Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 273063, 18 pages
doi:10.1155/2009/273063

Research Article

Positive Solutions for a Class of Coupled System of Singular Three-Point Boundary Value Problems

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Received 27 February 2009; Accepted 15 May 2009

Recommended by Juan J. Nieto

Existence of positive solutions for a coupled system of nonlinear three-point boundary value problems of the type $-x''(t) = f(t, x(t), y(t))$, $t \in (0,1)$, $-y''(t) = g(t, x(t), y(t))$, $t \in (0,1)$, $x(0) = y(0) = 0$, $x(1) = \alpha x(\eta)$, $y(1) = \alpha y(\eta)$, is established. The nonlinearities $f, g : (0,1) \times (0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and may be singular at $t = 0, t = 1, x = 0$, and/or $y = 0$, while the parameters η, α satisfy $\eta \in (0,1), 0 < \alpha < 1/\eta$. An example is also included to show the applicability of our result.

1. Introduction

Multipoint boundary value problems (BVPs) arise in different areas of applied mathematics and physics. For example, the vibration of a guy wire composed of N parts with a uniform cross-section and different densities in different parts can be modeled as a Multipoint boundary value problem [1]. Many problems in the theory of elastic stability can also be modeled as Multipoint boundary value problem [2].

The study of Multipoint boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev, [3, 4], and extended to nonlocal linear elliptic boundary value problems by Bitsadze et al. [5, 6]. Existence theory for nonlinear three-point boundary value problems was initiated by Gupta [7]. Since then the study of nonlinear three-point BVPs has attracted much attention of many researchers, see [8–11] and references therein for boundary value problems with ordinary differential equations and also [12] for boundary value problems on time scales. Recently, the study of singular BVPs has attracted the attention of many authors, see for example, [13–18] and the recent monograph by Agarwal et al. [19].

The study of system of BVPs has also fascinated many authors. System of BVPs with continuous nonlinearity can be seen in [20–22] and the case of singular nonlinearity can be seen in [8, 21, 23–26]. Wei [25], developed the upper and lower solutions method for the existence of positive solutions of the following coupled system of BVPs:

$$\begin{gathered}
-x''(t) = f(t, x(t), y(t)), \quad t \in (0,1), \\
-y''(t) = g(t, x(t), y(t)), \quad t \in (0,1), \\
x(0) = 0, \qquad x(1) = 0, \\
y(0) = 0, \qquad y(1) = 0,
\end{gathered} \tag{1.1}$$

where $f, g \in C((0,1) \times (0,\infty) \times (0,\infty), [0,\infty))$, and may be singular at $t = 0, t = 1, x = 0$ and/or $y = 0$.

By using fixed point theorem in cone, Yuan et al. [26] studied the following coupled system of nonlinear singular boundary value problem:

$$\begin{gathered}
x^{(4)}(t) = f(t, x(t), y(t)), \quad t \in (0,1), \\
-y''(t) = g(t, x(t), y(t)), \quad t \in (0,1), \\
x(0) = x(1) = x''(0) = x''(1) = 0, \\
y(0) = y(1) = 0,
\end{gathered} \tag{1.2}$$

f, g are allowed to be superlinear and are singular at $t = 0$ and/or $t = 1$. Similarly, results are studied in [8, 21, 23].

In this paper, we generalize the results studied in [25, 26] to the following more general singular system for three-point nonlocal BVPs:

$$\begin{gathered}
-x''(t) = f(t, x(t), y(t)), \quad t \in (0,1), \\
-y''(t) = g(t, x(t), y(t)), \quad t \in (0,1), \\
x(0) = 0, \qquad x(1) = \alpha x(\eta), \\
y(0) = 0, \qquad y(1) = \alpha y(\eta),
\end{gathered} \tag{1.3}$$

where $\eta \in (0,1)$, $0 < \alpha < 1/\eta$, $f, g \in C((0,1) \times (0,\infty) \times (0,\infty), [0,\infty))$. We allow f and g to be singular at $t = 0$, $t = 1$, and also $x = 0$ and/or $y = 0$. We study the sufficient conditions for existence of positive solution for the singular system (1.3) under weaker hypothesis on f and g as compared to the previously studied results. We do not require the system (1.3) to have lower and upper solutions. Moreover, the cone we consider is more general than the cones considered in [20, 21, 26].

