Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 156063, 10 pages
doi:10.1155/2009/156063

Research Article

On the Solvability of Superlinear and Nonhomogeneous Quasilinear Equations

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Received 14 January 2009; Accepted 6 September 2009

Recommended by Michel C. Chipot

Using Mountain Pass Lemma, we obtain the existence of nontrivial weak solutions for a class of superlinear and nonhomogeneous quasilinear equations. The key factor in this paper is to use the new idea of near p-homogeneity in conjunction with variational techniques to obtain a new multiplicity result for a vast set of nonlinear equations, such as the mean curvature equation and so on.

1. Introduction

In this paper, we consider the nonlinear elliptic equation

$$\begin{gathered} Qu = \lambda |u|^{p-2}u + f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

$$Qu = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, \xi_m(u)), \tag{1.2}$$

where $\Omega \subset R^N$ is a bounded open connected set, $N \ge 1$. Q is a quasilinear elliptic operator generalizing the p-Laplace, that is, $Qu = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

It appears that certain nonlinear mathematical models lead to nonlinear differential equations; one of them describes the behavior compressible fluid in a homogeneous isotropic rigid porous medium, such as the p-Laplace equation. And some purely mathematical properties of the p-Laplace seem to be a challenge for nonlinear analysis, and their study leads to the development of new methods and approaches.

These statements generalize certain homogeneous operators to a class of nonhomogeneous quasilinear elliptic operators. In particular, we get an equation involving the mean curvature [1, page 357], that is, $Qu = \sum_{|\alpha|=1}(-1)^{|\alpha|}D^{\alpha}(|\xi_1'(u)|^{p-2}+(1+|\xi_1'(u)|^2)^{-(2-r)/2})D^{\alpha}u,\ r = 1$, which is nonhomogeneous; see [2].

For nonhomogeneous quasilinear operators, there are many papers in literature describing the properties of the principle eigenvalue and corresponding principle eigenfunction. One can refer to [3–9]. It is the purpose of this paper to study the existence results of non-homogeneous quasilinear equations. Using Mountain Pass Theorem [10], the work of Leray and Lions (A-3 below) [11], and variational techniques of Euler and Lagrange, we obtain the nontrivial weak solution of (1.1).

In conclusion, we like to say that Theorem 2.1 in this paper extends and unifies the previous results of [2].

This paper is organized as follows. In Section 2, we introduce some preliminaries and state the main results in this paper. In Section 3, the proof of Theorem 2.1 is given.

2. Preliminaries and Basic Results

In this section, we introduce the assumptions and definitions necessary for the proof of the theorem to come in the next section.

Let $L^p(\Omega)$ denote the usual Lebesgue space endowed with the norm $|u|_p^p = \int_\Omega |u|^p dx$, and let $W_0^{m,p}(\Omega)$ denote the completion of the space C_0^∞ in the standard norm $\|u\|_{m,p} = (\int_\Omega \sum_{|\alpha|\le m}|D^\alpha u|^p dx)^{1/p}$.

Denote by D^α the differential operator

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}, \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index consisting of nonnegative integers, and $|\alpha| = \sum_{i=1}^N \alpha_i$ denotes the order of D^α. In order to write nonlinear partial differential operators in a convenient form, we introduce, as in [12], the vector space R^{S_m} whose elements are of the form $\xi_m(u)(x) = \{D^\alpha(u(x)) : |\alpha| \le m\}$, for each $u \in W_0^{m,p}(\Omega)$, where m is a positive integer (note $D^{(0,\ldots,0)}u = u$).

We will assume that the Q has a variational structure in the sense that there exists a potential function $\Gamma : \Omega \times R^{S_m} \to R$ satisfying the following.

(Q-1) The map $x \to \Gamma(x, \xi_m)$ is measurable for each $\xi_m \in R^{S_m}$, and the map $\xi_m \to \Gamma(x, \xi_m)$ is continuously differentiable for $a.e. x \in \Omega$.

(Q-2) There exist constants p and c_1, with $1 < p < \infty$ and $c_1 > 0$, and a nonnegative function $h \in L^1(\Omega)$ such that

$$|\Gamma(x, \xi_m)| \le h(x) + c_1|\xi_m|^p \tag{2.2}$$

for $a.e. x \in \Omega$ and all $\xi_m \in R^{S_m}$.

