Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 147308, 19 pages
doi:10.1155/2009/147308

Research Article

On Multiple Solutions of Concave and Convex Nonlinearities in Elliptic Equation on $\mathbb{R}^N$

Kuan-Ju Chen

Department of Applied Science, Naval Academy, 90175 Zuoying, Taiwan

Correspondence should be addressed to Kuan-Ju Chen, kuanju@mail.cna.edu.tw

Received 18 February 2009; Accepted 28 May 2009

Recommended by Martin Schechter

We consider the existence of multiple solutions of the elliptic equation on $\mathbb{R}^N$ with concave and convex nonlinearities.

1. Introduction

First, we look for positive solutions of the following problem:

$$\begin{gathered}-\Delta u + u = a(x)u^{p-1} + \lambda b(x)u^{q-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right),\end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a real parameter, $1 < p < 2 < q < 2^* = 2N/N-2$, $N \geq 3$. We will impose some assumptions on $a(x)$ and $b(x)$. Assume

(a1) $a(x) \geq 0, a(x) \in L^{\alpha/(\alpha-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $1 < \alpha < 2^*/p$,

(b1) $b(x) \in C(\mathbb{R}^N), b(x) \to b^\infty > 0$ as $|x| \to \infty, b(x) \geq b^\infty$ for all $x \in \mathbb{R}^N$,

Such problems occur in various branches of mathematical physics and population dynamics, and sublinear analogues or superlinear analogues of problem (1.1) have been considered by many authors in recent years (see [1–4]). Little information is known about the combination of sublinear and superlinear case of problem (1.1). In [5, 6], they deal with the analogue of problem (1.1) when $\mathbb{R}^N$ is replaced by a bounded domain Ω. For the $\mathbb{R}^N$ case, the existence of positive solutions for problem (1.1) was proved by few people.

In the present paper, we discuss the Nehari manifold and examine carefully the connection between the Nehari manifold and the fibrering maps, then using arguments similar to those used in [7], we will prove the existence of the two positive solutions by using Ekeland's Variational Principle [8].

In [5], Ambrosetti et al. showed that for $\lambda > 0$ small with respect to $\mu > 0$ there exist infinitely many solutions $u \in H_0^1(\Omega)$ of the semilinear elliptic problem:

$$\begin{aligned} -\Delta u &= \lambda |u|^{p-2}u + \mu |u|^{q-2}u, \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

with negative energy:

$$\psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{p}\int_\Omega |u|^p - \frac{\mu}{q}\int_\Omega |u|^q, \tag{1.3}$$

and infinitely many solutions with positive energy, where $\Omega \subset \mathbb{R}^N$ is an open bounded domain. In [9], Bartsch and Willem obtained infinitely many solutions of problem (1.2) with negative energy for every $\lambda > 0$. For the $\mathbb{R}^N$ case, the existence of multiple solutions was proved by few people.

Finally we propose herein a result similar to [9] or [10] for the existence of infinitely many solutions (possibly not positive) of

$$\begin{aligned} -\Delta u + u &= \mu a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \\ u &\in H^1\left(\mathbb{R}^N\right), \end{aligned} \tag{1.4}$$

by taking advantage of the oddness of the nonlinearity.

Our main results state the following.

Theorem 1.1. *Under the assumptions (a1) and (b1), there exists $\lambda^* > 0$, such that for all $\lambda \in (0, \lambda^*)$, problem* (1.1) *has at least two positive solutions u_0 and u_1, u_0 is a local minimizer of I_λ and $I_\lambda(u_0) < 0$, where I_λ is the energy functional of problem* (1.1).

Theorem 1.2. *Under the assumptions (a1) and (b1), for every $\lambda > 0$ and $\mu \in \mathbb{R}$, the problem* (1.4) *has infinitely many solutions with positive energy and for every $\mu > 0$ and $\lambda \in \mathbb{R}$, infinitely many solutions with negative energy.*

2. The Existence of Two Positive Solutions

The variational functional of problem (1.1) is

$$I_\lambda(u) = \frac{1}{2}\int \left(|\nabla u|^2 + u^2\right) - \frac{1}{p}\int a(x)|u|^p - \frac{\lambda}{q}\int b(x)|u|^q, \tag{2.1}$$

here and from now on, we omit "dx" and "$\mathbb{R}^N$" in all the integrations if there is no other indication.

Through this paper, we denote the universal positive constant by C unless some special statement is given. Let $\langle\cdot,\cdot\rangle$ denote the usual scalar product in $H^1(\mathbb{R}^N)$. Easy computations show that I_λ is bounded from below on the Nehari manifold,

$$\Lambda_\lambda = \left\{u \in H^1\left(\mathbb{R}^N\right) : \langle I'_\lambda(u), u\rangle = 0\right\}. \tag{2.2}$$

Thus $u \in \Lambda_\lambda$ if and only if

$$\|u\|^2 - \int a(x)|u|^p - \lambda \int b(x)|u|^q = 0. \tag{2.3}$$

In particular, on Λ_λ, we have

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{p}\right)\int b(x)|u|^q \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 - \left(\frac{1}{p} - \frac{1}{q}\right)\int a(x)|u|^p. \end{aligned} \tag{2.4}$$

The Nehari manifold is closely linked to the behavior of the functions of the form $\phi_u : t \to I_\lambda(tu)$ $(t > 0)$. Such maps are known as fibrering maps and were introduced by Drábek and Pohozaev in [11] and are discussed by Brown and Zhang [12]. If $u \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} \phi_u(t) &= \frac{t^2}{2}\|u\|^2 - \frac{t^p}{p}\int a(x)|u|^p - \lambda\frac{t^q}{q}\int b(x)|u|^q, \\ \phi'_u(t) &= t\|u\|^2 - t^{p-1}\int a(x)|u|^p - \lambda t^{q-1}\int b(x)|u|^q, \\ \phi''_u(t) &= \|u\|^2 - (p-1)t^{p-2}\int a(x)|u|^p - \lambda(q-1)t^{q-2}\int b(x)|u|^q. \end{aligned} \tag{2.5}$$

Similarly to the method used in [7], we split Λ_λ into three parts corresponding to local minima, local maxima, and points of inflection, and so we define

$$\begin{aligned} \Lambda_\lambda^+ &= \left\{u \in \Lambda_\lambda : \phi''_u(1) > 0\right\}, \\ \Lambda_\lambda^- &= \left\{u \in \Lambda_\lambda : \phi''_u(1) < 0\right\}, \\ \Lambda_\lambda^0 &= \left\{u \in \Lambda_\lambda : \phi''_u(1) = 0\right\}, \end{aligned} \tag{2.6}$$

and note that if $u \in \Lambda_\lambda$, that is, $\phi'_u(1) = 0$, then

$$\begin{aligned} \phi''_u(1) &= (2-p)\|u\|^2 - \lambda(q-p)\int b(x)|u|^q \\ &= (2-q)\|u\|^2 - (p-q)\int a(x)|u|^p. \end{aligned} \tag{2.7}$$

This section will be devoted to prove Theorem 1.2. To prove Theorem 1.2, several preliminary results are in order.