By singularity, we mean the functions $f(t, x, y)$ and $g(t, x, y)$ are allowed to be unbounded at $t = 0$, $t = 1$, $x = 0$, and/or $y = 0$. To the best of our knowledge, existence of positive solutions for a system (1.3) with singularity with respect to dependent variable(s) has not been studied previously. Moreover, our conditions and results are different from those studied in [21, 24–26]. Throughout this paper, we assume that $f, g : (0,1) \times (0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and may be singular at $t = 0$, $t = 1$, $x = 0$, and/or $y = 0$. We also assume that the following conditions hold:

(A_1) $f(\cdot,1,1)$, $g(\cdot,1,1) \in C((0,1),(0,\infty))$ and satisfy

$$a := \int_0^1 t(1-t) f(t,1,1)\,dt < +\infty, \qquad b := \int_0^1 t(1-t) g(t,1,1)\,dt < +\infty. \tag{1.4}$$

(A_2) There exist real constants α_i, β_i such that $\alpha_i \le 0 \le \beta_i < 1$, $i = 1,2$, $\beta_1 + \beta_2 < 1$ and for all $t \in (0,1)$, $x, y \in (0,\infty)$,

$$\begin{aligned} c^{\beta_1} f(t,x,y) \le f(t,cx,y) \le c^{\alpha_1} f(t,x,y), &\quad \text{if } 0 < c \le 1,\\ c^{\alpha_1} f(t,x,y) \le f(t,cx,y) \le c^{\beta_1} f(t,x,y), &\quad \text{if } c \ge 1,\\ c^{\beta_2} f(t,x,y) \le f(t,x,cy) \le c^{\alpha_2} f(t,x,y), &\quad \text{if } 0 < c \le 1,\\ c^{\alpha_2} f(t,x,y) \le f(t,x,cy) \le c^{\beta_2} f(t,x,y), &\quad \text{if } c \ge 1. \end{aligned} \tag{1.5}$$

(A_3) There exist real constants γ_i, ρ_i such that $\gamma_i \le 0 \le \rho_i < 1$, $i = 1,2$, $\rho_1 + \rho_2 < 1$ and for all $t \in (0,1)$, $x, y \in (0,\infty)$,

$$\begin{aligned} c^{\rho_1} g(t,x,y) \le g(t,cx,y) \le c^{\gamma_1} g(t,x,y), &\quad \text{if } 0 < c \le 1,\\ c^{\gamma_1} g(t,x,y) \le g(t,cx,y) \le c^{\rho_1} g(t,x,y), &\quad \text{if } c \ge 1,\\ c^{\rho_2} g(t,x,y) \le g(t,x,cy) \le c^{\gamma_2} g(t,x,y), &\quad \text{if } 0 < c \le 1,\\ c^{\gamma_2} g(t,x,y) \le g(t,x,cy) \le c^{\rho_2} g(t,x,y), &\quad \text{if } c \ge 1, \end{aligned} \tag{1.6}$$

for example, a function that satisfies the assumptions (A_2) and (A_3) is

$$h(t,x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}(t)\, x^{r_i} y^{s_j}, \tag{1.7}$$

where $p_{ij} \in C((0,1),(0,\infty))$, $r_i, s_j < 1$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$ such that

$$\max_{1\le i\le m} r_i + \max_{1\le j\le n} s_j < 1. \tag{1.8}$$

The main result of this paper is as follows.