(Q-3) $\Gamma(x, 0) = 0$, $a.e. x \in \Omega$, and for each α, $0 \le |\alpha| \le m$, $(x, \xi_m) \in \Omega \times R^{S_m}$,

$$\frac{\partial \Gamma}{\partial \xi_\alpha}(x, \xi_m) = A_\alpha(x, \xi_m). \tag{2.3}$$

The functions $A_\alpha : \Omega \times R^{S_m} \to R$ defined in (Q-3) will be assumed to satisfy the Caratheodory conditions (i.e., $A_\alpha(x, \xi_m)$, are measurable in x for all $\xi_m \in R^{S_m}$, and are continuous in ξ_m for $a.e.x \in \Omega$), as well as the following four conditions.

(A-1) There exists a constant c_2, with $c_2 > 0$, and a nonnegative function $h_0 \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ and p is as in (Q-2), such that

$$|A_\alpha(x, \xi_m)| \le h_0(x) + c_2 |\xi_m|^{p-1}, \quad 0 \le |\alpha| \le m \tag{2.4}$$

for $a.e.x \in \Omega$ and all $\xi_m \in R^{S_m}$.

(A-2) There exists a positive constant c_0 such that

$$\sum_{|\alpha| \le m} A_\alpha(x, \xi_m)\xi_\alpha \ge c_0 \left(\sum_{|\alpha| = m} |\xi_\alpha|^2 \right)^{p/2} \tag{2.5}$$

for $a.e.x \in \Omega$ and all $\xi_m \in R^{S_m}$.

(A-3) Let $\xi_m = (\eta_{m-1}, \varsigma_m)$ be the division of ξ_m into its mth order component and the corresponding $(m-1)$st order terms η_{m-1}, that is, $\eta_{m-1} = \{\xi_\beta : 0 \le |\beta| \le m-1\} \in R^{S_{m-1}}$, and $\varsigma_m = \{\xi_\alpha : |\alpha| = m\}$. Put $A_\alpha(x, \xi_m) = A_\alpha(x, \eta_{m-1}, \varsigma_m)$. Then for $a.e.x \in \Omega$ and each $\eta_{m-1} \in R^{S_{m-1}}$, $\varsigma_m \ne \varsigma_m^*$, we have

$$\sum_{|\alpha| = m} \left(A_\alpha(x, \eta_{m-1}, \varsigma_m) - A_\alpha(x, \eta_{m-1}, \varsigma_m^*) \right) > 0. \tag{2.6}$$

(A-4) (Near p-homogeneity). For $0 \le |\alpha| \le m$,

(i) $A_\alpha(x, t\xi_m)t\xi_\alpha \le |t|^p A_\alpha(x, \xi_m)\xi_\alpha, \ |t| \ge 1$,

(ii) $A_\alpha(x, t\xi_m)t\xi_\alpha \ge |t|^p A_\alpha(x, \xi_m)\xi_\alpha, \ |t| \le 1$,

for $t \in R$, $a.e.x \in \Omega$ and all $\xi_m \in R^{S_m}$.

We note that (A-4)(ii) and the Caratheodory conditions imply that $A_\alpha(x, 0) = 0$ for $0 \le |\alpha| \le m$ and $a.e.x \in \Omega$.

We define the following semilinear Dirichlet form:

$$Q(u, v) = \sum_{|\alpha| \le m} \int_\Omega A_\alpha(x, \xi_m(u)) D^\alpha v, \quad \forall u, \ v \in W_0^{m,p}(\Omega). \tag{2.7}$$

From the definition above and (A-2), we get

$$Q(u, u) \ge c_0 \left(\int_\Omega \sum_{|\alpha| = m} |D^\alpha u|^2 \right)^{p/2}, \quad \forall u \in W_0^{m,p}(\Omega). \tag{2.8}$$

Then it follows from [13, page 1822] that

$$\liminf_{\|u\|_{L^p} \to \infty} \frac{Q(u, u)}{\|u\|_{L^p}^p} < \infty. \tag{2.9}$$

So we define as in [13, page 1821]

$$\lambda_1 = \liminf_{\|u\|_{L^p} \to \infty} \frac{Q(u,u)}{\|u\|_{L^p}^p}. \tag{2.10}$$

Also $f(x,u) \in C(\overline{\Omega} \times R, R)$ will meet the following conditions.