Lemma 2.1. *Under the assumptions (a1), (b1), there exists $\lambda^* > 0$ such that when $0 < \lambda < \lambda^*$, for every $u \in H^1(\mathbb{R}^N)$, $u \not\equiv 0$, there exist unique $t^+ = t^+(u) > 0$, $t^- = t^-(u) > 0$ such that $t^+u \in \Lambda_\lambda^-$, $t^-u \in \Lambda_\lambda^+$. In particular, one has*

$$t^+ > \left(\frac{(2-q)\|u\|^2}{(p-q)\int a(x)|u|^p}\right)^{1/(p-2)} = t_{\max} > t^-, \tag{2.8}$$

$I_\lambda(t^-u) = \min_{t\in[0,t^+]} I_\lambda(tu) < 0$ *and* $I_\lambda(t^+u) = \max_{t\geq t^-} I_\lambda(tu)$.

Proof. Given $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, set $\varphi_u(t) = t^{2-q}\|u\|^2 - t^{p-q}\int a(x)|u|^p$. Clearly, for $t > 0$, $tu \in \Lambda_\lambda$ if and only if t is a solution of

$$\varphi_u(t) = \lambda \int b(x)|u|^q. \tag{2.9}$$

Moreover,

$$\varphi_u'(t) = (2-q)t^{1-q}\|u\|^2 - (p-q)t^{p-q-1}\int a(x)|u|^p, \tag{2.10}$$

easy computations show that φ_u is concave and achieves its maximum at

$$t_{\max} = \left(\frac{(2-q)\|u\|^2}{(p-q)\int a(x)|u|^p}\right)^{1/(p-2)}. \tag{2.11}$$

If $\lambda > 0$ is sufficiently large, (2.9) has no solution, and so $\phi_u(t) = I_\lambda(tu)$ has no critical points, in this case ϕ_u is a decreasing function, hence no multiple of u lies in Λ_λ.

If, on the other hand, $\lambda > 0$ is sufficiently small, then there exist exactly two solutions $t^+(u) > t^-(u) > 0$ of (2.9), where $t^+ = t^+(u)$, $t^- = t^-(u)$, $\varphi_u'(t^-) > 0$, and $\varphi_u'(t^+) < 0$.

It follows from (2.7) and (2.10) that $\phi_{tu}''(1) = t^{q+1}\varphi_u'(t)$, and so $t^+u \in \Lambda_\lambda^-$, $t^-u \in \Lambda_\lambda^+$; moreover ϕ_u is decreasing in $(0, t^-)$, increasing in (t^-, t^+), and decreasing in (t^+, ∞).

Next, we will discussion the sufficiently small λ^*, such that when $0 < \lambda < \lambda^*$, there exist exactly two solutions of problem (2.9) for all $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, that is,

$$\lambda \int b(x)|u|^q < \varphi_u(t_{\max}) = \left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} \left(\frac{p-2}{p-q}\right) \frac{\|u\|^{(2p-2q)/(p-2)}}{\left(\int a(x)|u|^p\right)^{(2-q)/(p-2)}}. \tag{2.12}$$

Since

$$\int a(x)|u|^p \le \|a\|_{L^{\alpha/(\alpha-1)}} \|u\|_{L^{\alpha p}}^p \le \|a\|_{L^{\alpha/(\alpha-1)}} S_{\alpha p}^p \|u\|^p, \tag{2.13}$$

where $S_{\alpha p}$ denotes the Sobolev constant of the embedding of $H^1(\mathbb{R}^N)$ into $L^{\alpha p}(\mathbb{R}^N)$, hence,

$$\begin{aligned}\varphi_u(t_{\max}) &\ge \left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} \left(\frac{p-2}{p-q}\right) \frac{\|u\|^{(2p-2q)/(p-2)}}{\left(\|a\|_{L^{\alpha/(\alpha-1)}} S_{\alpha p}^p \|u\|^p\right)^{(2-q)/(p-2)}} \\ &= \left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} \left(\frac{p-2}{p-q}\right) \frac{\|u\|^q}{\left(\|a\|_{L^{\alpha/(\alpha-1)}} S_{\alpha p}^p\right)^{(2-q)/(p-2)}},\end{aligned} \tag{2.14}$$

and then

$$\begin{aligned}\int b(x)|u|^q &\le M\|u\|_{L^q}^q \le M S_q^q \|u\|^q \\ &\le M S_q^q \left(\frac{p-q}{2-q}\right)^{(2-q)/(p-2)} \left(\frac{p-q}{p-2}\right) \left(\|a\|_{L^{\alpha/(\alpha-1)}} S_{\alpha p}^p\right)^{(2-q)/(p-2)} \varphi_u(t_{\max}) \\ &= c\varphi_u(t_{\max}),\end{aligned} \tag{2.15}$$

where S_q denotes the Sobolev constant of the embedding of $H^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$, c is independent of u, hence

$$\varphi_u(t_{\max}) - \lambda \int b(x)|u|^q \ge \varphi_u(t_{\max}) - \lambda c \varphi_u(t_{\max}) = \varphi_u(t_{\max})(1 - \lambda c), \tag{2.16}$$

and so $\lambda \int b(x)|u|^q < \varphi_u(t_{\max})$ for all $u \in H^1(R^N) \setminus \{0\}$ provided $\lambda < 1/2c = \lambda^*$.

Hence when $0 < \lambda < \lambda^*$, ϕ_u has exactly two critical points—a local minimum at $t^- = t^-(u)$ and a local maximum at $t^+ = t^+(u)$; moreover $I_\lambda(t^-u) = \min_{t\in[0,t^+]} I_\lambda(tu) < 0$ and $I_\lambda(t^+u) = \max_{t\ge t^-} I_\lambda(tu)$.

In particular, we have the following result. □

Corollary 2.2. *Under the assumptions (a1), (b1), when $0 < \lambda < \lambda^*$, for every $u \in \Lambda_\lambda$, $u \not\equiv 0$, one has*

$$(2-q)\|u\|^2 - (p-q) \int a(x)|u|^p \not\equiv 0 \tag{2.17}$$

(i.e., $\Lambda_\lambda^0 = \emptyset$).

