geq 0$, $\beta<0$, $T>0$, q is a positive integer, $\alpha_{2q+1}>0$, m, n are positive integers, $r\in L^1[0,T]$, $b_k\in R$, $0<t_1<\cdots<t_m<T$, $\alpha_k,\beta_k\in R$.

It is easy to see that

(i) (A_1') holds,

(ii) It is easy to see that

$$xJ_k(x,y)+yI_k(x,y)+\lambda I_k(x,y)J_k(x,y)=b_kx^6\geq 0 \tag{2.15}$$

if $b_k\geq 0$, then (A_2) holds,

(iii) (A_3) holds,

(iv) if $h(t,x)=\alpha_{2q+1}x^{2q+1}$ with $\alpha_{2q+1}>0$ and $g_k(t,x)=\beta_k x^{2q+1}$ for $k=1,\ldots,n$, $g_0(t,x)=\sum_{k=0}^{2q}\alpha_k x^k$, then (C) holds.

It follows from Theorem 2.2 that PBVP (2.14) has at least one solution if

$$\begin{gathered} b_k\geq 0,\ \ k=1,\ldots,m,\quad \sum_{k=1}^{m}|b_k|<1, \\ \alpha_{2q+1}>0,\quad \sum_{k=1}^{n}\beta_k+\alpha_{2q+1}\neq 0,\quad \sum_{k=1}^{n}\beta_k(n+1)^{(2q+1)/(2q+2)}<\alpha_{2q+1}. \end{gathered} \tag{2.16}$$

3. Proofs of theorems

In this section, we prove that main theorems are presented in Section 2. We define the linear operator $L:D(L)\subseteq X\to Y$ and the nonlinear operator $N:X\to Y$ by

$$Lx(t)=\begin{pmatrix} x''(t)+\alpha x'(t)+\beta x(t) \\ \Delta x(t_1) \\ \vdots \\ \Delta x(t_m) \\ \Delta x'(t_1) \\ \vdots \\ \Delta x'(t_m) \end{pmatrix}\quad \text{for } x\in D(L), \tag{3.1}$$

where

$$D(L)=\left\{u:[0,T]\longrightarrow R\ \middle|\ \begin{array}{l} u\in X \text{ is differentiable in } (t_k,t_{k+1})\ (k=0,1,\ldots,m),\\ \text{there exist the limits } \lim\limits_{t\to t_k^+} x'(t),\\ \lim\limits_{t\to t_k^-} x'(t)=x'(t_k)\ (k=0,1,\ldots,m),\\ \lim\limits_{t\to 0^+} x'(t)=x'(0),\ \lim\limits_{t\to T^-} x'(t)=x'(T),\\ x'\in X \text{ there exist the limits } \lim\limits_{t\to t_k^+} x''(t),\\ \lim\limits_{t\to t_k^-} x''(t)=x''(t_k)\ (k=0,1,\ldots,m),\\ \lim\limits_{t\to 0^+} x''(t)=x''(0),\ \lim\limits_{t\to T^-} x''(t)=x''(T),\\ x''\in X, \end{array}\right\},$$

$$Nx(t)=\begin{pmatrix} f(t,x(t),x(\alpha_1(t)),\ldots,x(\alpha_n(t)))\\ I_1(x(t_1),x'(t_1))\\ \vdots\\ I_m(x(t_m),x'(t_m))\\ J_1(x(t_1),x'(t_1))\\ \vdots\\ J_m(x(t_m),x'(t_m)) \end{pmatrix} \quad \text{for } x\in X. \tag{3.2}$$

Lemma 3.1. *Suppose $\alpha\neq 0$, $\beta\neq 0$, or $\alpha=0$ and $\beta<0$, $f:[0,T]\times R^{n+1}\to R$ is an impulsive Caratheodory function, $I_k, J_k: R^2\to R$ are continuous. Then, the following results hold.*

(i) $\operatorname{Ker} L=\{0\}$.