(f-1) There exist constants $b_0 > 0, b_1 > 0$, such that

$$\left|f(x,u)\right| \le b_0|u|^{q-1} + b_1|u|^{r-1}, \quad \forall x \in \Omega, \tag{2.11}$$

where $1 < r < p < q < p^*, p^* = Np/(N - mp)$.

(f-2) There exist constants $\theta > p, M > 0$, such that

$$0 < F(x,u) = \int_0^u f(x,s)ds \le \frac{1}{\theta} u f(x,u), \quad \forall x \in \Omega,\ |u| \ge M. \tag{2.12}$$

(f-3) $f(x,0) = 0, uf(x,u) \ge 0, u \in R$ and for $a.e. x \in \Omega$, $\lim_{t\to 0}(f(x,t))/|t|^{p-1} = 0$.

Now, we state our main theorem in this paper.

Theorem 2.1. *Assume that Q given by* (1.2) *satisfies (Q-1)–(Q-3), $A_\alpha(x,\xi_m)$ satisfies (A-1)–(A-4), $\lambda \in (0,\lambda_1)$, and f satisfies (f-1)–(f-3). Then problem* (1.1) *has at least one nontrivial weak solution.*

3. Proof of the Theorem

Define a functional $I : W_0^{m,p}(\Omega) \to R$ by

$$I(u) = \int_\Omega \Gamma(x, \xi_m(u))dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \int_\Omega F(x,u)dx. \tag{3.1}$$

Also we note that there are positive constants c_3 and c_4 such that

$$c_3\|u\|_{m,p} \le \|\xi_m(u)\|_{L^p} \le c_4\|u\|_{m,p}, \quad \forall u \in W_0^{m,p}(\Omega), \tag{3.2}$$

and from the Poincaré inequality, there is a positive constant c_5 such that

$$\|\xi_m(u)\|_{L^p}^p \le c_5\left(\int_\Omega \sum_{|\alpha|=m} |D^\alpha u|^2\right)^{p/2}, \quad \forall u \in W_0^{m,p}(\Omega). \tag{3.3}$$

Let $W_0^{m,p}(\Omega)^*$ be the dual of $W_0^{m,p}(\Omega)$. $I'(u)$ is the Frechet derivative of $I(u)$. So the weak solutions of problem (1.1) are equivalent to the critical points of $I(u)$. And (f-3) implies that $u = 0$ is a trivial solution to problem (1.1).

To derive out Theorem 2.1 we need the following lemma.

Lemma 3.1. *Assume that all the conditions in the hypothesis of Theorem 2.1 hold, then I satisfies the (PS) condition.*

Proof. (1) We have the boundedness of (PS) sequence of $I(u)$.

Suppose that $\{u_n\}$ is a (PS) sequence of $I(u)$; that is, there exists $C > 0$, such that

$$|I(u_n)| \leq C, \quad I'(u_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.4}$$

Let $E = W_0^{m,p}(\Omega)$, $S = \{u_n\}$. From (3.4) we obtain

$$Q(u_n, \varphi) - \lambda \int_\Omega |u_n|^{p-2} u_n \varphi dx - \int_\Omega f(x, u_n)\varphi dx = 0(1)\|\varphi\|_{m,p}, \quad \forall \varphi \in E. \tag{3.5}$$

By (Q-2), (A-4) and Fubini theorem, we have

$$\int_\Omega \Gamma(x, \xi_m) dx = \int_0^1 Q(tu, u) dt \geq \frac{Q(u,u)}{p}. \tag{3.6}$$

From (3.5) we have

$$\begin{aligned}
I(u_n) - \frac{1}{\theta} 0(1)\|u_n\|_{m,p} &= \int_\Omega \Gamma(x, \xi_m(u_n)) dx - \frac{1}{\theta} Q(u_n, u_n) \\
&\quad - \lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_\Omega |u_n|^p dx + \int_\Omega \left(\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\right) dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{\lambda_1}\right) Q(u_n, u_n) \\
&\quad + \int_{\Omega(|u_n| \geq M)} \left(\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\right) dx \\
&\quad + \int_{\Omega(|u_n| < M)} \left(\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\right) dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{\lambda_1}\right) Q(u_n, u_n) - C_1 \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{\lambda_1}\right) \frac{c_0 c_3^p}{c_5} \|u\|_{m,p}^p - C_1,
\end{aligned} \tag{3.7}$$