Proof. Let us argue by contradiction and assume that there exists $u \in \Lambda_\lambda \setminus \{0\}$ such that $(2-q)\|u\|^2 - (p-q)\int a(x)|u|^p = 0$, this implies

$$
\begin{aligned}
\lambda \int b(x)|u|^q &= \|u\|^2 - \int a(x)|u|^p \\
&= \left(\frac{p-2}{2-q}\right)\int a(x)|u|^p \\
&= \left(\frac{p-2}{2-q}\right)\left(\int a(x)|u|^p\right)^{(p-q)/(p-2)}\left(\int a(x)|u|^p\right)^{(q-2)/(p-2)} \\
&= \left(\frac{p-2}{2-q}\right)\left(\frac{1}{p-q}\right)^{(p-q)/(p-2)}\left((p-q)\int a(x)|u|^p\right)^{(p-q)/(p-2)}\left(\int a(x)|u|^p\right)^{(q-2)/(p-2)} \\
&= \left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)}\|u\|^{2(p-q)/(p-2)}\left(\int a(x)|u|^p\right)^{(q-2)/(p-2)} \\
&= \varphi_u\ (t_{\max})
\end{aligned}
\tag{2.18}
$$

which contradicts (2.12) for $0<\lambda<\lambda^*$. □

As a consequence of Corollary 2.2, we have the following lemma.

Lemma 2.3. *Under the assumptions (a1), (b1), if $0<\lambda<\lambda^*$, for every $u \in \Lambda_\lambda$, $u \not\equiv 0$, then there exist a $\epsilon > 0$ and a C^1-map $t = t(w) > 0$, $w \in H^1(\mathbb{R}^N)$, $\|w\| < \epsilon$ satisfying that*

$$
\begin{gathered}
t(0) = 1, \qquad t(w)(u-w) \in \Lambda_\lambda, \quad \text{for } \|w\| < \epsilon, \\
\langle t'(0), w\rangle = \frac{2\int(\nabla u \nabla w + uw) - p\int a(x)|u|^{p-2}uw - \lambda q\int b(x)|u|^{q-2}uw}{(2-q)\|u\|^2 - (p-q)\int a(x)|u|^p}.
\end{gathered}
\tag{2.19}
$$

Proof. We define $F : \mathbb{R}\times H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$
F(t,w) = t\|u-w\|^2 - t^{p-1}\int a(x)|u-w|^p - \lambda t^{q-1}\int b(x)|u-w|^q. \tag{2.20}
$$

Since $F(1,0) = 0$ and $F_t(1,0) = \|u\|^2 - (p-1)\int a(x)|u|^p - \lambda(q-1)\int b(x)|u|^q = (2-q)\|u\|^2 - (p-q)\int a(x)|u|^p \not\equiv 0$ (by Corollary 2.2), we can apply the implicit function theorem at the point $(1,0)$ and get the result. □

Apply Lemma 2.1, Corollary 2.2, Lemma 2.3, and Ekeland variational principle [8], we can establish the existence of the first positive solution.

Proposition 2.4. *If $0<\lambda<\lambda^*$, then the minimization problem:*

$$
c_0 = \inf_{\Lambda_\lambda} I_\lambda = \inf_{\Lambda_\lambda^+} I_\lambda \tag{2.21}
$$

is achieved at a point $u_0 \in \Lambda_\lambda^+$ which is a critical point for I_λ with $u_0 > 0$ and $I_\lambda(u_0) < 0$. Furthermore, u_0 is a local minimizer of I_λ.

Proof. First, we show that I_λ is bounded from below in Λ_λ. Indeed, for $u \in \Lambda_\lambda$, from (2.13), we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p}\int a(x)|u|^p - \frac{\lambda}{q}\int b(x)|u|^q \\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \left(\frac{1}{p}-\frac{1}{q}\right)\int a(x)|u|^p \\ &\geq \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \left(\frac{1}{p}-\frac{1}{q}\right)\|a\|_{L^{\alpha/(\alpha-1)}} S_{\alpha p}^p \|u\|^p \end{aligned} \tag{2.22}$$

and so I_λ is bounded from below in Λ_λ.

Then we will claim that $c_0 < 0$, indeed if $v \in H^1(\mathbb{R}^N)\setminus\{0\}$, from Lemma 2.1, there exist $0 < t^-(v) < t^+(v)$ such that $t^-(v)v \in \Lambda_\lambda$. Thus,

$$c_0 \leq I_\lambda\left(t^-(v)v\right) = \min_{t\in[0,t^+(v)]} I_\lambda(tv) < 0. \tag{2.23}$$

By Ekeland's Variational Principle [8], there exists a minimizing sequence $\{u_n\} \subset \Lambda_\lambda$ of the minimization problem (2.21) such that

$$c_0 \leq I_\lambda(u_n) < c_0 + \frac{1}{n}, \tag{2.24}$$

$$I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}\|v - u_n\|, \quad \forall\, v \in \Lambda_\lambda. \tag{2.25}$$

Taking n large enough, from (2.7) we have

$$I_\lambda(u_n) = \left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|^2 - \left(\frac{1}{p}-\frac{1}{q}\right)\int a(x)|u_n|^p < c_0 + \frac{1}{n} < 0, \tag{2.26}$$

from which we deduce that for n large

$$\int a(x)|u_n|^p \geq \frac{pq}{p-q}c_0, \qquad \|u_n\|^2 \leq \frac{2(q-p)}{p(q-2)}\int a(x)|u_n|^p, \tag{2.27}$$

which yields

$$b_1 \leq \|u_n\| \leq b_2 \tag{2.28}$$

for suitable $b_1, b_2 > 0$.

Now we will show that

$$\|I'_\lambda(u_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.29}$$

Since $u_n \in \Lambda_\lambda$, by Lemma 2.3, we can find a $\epsilon_n > 0$ and a C^1-map $t_n = t_n(w) > 0$, $w \in H^1(\mathbb{R}^N)$, $\|w\| < \epsilon_n$ satisfying that

$$v_n = t_n(w)(u_n - w) \in \Lambda_\lambda, \quad \text{for } \|w\| < \epsilon_n. \tag{2.30}$$

By the continuity of $t_n(w)$ and $t_n(0) = 1$, without loss of generality, we can assume ϵ_n satisfies that $1/2 \le t_n(w) \le 3/2$ for $\|w\| < \epsilon_n$.