(ii) *L is a Fredholm operator of index zero.*

(iii) *N is L-compact on $\overline{\Omega}$ with Ω being open and bounded.*

Lemma 3.2 ([15]). *Let X and Y be Banach spaces. Suppose $L: D(L)\subset X\to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L=\{0\}$, $N: X\to Y$ is L-compact on any open-bounded subset of X. If $0\in\Omega\subset X$ is an open-bounded subset and $Lx\neq\lambda Nx$ for all $x\in D(L)\cap\partial\Omega$ and $\lambda\in[0,1]$, then there exists at least one $x\in\Omega$ such that $Lx=Nx$.*

Lemma 3.3. *Suppose $\beta<0$ and $\alpha\geq 0$ and (A_1), (A_2), (A_3), and (C) hold. Let $\Omega_1=\{x\in D(L): Lx=\lambda Nx,\ \exists\lambda\in(0,1)\}$. Then, Ω_1 is bounded if (2.11) holds.*

Proof. Suppose $x\in\Omega_1$, then

$$\begin{gathered} x''(t)+\alpha x'(t)+\beta x(t)=\lambda f(t,x(t),x(\alpha_1(t)),\ldots,x(\alpha_n(t))),\quad \text{a.e. } t\in[0,T],\\ \Delta x(t_k)=\lambda I_k(x(t_k),x'(t_k)),\quad k=1,\ldots,m,\\ \Delta x'(t_k)=\lambda J_k(x(t_k),x'(t_k)),\quad k=1,\ldots,m,\\ x(0)=x(T),\qquad x'(0)=x'(T). \end{gathered} \tag{3.3}$$

Step 1. Prove that there exists a constant $M_1 > 0$ so that $\int_0^T |x(s)|^{q+1} ds \le M_1$ for each $x \in \Omega_1$.

Multiplying both sides of the first equation of (3.3) by $x(t)$, integrating it from 0 to T, we get from (C) that

$$\begin{aligned}
&x'(T)x(T) - x'(0)x(0) - \sum_{k=1}^{m}\left[x'\left(t_k^+\right)x\left(t_k^+\right) - x'(t_k)x(t_k)\right] - \int_0^T \left[x'(s)\right]^2 ds \\
&\quad + \frac{\alpha}{2}\left[(x(T))^2 - (x(0))^2\right] - \frac{\alpha}{2}\sum_{k=1}^{m}\left[\left(x\left(t_k^+\right)\right)^2 - (x(t_k))^2\right] + \beta\int_0^T |x(t)|^2 dt \\
&= \lambda\int_0^T f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)ds \\
&= \lambda\Bigg(\int_0^T h(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds \\
&\qquad + \sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s)))x(s)ds + \int_0^T r(s)x(s)ds\Bigg).
\end{aligned} \tag{3.4}$$

It follows from (A_1) that

$$\begin{aligned}
\sum_{k=1}^{m}\left[\left(x\left(t_k^+\right)\right)^2 - (x(t_k))^2\right] &= \sum_{k=1}^{m}\left(x\left(t_k^+\right) - x(t_k)\right)\left(x\left(t_k^+\right) + x(t_k)\right) \\
&= \sum_{k=1}^{m}\Delta x(t_k)\left(2x(t_k) + \Delta x\left(t_k^-\right)\right) \\
&= \lambda\sum_{k=1}^{m} I_k(x(t_k), x'(t_k))(2x(t_k) + \lambda I_k(x(t_k), x'(t_k))) \\
&\ge 2\lambda\sum_{k=1}^{m} I_k(x(t_k), x'(t_k))x(t_k) \ge 0.
\end{aligned} \tag{3.5}$$

On the other hand, (A_2) implies that

$$\begin{aligned}
\sum_{k=1}^{m}\left(x'\left(t_k^+\right)x\left(t_k^+\right) - x'(t_k)x(t_k)\right) &= \sum_{k=1}^{m}\left[x'\left(t_k^+\right)\left(x\left(t_k^+\right) - x(t_k)\right) + \left(x'\left(t_k^+\right) - x'(t_k)\right)x(t_k)\right] \\
&= \lambda\sum_{k=1}^{m}\left(x'(t_k)I_k(x(t_k), x'(t_k)) + x(t_k)J_k(x(t_k), x'(t_k))\right) \\
&\quad + \lambda^2\sum_{k=1}^{m} I_k(x(t_k), x'(t_k))J_k(x(t_k), x'(t_k)) \\
&\ge 0.
\end{aligned} \tag{3.6}$$

We get

$$\begin{aligned}
&\int_0^T h(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds \\
&\quad + \sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s)))x(s)ds + \int_0^T r(s)x(s)ds \le 0.
\end{aligned} \tag{3.7}$$

It follows from (C) that

$$\begin{aligned}\theta\int_0^T|x(s)|^{q+1}ds &\le -\int_0^T g_0(s,x(s))x(s)ds-\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s)))x(s)ds\\ &\quad+\int_0^T r(s)x(s)ds+\beta\int_0^T|x(t)|^2dt\\ &\le \int_0^T|g_0(s,x(s))||x(s)|ds+\sum_{i=1}^n\int_0^T|g_i(s,x(\alpha_i(s)))||x(s)|ds\\ &\quad+\int_0^T|r(s)||x(s)|ds.\end{aligned}\tag{3.8}$$