where C_1 is a constant independent of u_n. The above estimates imply that

$$\|u_n\|_{m,p} \leq C. \tag{3.8}$$

Since $W^{m,p}(\Omega)$ is a separable Banach space, from Sobolev compact imbedding theorem [14, page 144] and the weak convergence theorem [15, page 8] we obtain that there exists a subsequence (still denoted by $\{u_n\}$) and a function $u \in W_0^{m,p}(\Omega)$, such that

$$u_n \longrightarrow u, \ a.e. \ \text{in} \ \Omega, \ (n \longrightarrow \infty). \tag{3.9}$$

$$\lim_{n\to\infty} \|D^\alpha u_n - D^\alpha u\|_p = 0 \quad \text{for } |\alpha| \le m-1. \tag{3.10}$$

$$\lim_{n\to\infty} \int_\Omega D^\alpha u_n w = \int_\Omega D^\alpha u w, \quad \forall w \in L^{p'}, \ |\alpha| = m. \tag{3.11}$$

$$\lim_{n\to\infty} \eta_{m-1}(u_n(x)) = \eta_{m-1}(u(x)) \quad a.e. \ \ x \in \Omega. \tag{3.12}$$

(2) Next, for the above $\{u_n\}$ we claim that

$$\lim_{n\to\infty} Q(u_n, u_n - u) = 0. \tag{3.13}$$

Let $\varphi = u_n - u$ in (3.5), we see that

$$Q(u_n, u_n - u) = \lambda \int_\Omega |u_n|^{p-2} u_n (u_n - u) dx + \int_\Omega f(x, u_n)(u_n - u) dx + 0(1)\|u_n - u\|_{m,p}. \tag{3.14}$$

We conclude from (f-1) and Sobolev compact imbedding theorem that

$$\begin{aligned} \left| \int_\Omega f(x, u_n)(u_n - u) dx \right| &\le \|f(x, u_n)\|_{q'} \|u_n - u\|_q \\ &\le \left(b_0 \left\| u_n^{q-1} \right\|_{q'} + b_1 \left\| u_n^{r-1} \right\|_{q'} \right) \|u_n - u\|_q \\ &\le c\left(\|u_n\|_{m,p}^{q-1} + \|u_n\|_{m,p}^{r-1} \right) \|u_n - u\|_q \\ &\longrightarrow 0, \quad (n \longrightarrow \infty), \end{aligned} \tag{3.15}$$

where $q' = q/(q-1)$.

Also we see from (3.8) that

$$\lambda \left| \int_\Omega |u_n|^{p-2} u_n (u_n - u) dx \right| \le \lambda \|u_n\|_p^{p-1} \|u_n - u\|_p \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.16}$$

By (3.14), (3.15), (3.16) we obtain that (3.13) holds.

(3) There exists a subsequence $\{u_{n_k}\}_{k=1}^\infty \subset \{u_n\}$ satisfying

$$\lim_{k\to\infty} \zeta_m(u_{n_k}(x)) = \zeta_m(u(x)), \quad a.e. \ \ x \in \Omega, \tag{3.17}$$

where $\zeta_m(u(x)) = \{D^\alpha u(x) : |\alpha| = m\}$.

To establish (3.17), it is sufficient to establish that subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ satisfies the following two facts.

(c1) One has

$$\lim_{k\to\infty}\sum_{|\alpha|=m}\left(A_\alpha\left(x,\eta_{m-1}(u_{n_k}),\zeta_m(u_{n_k})\right)-A_\alpha\left(x,\eta_{m-1}(u_{n_k}),\zeta_m(u)\right)\right) \times \left(D^\alpha u_{n_k}(x)-D^\alpha u(x)\right)=0, \quad a.e.\ \ x\in\Omega. \tag{3.18}$$

(c2) With $\{u_{n_k}\}_{k=1}^{\infty}$ designating the same subsequence as in (3.20),

$$\{|\zeta_m(u_{n_k})|\}_{k=1}^{\infty} \text{ is pointwise bounded for } a.e.\ \ x\in\Omega. \tag{3.19}$$