It follows from (2.25) that

$$I_\lambda(t_n(w)(u_n - w)) - I_\lambda(u_n) \ge -\frac{1}{n}\|t_n(w)(u_n - w) - u_n\|; \tag{2.31}$$

that is,

$$\begin{aligned} &\langle I'_\lambda(u_n), t_n(w)(u_n - w) - u_n\rangle + o(\|t_n(w)(u_n - w) - u_n\|) \\ &\quad \ge -\frac{1}{n}\|t_n(w)(u_n - w) - u_n\|. \end{aligned} \tag{2.32}$$

Consequently,

$$\begin{aligned} &t_n(w)\langle I'_\lambda(u_n), w\rangle + (1 - t_n(w))\langle I'_\lambda(u_n), u_n\rangle \\ &\quad \le \frac{1}{n}\|(t_n(w) - 1)u_n - t_n(w)w\| + o(\|t_n(w)(u_n - w) - u_n\|). \end{aligned} \tag{2.33}$$

By the choice of ϵ_n, we obtain

$$\begin{aligned} \langle I'_\lambda(u_n), w\rangle \le{}& \frac{C}{n}\left|\langle t'_n(0), w\rangle\right| + o(\|w\|) + \frac{C}{n}\|w\| \\ &+ o\left(\left|\langle t'_n(0), w\rangle\right|(\|u_n\| + \|w\|)\right). \end{aligned} \tag{2.34}$$

By Lemma 2.3, Corollary 2.2, and the estimate (2.28), we have

$$\begin{aligned} \langle t'_n(0), w\rangle &= \frac{2\int(\nabla u_n \nabla w + u_n w) - p\int a(x)|u_n|^{p-2}u_n w - \lambda q\int b(x)|u_n|^{q-2}u_n w}{(2-q)\|u_n\|^2 - (p-q)\int a(x)|u_n|^p} \\ &\le C\|w\|, \end{aligned} \tag{2.35}$$

then from (2.34) we get

$$\langle I'_\lambda(u_n), w\rangle \le \frac{C}{n}\|w\| + \frac{C}{n}\|w\| + o(\|w\|), \quad \text{for } \|w\| \le \epsilon_n. \tag{2.36}$$

Hence, for any $\epsilon \in (0, \epsilon_n)$, we have

$$\left\| I'_\lambda(u_n) \right\| = \frac{1}{\epsilon} \sup_{\|w\|=\epsilon} \left\langle I'_\lambda(u_n), w \right\rangle \leq \frac{C}{n} + \frac{1}{\epsilon} o(\epsilon), \tag{2.37}$$

for some $C > 0$ independent of ϵ and n. Taking $\epsilon \to 0$, we obtain (2.29).

Let $u_0 \in H^1(\mathbb{R}^N)$ be the weak limit in $H^1(\mathbb{R}^N)$ of u_n. From (2.29),

$$\left\langle I'_\lambda(u_0), w \right\rangle = 0, \quad \forall w \in H^1\left(\mathbb{R}^N\right); \tag{2.38}$$

that is, u_0 is a weak solution of problem (1.1) and consequently $u_0 \in \Lambda_\lambda$. Therefore,

$$c_0 \leq I_\lambda(u_0) \leq \lim_{n \to \infty} I_\lambda(u_n) = c_0; \tag{2.39}$$

that is,

$$c_0 = I_\lambda(u_0) = \inf_{\Lambda_\lambda} I_\lambda. \tag{2.40}$$

Moreover, we have $u_0 \in \Lambda_\lambda^+$. In fact, if $u_0 \in \Lambda_\lambda^-$, by Lemma 2.1, there exists only one $t^+ > 0$ such that $t^+ u_0 \in \Lambda_\lambda^-$, we have $t^+ = t^+(u_0) = 1$, $t^- = t^-(u_0) < 1$. Since

$$\frac{dI_\lambda(t^- u_0)}{dt} = 0, \qquad \frac{d^2 I_\lambda(t^- u_0)}{dt^2} > 0, \tag{2.41}$$

there exists $t^+ \geq \bar{t} > t^-$ such that $I_\lambda(\bar{t} u_0) > I_\lambda(t^- u_0)$. By Lemma 2.1,

$$I_\lambda(t^- u_0) < I_\lambda\left(\bar{t} u_0\right) \leq I_\lambda(t^+ u_0) = I_\lambda(u_0); \tag{2.42}$$

this is a contradiction.

To conclude that u_0 is a local minimizer of I_λ, notice that for every $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we have from Lemma 2.1,

$$I_\lambda(su) \geq I_\lambda(t^- u) \quad \forall 0 < s < \left(\frac{(2-q)\|u\|^2}{(p-q) \int a(x)|u|^p} \right)^{1/(p-2)}. \tag{2.43}$$

In particular, for $u = u_0 \in \Lambda_\lambda^+$, we have

$$t^-(u_0) = 1 < \left(\frac{(2-q)\|u_0\|^2}{(p-q) \int a(x)|u_0|^p} \right)^{1/(p-2)}. \tag{2.44}$$

Let $\epsilon > 0$ sufficiently small to have

$$1 < \left(\frac{(2-q)\|u_0 - w\|^2}{(p-q)\int a(x)|u_0 - w|^p} \right)^{1/(p-2)}, \quad \text{for } \|w\| < \epsilon. \tag{2.45}$$

From Lemma 2.3, let $t(w) > 0$ satisfy $t(w)(u_0 - w) \in \Lambda_\lambda$ for every $\|w\| < \epsilon$. By the continuity of $t(w)$ and $t(0) = 1$, we can always assume that

$$t(w) < \left(\frac{(2-q)\|u_0 - w\|^2}{(p-q)\int a(x)|u_0 - w|^p} \right)^{1/(p-2)}, \quad \text{for } \|w\| < \epsilon. \tag{2.46}$$

Namely, $t(w)(u_0 - w) \in \Lambda_\lambda^+$ and for

$$0 < s < \left(\frac{(2-q)\|u_0 - w\|^2}{(p-q)\int a(x)|u_0 - w|^p} \right)^{1/(p-2)}, \tag{2.47}$$

we have

$$I_\lambda(s(u_0 - w)) \geq I_\lambda(t(w)(u_0 - w)) \geq I_\lambda(u_0). \tag{2.48}$$

Taking $s = 1$, we conclude

$$I_\lambda(u_0 - w) \geq I_\lambda(t(w)(u_0 - w)) \geq I_\lambda(u_0), \quad \text{for } \|w\| < \epsilon, \tag{2.49}$$

which means that u_0 is a local minimizer of I_λ.