Let $\epsilon>0$ satisfy

$$\theta>(r_0+\epsilon)+\sum_{k=1}^n(r_k+\epsilon)\|\beta_k'\|_\infty^{q/(q+1)}.\tag{3.9}$$

For such $\epsilon>0$, there exists $\delta>0$ such that for every $i=0,1,\dots,n$,

$$|g_i(t,x)|<(r_i+\epsilon)|x|^q\quad\text{for a.e. } t\in[0,T] \text{ and all } x \text{ such that } |x|>\delta.\tag{3.10}$$

Let, for $i=1,\dots,n$, $\Delta_{1,i}=\{t:\ t\in[0,T],\ |x(\alpha_i(t))|\le\delta\}$, $\Delta_{2,i}=\{t:\ t\in[0,T],\ |x(\alpha_i(t))|>\delta\}$, $g_{\delta,i}=\max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|$, and $\Delta_1=\{t\in[0,T],\ |x(t)|\le\delta\}$, $\Delta_2=\{t\in[0,T],\ |x(t)|>\delta\}$, and $\delta'=\max\{g_{\delta,k}:k=0,\dots,n\}$. Then, we get

$$\begin{aligned}&\theta\int_0^T|x(s)|^{q+1}ds\\ &\quad=\int_{\Delta_1}|g_0(s,x(s))||x(s)|ds+\int_{\Delta_2}|g_0(s,x(s))||x(s)|ds+\sum_{i=1}^n\int_{\Delta_{1,i}}|g_i(s,x(\alpha_i(s)))||x(s)|ds\\ &\qquad+\sum_{i=1}^n\int_{\Delta_{2,i}}|g_i(s,x(\alpha_i(s)))||x(s)|ds+\int_0^T|r(s)||x(s)|ds\\ &\quad\le(r_0+\epsilon)\int_0^T|x(s)|^{q+1}ds+\sum_{k=1}^n(r_k+\epsilon)\int_0^T|x(\alpha_k(s))|^q|x(s)|ds\\ &\qquad+\int_0^T|r(s)||x(s)|ds+g_{\delta,0}\int_0^T|x(s)|ds+\sum_{k=1}^n g_{\delta,k}\int_0^T|x(s)|ds\\ &\quad\le(r_0+\epsilon)\int_0^T|x(s)|^{q+1}ds\\ &\qquad+\sum_{k=1}^n(r_k+\epsilon)\left(\int_0^T|x(\alpha_k(s))|^{q+1}ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}ds\right)^{1/(q+1)}\\ &\qquad+\left(\int_0^T|r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}ds\right)^{1/(q+1)}+(n+1)\delta'\int_0^T|x(s)|ds\\ &\quad=(r_0+\epsilon)\int_0^T|x(s)|^{q+1}ds+\sum_{k=1}^n(r_k+\epsilon)\left|\int_{\alpha_k(0)}^{\alpha_k(T)}|x(u)|^{q+1}|\beta_k'(u)|du\right|^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}ds\right)^{1/(q+1)}\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\
&+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\
\le{}& (r_0+\epsilon)\int_0^T |x(s)|^{q+1}ds\\
&+\sum_{k=1}^{n}(r_k+\epsilon)\|\beta_k'\|^{q/(q+1)}\left(\int_0^T |x(u)|^{1+q}du\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\
&+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\
&+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\
={}&\left((r_0+\epsilon)+\sum_{k=1}^{n}(r_k+\epsilon)\|\beta_k'\|^{q/(q+1)}\right)\int_0^T |x(s)|^{q+1}ds\\
&+\left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\\
&+(n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}.
\end{aligned}
\tag{3.11}
$$

Then, (3.9) implies that there exists a constant $M_1 > 0$ such that $\int_0^T |x(s)|^{q+1}ds \le M_1$.

Step 2. Prove that there exists a constant $M_2 > 0$ such that $\|x\|_\infty \le M_2$ for each $x \in \Omega_1$.

It follows from Step 1 that there exists $\xi \in [0,T]$ such that $|x(\xi)| \le (M_1/T)^{1/(q+1)}$.