To see that (c1) and (c2) together imply (3.17), let $\Omega_1 = \{x \in \Omega, \text{(c1)}, \text{(c2)}, \text{(A-1)-(A-2)}$ hold simultaneously$\}$. We have $\operatorname{meas}\Omega = \operatorname{meas}\Omega_1$. If (3.17) does not hold, there must exist a point $x_0 \in \Omega_1$, and further a subsequence $\{\zeta_m(u_{n_{k_l}}(x_0))\}_{l=1}^{\infty}$ and $\zeta_m^* \in R^{S_m - S_{m-1}}$, where $\zeta_m^* \neq \zeta_m(u(x_0))$, such that

$$\lim_{l\to\infty}\zeta_m\left(u_{n_{k_l}}(x_0)\right)=\zeta_m^*. \tag{3.20}$$

Therefore (3.12) produces

$$\begin{aligned}\lim_{l\to\infty}\sum_{|\alpha|=m}&\left(A_\alpha\left(x_0,\eta_{m-1}\left(u_{n_{k_l}}\right),\zeta_m\left(u_{n_{k_l}}\right)\right)-A_\alpha\left(x_0,\eta_{m-1}\left(u_{n_{k_l}}\right),\zeta_m(u)\right)\right)\\ &\times\left(D^\alpha u_{n_{k_l}}(x_0)-D^\alpha u(x_0)\right)\\ =&\sum_{|\alpha|=m}\left(A_\alpha\left(x_0,\eta_{m-1}(u),\zeta_m^*\right)-A_\alpha\left(x_0,\eta_{m-1}(u),\zeta_m(u)\right)\right)\times\left(\zeta_m^*-D^\alpha u(x_0)\right).\end{aligned} \tag{3.21}$$

It is easy to see from (3.20) and (A-3) that the right side of the equality in (3.21) is strictly positive and so the left side. This is contrary with $x_0 \in \Omega_1$ and (c1). Therefore there is no such a point x_0 in Ω_1. Hence (3.17) is established.

Now we need to prove that (c1) and (c2) hold. Set

$$p_k(x)=\sum_{|\alpha|=m}\left(A_\alpha\left(x,\eta_{m-1}(u_{n_k}),\zeta_m(u_{n_k})\right)-A_\alpha\left(x,\eta_{m-1}(u_{n_k}),\zeta_m(u)\right)\right) \times\left(D^\alpha u_{n_k}(x)-D^\alpha u(x)\right)=0. \tag{3.22}$$

From (A-3) we see that $p_k(x) \geq 0$, then

$$\begin{aligned}
I_k &= \int_\Omega p_k(x)dx \\
&= -\int_\Omega \sum_{|\alpha|=m} \left(A_\alpha\left(x, \eta_{m-1}(u_{n_k}), \zeta_m(u)\right) - A_\alpha\left(x, \eta_{m-1}(u), \zeta_m(u)\right)\right)\left(D^\alpha u_{n_k}(x) - D^\alpha u(x)\right) \\
&\quad - \int_\Omega \sum_{|\alpha|=m} A_\alpha\left(x, \eta_{m-1}(u), \zeta_m(u)\right)\left(D^\alpha u_{n_k}(x) - D^\alpha u(x)\right) \\
&\quad + \int_\Omega \sum_{|\alpha|=m} \left(A_\alpha\left(x, \eta_{m-1}(u_{n_k}), \zeta_m(u_{n_k})\right)\right)\left(D^\alpha u_{n_k}(x) - D^\alpha u(x)\right) \\
&= I_k^{(1)} + I_k^{(2)} + I_k^{(3)}.
\end{aligned} \tag{3.23}$$

By [16, page 70], we obtain that $p_k(x) \to 0$, *a.e.* $x \in \Omega$, if $I_k \to 0$.

From (3.11), (A-1), (A-2) and $u \in W_0^{m,p}$, it follows that $I_k^{(2)} \to 0$.