Furthermore, taking $t^-(|u_0|) > 0$ with $t^-(|u_0|)|u_0| \in \Lambda_\lambda^+$, therefore,

$$I_\lambda(u_0) \leq I_\lambda\left(t^-(|u_0|)|u_0|\right) \leq I_\lambda(|u_0|) \leq I_\lambda(u_0). \tag{2.50}$$

So we can always take $u_0 \geq 0$. By the maximum principle for weak solutions (see [13]), we can show that $u_0 > 0$ in $\mathbb{R}^N$.

Since $u_0 \in \Lambda_\lambda^+$ and $c_0 = \inf_{\Lambda_\lambda} I_\lambda = \inf_{\Lambda_\lambda^+} I_\lambda$, thus, in the search of our second positive solution, it is natural to consider the second minimization problem:

$$c_1 = \inf_{\Lambda_\lambda^-} I_\lambda. \tag{2.51}$$

Let us now introduce the problem at infinity associated with (1.1):

$$\begin{gathered} -\Delta u + u = \lambda b^\infty u^{q-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right). \end{gathered} \tag{2.52}$$

We state here some known results for problem (2.52). First of all, we recall that Lions [14] has studied the following minimization problem closely related to problem (2.52): $S_\lambda^\infty = \inf\{I_\lambda^\infty(u) : u \in H^1(\mathbb{R}^N),\ u \neq 0\ ,\ I_\lambda^{\infty\prime}(u) = 0\} > 0$, where $I_\lambda^\infty(u) = (1/2)\|u\|^2 - (1/q)\lambda b^\infty \int |u|^q$. For future reference, note also that a minimum exists and is realized by a ground state $\omega > 0$ in $\mathbb{R}^N$ such that $S_\lambda^\infty = I_\lambda^\infty(\omega) = \sup_{s\geq 0} I_\lambda^\infty(s\omega)$. Gidas et al. [15] showed that there exist a_1,$a_2 > 0$ such that for all $x \in \mathbb{R}^N$,

$$a_1(|x|+1)^{-(N-1)/2}e^{-|x|} \leq \omega(x) \leq a_2(|x|+1)^{-(N-1)/2}e^{-|x|}. \tag{2.53}$$

□

Lemma 2.5. *Let $a(x) \in L^{\alpha/(\alpha-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $1 < \alpha < 2^*/p$ and $1 < p < 2$. If $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, satisfies*

$$\lim_{n\to\infty} \int a(x)|u_n - u|^p = 0. \tag{2.54}$$

Proof. Since $a(x) \in L^{\alpha/(\alpha-1)}(\mathbb{R}^N)$, then for every $\epsilon > 0$, there exists $R_0 > 0$ such that

$$\left(\int_{|x|>R_0} |a(x)|^{\alpha/(\alpha-1)}dx\right)^{(\alpha-1)/\alpha} < \epsilon. \tag{2.55}$$

Since $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^s_{loc}(\mathbb{R}^N)$, $1 \leq s < 2N/N-2$, then we have

$$\left(\int_{|x|\leq R_0} |u_n - u|^{\alpha p}dx\right)^{1/\alpha p} < \epsilon. \tag{2.56}$$

Observe that by Hölder inequality we have

$$\int a(x)|u_n - u|^p dx = \int_{|x|\leq R_0} a(x)|u_n - u|^p dx + \int_{|x|>R_0} a(x)|u_n - u|^p dx \leq C\epsilon, \tag{2.57}$$

hence $\lim_{n\to\infty} \int a(x)|u_n - u|^p = 0$. □

Our first task is to locate the levels free from this noncompactness effect.

Proposition 2.6. *Every sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$, satisfying*

(a) $I_\lambda\ (u_n) = c + o(1)$ *with* $c < c_0 + S_\lambda^\infty$,

(b) $I'_\lambda(u_n) = o(1)$ *strongly in* $H^{-1}(\mathbb{R}^N)$,

has a convergent subsequence.

Proof. It is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, so we can find a $\overline{u} \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup \overline{u}$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to \overline{u}$ almost every in $\mathbb{R}^N$, $u_n \to \overline{u}$ strongly in $L^s_{loc}(\mathbb{R}^N)$, $1 \le s < 2N/N-2$. From condition (b), we have

$$\langle I'_\lambda(\overline{u}), w\rangle = 0, \quad \forall w \in H^1\left(\mathbb{R}^N\right); \tag{2.58}$$

that is, $\overline{u}$ is a weak solution of problem (1.1) and $\overline{u} \in \Lambda_\lambda$. Set $v_n = u_n - \overline{u}$ to get $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$, $v_n \to 0$ almost every in $\mathbb{R}^N$, $v_n \to 0$ strongly in $L^s_{loc}(\mathbb{R}^N)$, $1 \le s < 2N/N-2$, we can prove that there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) satisfying $v_n \to 0$ strongly in $H^1(\mathbb{R}^N)$. Arguing by contradiction, we assume that there exists a constant $\beta > 0$ such that $\|v_n\| \ge \beta > 0$. Apply the Brezis-Lieb theorem (see [16]) and Lemma 2.5,

$$\begin{aligned}
I_\lambda(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\int a(x)|u_n|^p - \frac{\lambda}{q}\int b(x)|u_n|^q \\
&= I_\lambda(\overline{u}) + \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int a(x)|v_n|^p - \frac{\lambda}{q}\int b(x)|v_n|^q + o(1) \\
&= I_\lambda(\overline{u}) + \frac{1}{2}\|v_n\|^2 - \frac{\lambda b^\infty}{q}\int |v_n|^q - \frac{\lambda}{q}\int (b(x)-b^\infty)|v_n|^q + o(1).
\end{aligned} \tag{2.59}$$

Moreover, taking into account (2.58),

$$\begin{aligned}
o(1) = \langle I'_\lambda(u_n), u_n\rangle &= \|u_n\|^2 - \int a(x)|u_n|^p - \lambda\int b(x)|u_n|^q \\
&= \langle I'_\lambda(\overline{u}), \overline{u}\rangle + \|v_n\|^2 - \int a(x)|v_n|^p - \lambda\int b(x)|v_n|^q + o(1) \\
&= \|v_n\|^2 - \lambda b^\infty\int |v_n|^q - \lambda\int (b(x)-b^\infty)|v_n|^q + o(1).
\end{aligned} \tag{2.60}$$

By (b1), for any $\epsilon > 0$, there exist: $R_0 > 0$ such that $|b(x) - b^\infty| < \epsilon$ for $|x| \ge R_0$. Since $v_n \to 0$ strongly in $L^s_{loc}(\mathbb{R}^N)$ for $1 \le s < 2N/N-2$, $\{v_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, therefore, $\int (b(x)-b^\infty)|v_n|^q \le C\int_{B_{R_0}} |v_n|^q + \epsilon C$. Setting $n \to \infty$, then $\epsilon \to 0$, we have

$$\int (b(x)-b^\infty)|v_n|^q = o(1). \tag{2.61}$$

Combining (2.60) and (2.59), we obtain

$$\|v_n\|^2 - \lambda b^\infty\int |v_n|^q = o(1), \qquad I_\lambda(u_n) \ge c_0 + \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{q}b^\infty\int |v_n|^q + o(1). \tag{2.62}$$