Multiplying both sides of the first equation of (3.3) by $x(t)$, integrating it from 0 to T, we get, using (A_1), (A_2), and (C),

$$
\begin{aligned}
\int_0^T [x'(s)]^2 ds ={}& -\frac{1}{2}\sum_{k=1}^{m}\left[(x(t_k^+))^2-(x(t_k))^2\right]-\sum_{k=1}^{m}\left[x'(t_k^+)x(t_k^+)-x'(t_k)x(t_k)\right]\\
&-\lambda\int_0^T f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)ds+\lambda\beta\int_0^T |x(t)|^2dt\\
\le{}& -\lambda\int_0^T f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)ds+\lambda\beta\int_0^T |x(t)|^2dt\\
\le{}& -\lambda\left(\int_0^T h(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)ds+\int_0^T g_0(s,x(s))x(s)ds\right.\\
&\left.+\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s)))x(s)ds+\int_0^T r(s)x(s)ds\right)+\lambda\beta\int_0^T |x(t)|^2dt
\end{aligned}
$$

$$\begin{aligned}
&\leq -\lambda\int_0^T g_0(s,x(s))x(s)ds - \lambda\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s)))x(s)ds - \lambda\int_0^T r(s)x(s)ds \\
&\leq \int_0^T |g_0(s,x(s))||x(s)|ds + \sum_{i=1}^n\int_0^T |g_i(s,x(\alpha_i(s)))||x(s)|ds + \int_0^T |r(s)||x(s)|ds \\
&\leq \left[\left((r_0+\epsilon) + \sum_{k=1}^n (r_k+\epsilon)\|\beta_k'\|_\infty^{q/(1+q)}\right)\int_0^T |x(s)|^{q+1}ds\right. \\
&\quad\left. + \left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\right] \\
&\quad + (n+1)\delta' T^{q/(q+1)}\left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}.
\end{aligned} \tag{3.12}$$

Then,

$$\begin{aligned}
\int_0^T [x'(s)]^2 ds &\leq \left[\left((r_0+\epsilon) + \sum_{k=1}^n (r_k+\epsilon)\|\beta_k'\|_\infty^{q/(1+q)}\right)M_1 + \left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)} M_1^{1/(q+1)}\right] \\
&\quad + (n+1)\delta' T^{q/(q+1)} M_1^{1/(q+1)} \\
&=: M_2.
\end{aligned} \tag{3.13}$$

Due to (A_3), one sees that

$$\begin{aligned}
|x(t)| &= \begin{cases} \left| x(\xi) + \lambda \sum_{\xi\leq t_k<t} I_k(x(t_k),x'(t_k)) + \int_\xi^t x'(s)ds\right| & \text{if } t\geq \xi, \\ \left| x(\xi) - \lambda \sum_{t\leq t_k<\xi} I_k(x(t_k),x'(t_k)) - \int_t^\xi x'(s)ds\right| & \text{if } t< \xi, \end{cases} \\
&\leq \left(\frac{M_1}{T}\right)^{1/(q+1)} + \sum_{k=1}^m \theta_k\|x\| + \int_0^T |x'(s)|ds \\
&\leq \left(\frac{M_1}{T}\right)^{1/(q+1)} + \sum_{k=1}^m \theta_k\|x\| + T^{1/2}\left(\int_0^T |x'(s)|^2ds\right)^{1/2} \\
&\leq \left(\frac{M_1}{T}\right)^{1/(q+1)} + \sum_{k=1}^m \theta_k\|x\| + T^{1/2}M_2^{1/2}.
\end{aligned} \tag{3.14}$$

It follows from (A_3) that

$$\|x\|_\infty \leq \frac{1}{1-\sum_{k=1}^m \theta_k}\left(\left(\frac{M_1}{T}\right)^{1/(q+1)} + T^{1/2}M_2^{1/2}\right). \tag{3.15}$$

It follows that Ω_1 is bounded. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Suppose $\beta < 0$ and $\alpha > 0$ and (A_1'), (A_2), (A_3), and (C) hold. Let $\Omega_1 = \{x \in D(L) : Lx = \lambda Nx, \exists\lambda \in (0,1)\}$. Then, Ω_1 is bounded if* (2.11) *holds.*

Proof. The proof is similar to that of Lemma 3.3. □

Lemma 3.5. *Suppose $\beta > 0$ and $\alpha > 0$ and (A_1'), (A_2'), (A_3), and (C') hold. Let $\Omega_1 = \{x \in D(L) : Lx = \lambda Nx, \exists \lambda \in (0,1)\}$. Then, Ω_1 is bounded if (2.11) holds.*

Proof. Suppose $x \in \Omega_1$, then we get (3.3).

Step 1. Prove that there exists a constant $M_1 > 0$ so that $\int_0^T |x(s)|^{q+1} ds \le M_1$ for each $x \in \Omega_1$.