By (3.10) and (A-1), for all $\varepsilon > 0$, there are $\delta > 0$, $\Omega' \subset \Omega$, with meas $\Omega' < \delta$, such that

$$\int_{\Omega'} \left|A_\alpha\left(x, \eta_{m-1}(u_{n_k}), \zeta_m(u)\right) - A_\alpha\left(x, \eta_{m-1}(u), \zeta_m(u)\right)\right|^{p'} < \varepsilon. \tag{3.24}$$

We have from Egoroff theorem and (3.11) $I_k^{(1)} \to 0$. On the other hand, we can conclude from Hölder's inequality that

$$\lim_{k\to\infty} \int_\Omega A_\alpha(x, \xi_m(u_{n_k}))(D^\alpha u_{n_k}(x) - D^\alpha u(x)) = 0, \quad \text{for } 0 \leq |\alpha| \leq m-1. \tag{3.25}$$

From the above and (3.13) we have that $I_k^{(3)} \to 0$ and $I_k \to 0$. Hence (c1) is established.

Using the same methods as in [13, pages 1835-1836], we obtain that (c2) holds. The proof of Lemma 3.1 is complete. □

Proof of Theorem 2.1. We will verify the geometric assumptions of the Mountain Pass Lemma.

(i) There exists $\rho > 0, \beta > 0 : \|u\|_{m,p} = \rho \Rightarrow I(u) \geq \beta$.

Let $\lambda = \lambda_1 - 2\delta > 0, \delta > 0$. For all $u \in W_0^{m,p}(\Omega)$, there holds

$$\begin{aligned}
I(u) &= \int_\Omega \Gamma(x, \xi_m(u))dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \int_\Omega F(x,u)dx \\
&\geq \frac{\delta}{p}\|u\|_p^p + \frac{\delta c_0 c_3^p}{\lambda_1 p c_5}\|u\|_{m,p}^p - \int_\Omega F(x,u)dx.
\end{aligned} \tag{3.26}$$

From (f-3), for all $\varepsilon > 0, \exists \rho_0 = \rho_0(\varepsilon)$ such that if $0 < \rho = \|u\|_{m,p} < \rho_0$, then

$$\left|f(x,u)\right| < \varepsilon |u|^{p-1}. \tag{3.27}$$

Thus,

$$\int_{\Omega} F(x,u)dx = \int_{\Omega}\int_0^u f(x,t)dt\,dx \le \frac{\varepsilon}{p}\int_{\Omega}|u|^p dx \le \frac{C_2\varepsilon}{p}\|u\|_{m,p}. \tag{3.28}$$

Taking $C_2 = (\delta c_0 c_3^p)/(2\lambda_1 c_5)$, from (3.26) and (3.28), we have

$$I(u) \ge \beta > 0. \tag{3.29}$$

(ii) There exists $u_0 \in W_0^{m,p}(\Omega) : \|u_0\|_{m,p} \ge \rho$ and $I(u_0) < 0$.

In fact, from (f-2), (f-3), we can deduce that there exist constants c_3', c_4' such that

$$F(x,u) \ge c_3'|u|^{\theta} - c_4', \quad \forall u \in W_0^{m,p}(\Omega). \tag{3.30}$$

Since $\theta > \rho$, a simple calculation shows that

$$I(tu_0) \le t^p \int_0^1 Q(tu_0,u_0)dt - \frac{\lambda t^p}{p}\int_{\Omega}|u_0|^p dx - c_3' t^{\theta}\int_{\Omega}|u_0|^{\theta}dx + c_4'|\Omega|. \tag{3.31}$$

The above implies that $I(tu_0) \to -\infty$, as $t \to \infty$.

Thus by the Mountain Pass Lemma, $I(u)$ possesses a nontrivial critical point, and the proof of the Theorem 2.1 is complete. □

Corollary 3.2. *Assume that Q given by* (1.2) *satisfies (Q-1)–(Q-3), $A_\alpha(x,\xi_m)$ satisfies (A-1)–(A-4), $\lambda \in (0,\lambda_1)$, and f satisfies (f-2), (f-3), and the condition (f-4). There exist constants $c_1' >, c_2' >$, for all $x \in \Omega$, such that*

$$|f(x,u)| \le c_1'|u|^{q-1} + c_2', \tag{3.32}$$

where $1 < q < p^*, p^* = Np/(N-mp)$. Then problem (2.7) *has nontrivial weak solutions.*

Acknowledgments

The authors express their sincere thanks to the referees for their valuable suggestions. This work was supporrted by Shanhai Leading Academic Discipline Project (S30501) and Innovation Programm of Shanghai Municipal Education Commission (08YZ94).

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