Since $\|v_n\| \geq \beta > 0$, we can find a sequence $\{s_n\}$, $s_n > 0$, $s_n \to 1$ as $n \to \infty$, such that $t_n = s_n v_n$ satisfying $\|t_n\|^2 - \lambda b^\infty \int |t_n|^q = 0$. Hence

$$I_\lambda(u_n) \geq c_0 + \frac{1}{2}\|t_n\|^2 - \frac{\lambda}{q} b^\infty \int |t_n|^q + o(1) \geq c_0 + S_\lambda^\infty + o(1); \tag{2.63}$$

that is, $c = \lim_{n\to\infty} I_\lambda(u_n) \geq c_0 + S_\lambda^\infty$, contradicting condition (a). Consequently, $u_n \to \overline{u}$ strongly.

Let $\mathbf{e} = (1, 0, \ldots, 0)$ be a fixed unit vector in $\mathbb{R}^N$ and ω be a ground state of problem (2.52). Here we use an interaction phenomenon between u_0 and ω. □

Proposition 2.7. *Under the assumptions (a1) and (b1), Then*

$$I_\lambda(u_0 + t\omega) < c_0 + I_\lambda^\infty(\omega) \quad \forall t > 0. \tag{2.64}$$

Proof.

$$\begin{aligned} I_\lambda(u_0 + t\omega) &= \frac{1}{2}\|u_0 + t\omega\|^2 - \frac{1}{p}\int a(x)|u_0 + t\omega|^p - \frac{\lambda}{q}\int b(x)|u_0 + t\omega|^q \\ &< I_\lambda(u_0) + \frac{1}{2}\|t\omega\|^2 - \frac{\lambda}{q} t^q \int b(x)|\omega|^q \\ &\leq I_\lambda(u_0) + \frac{1}{2}\|t\omega\|^2 - \frac{\lambda}{q} b^\infty \int |t\omega|^q \\ &= I_\lambda(u_0) + I_\lambda^\infty(t\omega) \\ &\leq c_0 + I_\lambda^\infty(\omega). \end{aligned} \tag{2.65}$$

□

Proposition 2.8. *If $0 < \lambda < \lambda^*$, for $c_1 = \inf_{\Lambda_\lambda^-} I_\lambda$, one can find a minimizing sequence $\{u_n\} \subset \Lambda_\lambda^-$ such that*

(a) $I_\lambda(u_n) = c_1 + o(1)$,

(b) $I'_\lambda(u_n) = o(1)$ *strongly in* $H^{-1}(\mathbb{R}^N)$,

(c) $c_1 < c_0 + S_\lambda^\infty$.

Proof. Set $\Sigma = \{u \in H^1(\mathbb{R}^N) : \|u\| = 1\}$ and define the map $\Psi : \Sigma \to \Lambda_\lambda^-$ given by $\Psi(u) = t^+(u)u$. Since the continuity of $t^+(u)$ follows immediately from its uniqueness and extremal property, thus Ψ is continuous with continuous inverse given by $\Psi^{-1}(u) = u/\|u\|$. Clearly Λ_λ^- disconnects $H^1(\mathbb{R}^N)$ in exactly two components:

$$\begin{aligned} U_1 &= \left\{u = 0 \text{ or } u : \|u\| < t^+\left(\frac{u}{\|u\|}\right)\right\}, \\ U_2 &= \left\{u : \|u\| > t^+\left(\frac{u}{\|u\|}\right)\right\}, \end{aligned} \tag{2.66}$$

and $\Lambda_\lambda^+ \subset U_1$.

We will prove that there exists t_1 such that $u_0+t_1\omega \in U_2$. Denote $t_0 = t^+((u_0+t\omega)/\|u_0+t\omega\|)$. Since $t^+((u_0+t\omega)/\|u_0+t\omega\|)((u_0+t\omega)/\|u_0+t\omega\|) \in \Lambda_\lambda^-$, we have

$$t_0^2 - \frac{t_0^q \lambda \int b(x)|u_0+t\omega|^q}{\|u_0+t\omega\|^q} = \frac{t_0^p \int a(x)|u_0+t\omega|^p}{\|u_0+t\omega\|^p} \geq 0. \tag{2.67}$$

Thus

$$\begin{aligned} t_0 &\leq \left[\frac{\|u_0+t\omega\|}{\left(\lambda \int b(x)|u_0+t\omega|^q\right)^{1/q}}\right]^{q/(q-2)} = \left[\frac{\|u_0/t+\omega\|}{\left(\lambda \int b(x)|u_0/t+\omega|^q\right)^{1/q}}\right]^{q/(q-2)} \\ &\leq \left[\frac{\|u_0/t+\omega\|}{\left(\lambda \int b^\infty|u_0/t+\omega|^q\right)^{1/q}}\right]^{q/(q-2)} \\ &\longrightarrow \|\omega\| < \infty \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{2.68}$$

Therefore, there exists $t_2 > 0$ such that $t_0 = t^+((u_0+t\omega)/\|u_0+t\omega\|) < 2\|\omega\|$, for $t \geq t_2$. Set $t_1 > t_2 + 2$, then

$$\begin{aligned} \|u_0+t_1\omega\|^2 &= \|u_0\|^2 + t_1^2\|\omega\|^2 + 2t_1 \int (\nabla u_0 \nabla \omega + u_0\omega) \\ &= \|u_0\|^2 + t_1^2\|\omega\|^2 + 2t_1\lambda b^\infty \int |\omega|^{q-1}u_0 \\ &> t_1^2\|\omega\|^2 > 4\|\omega\|^2 > t_0^2, \end{aligned} \tag{2.69}$$

hence $u_0 + t_1\omega \in U_2$.

However, Λ_λ^- disconnects $H^1(\mathbb{R}^N)$ in exactly two components, so we can find a $s \in (0,1)$ such that $u_0 + st_1\omega \in \Lambda_\lambda^-$. Therefore, $c_1 \leq I_\lambda(u_0+st_1\omega) < c_0 + S_\lambda^\infty$, which follows from Proposition 2.7.