Multiplying both sides of the first equation of (3.3) by $x(t)$, integrating it from 0 to T, we get from (C')

$$\begin{aligned} x'(T)x(T) - x'(0)x(0) &- \sum_{k=1}^{m}\left[x'(t_k^+)x(t_k^+) - x'(t_k)x(t_k)\right] - \int_0^T \left[x'(s)\right]^2 ds \\ &+ \frac{\alpha}{2}\left[(x(T))^2 - (x(0))^2\right] - \frac{\alpha}{2}\sum_{k=1}^{m}\left[(x(t_k^+))^2 - (x(t_k))^2\right] + \beta\int_0^T |x(t)|^2 dt \\ &= \lambda\left(\int_0^T h(s, x(s), x(\alpha_1(s)), \dots, x(\alpha_n(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds \right. \\ &\left. \quad + \sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T r(s)x(s)ds\right). \end{aligned} \tag{3.16}$$

It follows from (A_1') that

$$\begin{aligned} \sum_{k=1}^{m}\left[(x(t_k^+))^2 - (x(t_k))^2\right] &= \sum_{k=1}^{m}(x(t_k^+) - x(t_k))(x(t_k^+) + x(t_k)) \\ &= \sum_{k=1}^{m}\Delta x(t_k)(2x(t_k) + \Delta x(t_k^-)) \\ &= \lambda\sum_{k=1}^{m} I_k(x(t_k), x'(t_k))(2x(t_k) + \lambda I_k(x(t_k), x'(t_k))) \\ &\le \lambda\sum_{k=1}^{m} I_k(x(t_k), x'(t_k))(2x(t_k) + I_k(x(t_k), x'(t_k))) \\ &\le 0. \end{aligned} \tag{3.17}$$

On the other hand, (A_2') implies that

$$\begin{aligned} \sum_{k=1}^{m}(x'(t_k^+)x(t_k^+) - x'(t_k)x(t_k)) &= \sum_{k=1}^{m}\left[x'(t_k^+)(x(t_k^+) - x(t_k)) + (x'(t_k^+) - x'(t_k))x(t_k))\right] \\ &= \lambda\sum_{k=1}^{m}(x'(t_k)I_k(x(t_k), x'(t_k)) + x(t_k)J_k(x(t_k), x'(t_k))) \\ &\quad + \lambda^2\sum_{k=1}^{m} I_k(x(t_k), x'(t_k))J_k(x(t_k), x'(t_k)) \\ &\le 0. \end{aligned} \tag{3.18}$$

We get

$$\lambda\left(\int_0^T h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)ds+\int_0^T g_0(s,x(s))x(s)ds\right.$$
$$\left.+\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T r(s)x(s)ds\right)\geq 0. \tag{3.19}$$

It follows from (C') that

$$\begin{aligned}\theta\int_0^T |x(s)|^{q+1}ds &\leq \int_0^T g_0(s,x(s))x(s)ds-\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s))x(s)ds\\ &\quad+\int_0^T r(s)x(s)ds+\beta\int_0^T |x(t)|^2dt\\ &\leq \int_0^T |g_0(s,x(s))||x(s)|ds+\sum_{i=1}^n\int_0^T |g_i(s,x(\alpha_i(s))||x(s)|ds\\ &\quad+\int_0^T |r(s)||x(s)|ds.\end{aligned} \tag{3.20}$$

The remainder of the proof is similar to that of the corresponding part of the proof of Lemma 3.3. □

Lemma 3.6. *Suppose $\beta > 0$ and $\alpha < 0$ and (A_1), (A_2), (A_3), and (C) hold. Let $\Omega_1 = \{x \in D(L) : Lx = \lambda Nx, \exists\lambda \in (0,1)\}$. Then, Ω_1 is bounded if (2.11) holds.*

Proof. The proof is similar to that of Lemma 3.5. □

Proof of Theorem 2.1. It is easy to show that $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L = \{0\}$, $N : X \to Y$ is L-compact on any open-bounded subset of X. Let Ω satisfy $0 \in \overline{\Omega_1} \subset \Omega \subset X$ which is a open-bounded subset, where Ω_1 is given in Lemma 3.3. It follows from Lemma 3.3 that $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0,1]$, then there exists at least one $x \in \Omega$ such that $Lx = Nx$. Hence, x is a solution of PBVP (1.13). □

Proof of Theorem 2.2. Following Lemma 3.4, the proof is similar to that of Theorem 2.1. □

Proof of Theorem 2.3. Following Lemma 3.5, the proof is similar to that of Theorem 2.1. □

Proof of Theorem 2.4. Following Lemma 3.6, the proof is similar to that of Theorem 2.1. □

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