Analogously to the proof of Proposition 2.4, one can show that the Ekeland variational principle [8] gives a sequence $\{u_n\} \subset \Lambda_\lambda^-$ satisfying the conditions (a), (b), and (c). $\square$

Proposition 2.9. *If $0 < \lambda < \lambda^*$, then the minimization problem $c_1 = \inf_{\Lambda_\lambda^-} I_\lambda$ is achieved at a point $u_1 \in \Lambda_\lambda^-$ which is a critical point for I_λ and $u_1 > 0$.*

Proof. Applying Propositions 2.6 and 2.8, we have $u_n \to u_1$ strongly in $H^1(\mathbb{R}^N)$. Consequently, u_1 is a critical point for I_λ, $u_1 \in \Lambda_\lambda^-$ (since Λ_λ^- is closed) and $I_\lambda(u_1) = c_1$.

Let $t^+(|u_1|) > 0$ satisfy $t^+(|u_1|)|u_1| \in \Lambda_\lambda^-$. Since $u_1 \in \Lambda_\lambda^-$, $t^+(u_1) = 1$. From Lemma 2.1, we conclude that

$$t^+(|u_1|) \geq t_{\max}(|u_1|) = t_{\max}(u_1), \tag{2.70}$$

$$c_1 = I_\lambda(u_1) = \max_{t \geq t_{\max}(u_1)} I_\lambda(tu_1) \geq I_\lambda(t^+(|u_1|)u_1) \geq I_\lambda(t^+(|u_1|)|u_1|) \geq c_1. \tag{2.71}$$

Hence, $I(t^+(|u_1|)|u_1|) = c_1$, So we can always take $u_1 \geq 0$. By standard regularity method and the maximum principle for weak solutions (see [13]), we can show that $u_1 > 0$ in $\mathbb{R}^N$. □

Proof of Theorem 1.1. Applying Propositions 2.4 and 2.9, we can obtain the conclusion of Theorem 1.1. □

3. Proof of Theorem 1.2

In the sequel, $X := H^1(\mathbb{R}^N)$, (e_k) denotes an orthonormal base of X,

$$X(j) := \operatorname{span}(e_1, \ldots, e_j), \qquad X_k := \oplus_{j \geq k} X(j), \qquad X^k := \oplus_{j \leq k} X(j), \tag{3.1}$$

and $C, C_1, C_2, \ldots$, denote (possibly different) positive constants.

If $u \in X$, we let the variational functional of problem (1.4) be

$$I(u) = \frac{1}{2} \int \left(|\nabla u|^2 + u^2 \right) - \frac{\mu}{p} \int a(x)|u|^p - \frac{\lambda}{q} \int b(x)|u|^q. \tag{3.2}$$

Proposition 3.1. *Under the assumptions (a1) and (b1), for every $\lambda > 0$ and $\mu \in \mathbb{R}$, the problem* (1.4) *has infinitely many solutions with positive energy $I(u)$.*

Proof. We will show that the energy functional $I(u)$ satisfies the assumptions of Fountain theorem in [17]. These assumptions are as follows.

(A1) The energy functional $I \in C^1(X, R)$ and is even.

(A2) Every sequence $u_n \in X$ with $C := \sup_n I(u_n) < \infty$ and $I'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

(A3) $\inf_{\rho > 0} \sup_{u \in X^k, \|u\| \geq \rho} I(u) \leq 0$, for every $k \in N$.

(A4) $\sup_{r > 0} \inf_{u \in X_k, \|u\| = r} I(u) \to \infty$ as $k \to \infty$.

We define

$$\lambda_k = \sup_{u \in X_k - \{0\}} \frac{\left(\int b(x)|u|^q \right)^{1/q}}{\|u\|}, \tag{3.3}$$

then

$$\lambda_k \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.4}$$

Indeed, clearly we have

$$0 < \lambda_{k+1} \leq \lambda_k. \tag{3.5}$$

Assume that $\lambda_k \to \lambda_0 > 0$, as $k \to \infty$. Then for every $k \geq 1$, there exists $u_k \in X_k$ such that $\|u_k\| = 1$ and

$$\frac{\lambda_0}{2} < \int b(x)|u_k|^q. \tag{3.6}$$

By definition, $u_k \rightharpoonup 0$ in X, this contradicts with $\lambda_0 > 0$. Now, let us prove (A1)–(A4). The (PS)-condition (A2) has be shown as in Proposition 2.6. In order to prove (A3), since the subspace X^k is finite dimensional, all norms on X^k are equivalent, hence, we obtain

$$I(u) \leq \frac{1}{2}\|u\|^2 - C_1\|u\|^p - C_2\lambda\|u\|^q \leq \frac{1}{2}\|u\|^2 - C_2\lambda\|u\|^q. \tag{3.7}$$

Therefore, the term $-C_2\lambda\|u\|^q$ dominates for $\|u\|$ sufficiently large, and (A3) follows. To show (A4), since $p < 2$, there exists $R > 0$ large enough so that

$$\frac{\mu}{p}\|a\|_{L^{\alpha/(\alpha-1)}} S_{\alpha p}^p \|u\|^p \leq \frac{1}{4}\|u\|^2 \tag{3.8}$$

for $\|u\| \geq R$. Then, for $u \in X_k$, it follows from (2.13), (3.8), and (3.3) that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{p}\int a(x)|u|^p - \frac{\lambda}{q}\int b(x)|u|^q \\ &\geq \frac{1}{4}\|u\|^2 - \frac{\lambda}{q}\lambda_k^q\|u\|^q. \end{aligned} \tag{3.9}$$

Now we set $r_k = (8\lambda\lambda_k^q/q)^{1/(2-q)}$ so that

$$\frac{1}{8}r_k^2 = \frac{\lambda}{q}\lambda_k^q r_k^q. \tag{3.10}$$

Clearly

$$r_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \tag{3.11}$$

(A4) follows. Since the energy functional $I(u)$ is even, then, by Fountain theorem, there exist a sequence of critical points (v_k) such that $I(v_k) \to \infty$ as $k \to \infty$. □

Proposition 3.2. *Under the assumptions (a1) and (b1), for every $\mu > 0$ and $\lambda \in \mathbb{R}$, the problem* (1.4) *has infinitely many solutions with negative energy $I(u)$.*

Proof. We will show that the energy functional $I(u)$ satisfies the assumptions of in [9, Theorem 2]. These assumptions are as follows.

(B1) The energy functional $I \in C^1(X, \mathbb{R})$ and is even.

(B2) There exists k_0 such that for every $k \geq k_0$ there exists $R_k > 0$ such that $I(u) \geq 0$ for every $u \in X_k$ with $\|u\| = R_k$.

(B3) $b_k := \inf_{B_k} I(u) \to 0$ as $k \to \infty$, where $B_k = \{u \in X_k : \|u\| \leq R_k\}$.

(B4) For every $k \geq 1$, there exist $r_k \in (0, R_k)$ and $d_k < 0$ such that $I(u) \leq d_k$ for every $u \in X^k$ with $\|u\| = r_k$.

(B5) Every sequence $u_n \in X_{-n}^n := \oplus_{j=-n}^n X(j)$ with $I(u_n) < 0$ bounded and $(I|'_{X_{-n}^n})(u_n) \to 0$ as $n \to \infty$ has a subsequence which converges to a critical point of I.

We define

$$\mu_k = \sup_{u \in X_k - \{0\}} \frac{\left(\int a(x)|u|^p\right)^{1/p}}{\|u\|}. \tag{3.12}$$

By Lemma 2.5,

$$\mu_k \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.13}$$

Indeed, clearly we have

$$0 < \mu_{k+1} \leq \mu_k. \tag{3.14}$$

Assume that $\mu_k \to \mu_0 > 0$, as $k \to \infty$. Then for every $k \geq 1$, there exists $u_k \in X_k$ such that $\|u_k\| = 1$ and

$$\frac{\mu_0}{2} < \int a(x)|u_k|^p. \tag{3.15}$$

By definition, $u_k 0$ in X. By Lemma 2.5, this contradicts (3.15). Now, let us prove (B1)–(B5). Since $q > 2$, there exists $R > 0$ small enough so that

$$\frac{\lambda}{q} b^\infty S_q^q \|u\|^q \leq \frac{1}{4}\|u\|^2 \tag{3.16}$$

for $\|u\| \leq R$. Then, for $u \in X_k$, it follows from (3.16) and (3.12) that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\mu}{p}\int a(x)|u|^p - \frac{\lambda}{q} b^\infty S_q^q \|u\|^q \\ &\geq \frac{1}{4}\|u\|^2 - \frac{\mu}{p}\mu_k^p \|u\|^p. \end{aligned} \tag{3.17}$$

Now we set $R_k = (4\mu\mu_k^p/p)^{1/(2-p)}$ so that

$$\frac{1}{4}R_k^2 = \frac{\mu}{p}\mu_k^p R_k^p. \tag{3.18}$$

Clearly

$$R_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.19}$$

so there exists k_0 such that $R_k \leq R$ when $k \geq k_0$. Thus if $u \in X_k$, $k \geq k_0$ satisfies $\|u\| = R_k$, we have

$$I(u) \geq \frac{1}{4}\|u\|^2 - \frac{\mu}{p}\mu_k^p \|u\|^p = 0. \tag{3.20}$$

This proves (B2). Next, (B3) follows immediately from (3.19). On the other hand, since the subspace X^k is finite dimensional, all norms on X^k are equivalent, hence, we obtain

$$I(u) \leq \frac{1}{2}\|u\|^2 - C_1\mu\|u\|^p - C_2\|u\|^q \leq \frac{1}{2}\|u\|^2 - C_1\mu\|u\|^p. \tag{3.21}$$

Therefore, the term $-C_1\mu\|u\|^p$ dominates near 0, and (B4) follows. This is precisely the point where $\mu > 0$ enters. Finally, the (PS) condition (B5) has been shown as in Proposition 2.6. Since the energy functional $I(u)$ is even, all the assumptions of in [9, Theorem 2] are satisfied. Then, there exists k_0 such that for each $k \geq k_0$,

$$I(u) \text{ has a critical value } c_k \in [b_k, d_k], \text{ so that } c_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.22}$$

This completes the proof of Theorem 1.2, since observe that (B3) and (B4) imply $b_k \leq d_k < 0$. □

Proof of Theorem 1.2. The proof follows from Propositions 3.1 and 3.2. □

References

[1] H. Brezis and S. Kamin, "Sublinear elliptic equations in $\mathbb{R}^N$," *Manuscripta Mathematica*, vol. 74, no. 1, pp. 87–106, 1992.

[2] H. Berestycki and P. L. Lion, "Nonlinear scalar field equations—I," *Archive for Rational Mechanics and Analysis*, vol. 82, pp. 313–345, 1983.

[3] H. Berestycki and P. L. Lion, "Nonlinear scalar field equations—II," *Archive for Rational Mechanics and Analysis*, vol. 82, pp. 347–375, 1983.

[4] Y. Li and W.-M. Ni, "On conformal scalar curvature equations in $\mathbb{R}^N$," *Duke Mathematical Journal*, vol. 57, no. 3, pp. 895–924, 1988.

[5] A. Ambrosetti, H. Brezis, and G. Cerami, "Combined effects of concave and convex nonlinearities in some elliptic problems," *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.

[6] K. J. Brown and T.-F. Wu, "A fibering map approach to a semilinear elliptic boundary value problem," *Electronic Journal of Differential Equations*, vol. 2007, no. 69, pp. 1–9, 2007.

[7] G. Tarantello, "On nonhomogeneous elliptic equations involving critical Sobolev exponent," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 9, no. 3, pp. 281–304, 1992.

[8] I. Ekeland, "On the variational principle," *Journal of Mathematical Analysis and Applications*, vol. 47, pp. 324–353, 1974.

[9] T. Bartsch and M. Willem, "On an elliptic equation with concave and convex nonlinearities," *Proceedings of the American Mathematical Society*, vol. 123, no. 11, pp. 3555–3561, 1995.

[10] S. B. Tshinanga, "On multiple solutions of semilinear elliptic equation on unbounded domains with concave and convex nonlinearities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 5, pp. 809–814, 1997.

[11] P. Drábek and S. I. Pohozaev, "Positive solutions for the p-Laplacian: application of the fibering method," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 127, no. 4, pp. 703–726, 1997.

[12] K. J. Brown and Y. Zhang, "The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function," *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, Germany, 2nd edition, 1993.

[14] P.-L. Lions, "On positive solutions of semilinear elliptic equations in unbounded domains," in *Nonlinear Diffusion Equations and Their Equilibrium States, II (Berkeley, CA, 1986)*, vol. 13 of *Math. Sci. Res. Inst. Publ.*, pp. 85–122, Springer, New York, NY, USA, 1988.

[15] B. Gidas, W.-M. Ni, and L. Nirenberg, "Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$," in *Mathematical Analysis and Applications, Part A*, L. Nachbin, Ed., Advances in Math. Suppl. Stud., 7A, pp. 369–402, Academic Press, New York, NY, USA, 1981.

[16] H. Brezis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.

[17] T. Bartsch, "Infinitely many solutions of a symmetric Dirichlet problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 20, no. 10, pp. 1205–1216, 1993.