Theorem 1.1. *Assume that* (A_1)–(A_3) *hold. Then the system* (1.3) *has at least one positive solution.*

2. Preliminaries

For each $u \in E := C[0,1]$, we write $\|u\| = \max\{u(t) : t \in [0,1]\}$. Let $P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$. Clearly, $(E, \|\cdot\|)$ is a Banach space and P is a cone. Similarly, for each $(x,y) \in E \times E$, we write $\|(x,y)\|_1 = \|x\| + \|y\|$. Clearly, $(E \times E, \|\cdot\|_1)$ is a Banach space and $P \times P$ is a cone in $E \times E$. For any real constant $r > 0$, define $\Omega_r = \{(x,y) \in E \times E : \|(x,y)\|_1 < r\}$. By a positive solution of (1.3), we mean a vector $(x,y) \in C[0,1] \cap C^2(0,1) \times C[0,1] \cap C^2(0,1)$ such that (x,y) satisfies (1.3) and $x > 0$, $y > 0$ on $(0,1)$. The proofs of our main result (Theorem 1.1) is based on the Guo's fixed-point theorem.

Lemma 2.1 (Guo's Fixed-Point Theorem [27]). *Let K be a cone of a real Banach space E, Ω_1, Ω_2 be bounded open subsets of E and $\theta \in \overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous such that one of the following condition hold:*

(i) $\|Tx\| \leq \|x\|$ *for* $x \in \partial\Omega_1 \cap K$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in \partial\Omega_2 \cap K$;

(ii) $\|Tx\| \leq \|x\|$ *for* $x \in \partial\Omega_2 \cap K$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in \partial\Omega_1 \cap K$.

Then, T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The following result can be easily verified.

Result 1. Let $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$. Let $x \in C[t_1, t_2]$, $x \geq 0$ and concave on $[t_1, t_2]$. Then, $x(t) \geq \min\{t - t_1, t_2 - t\} \max_{s \in [t_1, t_2]} x(s)$ for all $t \in [t_1, t_2]$.

Choose $n_0 \in \{3, 4, 5, \ldots\}$ such that $n_0 > \max\{1/\eta, 1/(1-\eta), (2-\alpha)/(1-\alpha\eta)\}$. For fixed $n \in \{n_0, n_0+1, n_0+2, \ldots\}$ and $z \in C[0,1]$, the linear three-point BVP

$$\begin{gathered} -u''(t) = z(t), \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\ u\left(\frac{1}{n}\right) = 0, \qquad u\left(1 - \frac{1}{n}\right) = \alpha u(\eta), \end{gathered} \tag{2.1}$$

has a unique solution

$$u(t) = \int_{1/n}^{1-1/n} H_n(t,s) z(s)\, ds, \tag{2.2}$$

where $H_n : [1/n, 1-1/n] \times [1/n, 1-1/n] \to [0, \infty)$ is the Green's function and is given by

$$H_n(t,s) = \begin{cases} \dfrac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} - \dfrac{\alpha(t-1/n)(\eta-s)}{1-2/n+\alpha/n-\alpha\eta} - (t-s), & \dfrac{1}{n} \leq s \leq t \leq 1 - \dfrac{1}{n},\ s \leq \eta, \\ \dfrac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} - \dfrac{\alpha(t-1/n)(\eta-s)}{1-2/n+\alpha/n-\alpha\eta}, & \dfrac{1}{n} \leq t \leq s \leq 1 - \dfrac{1}{n},\ s \leq \eta, \\ \dfrac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta}, & \dfrac{1}{n} \leq t \leq s \leq 1 - \dfrac{1}{n},\ s \geq \eta, \\ \dfrac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} - (t-s), & \dfrac{1}{n} \leq s \leq t \leq 1 - \dfrac{1}{n},\ s \geq \eta. \end{cases} \tag{2.3}$$

We note that $H_n(t,s) \to H(t,s)$ as $n \to \infty$, where

$$H(t,s) = \begin{cases} \dfrac{t(1-s)}{1-\alpha\eta} - \dfrac{\alpha t(\eta-s)}{1-\alpha\eta} - (t-s), & 0 \le s \le t \le 1,\ s \le \eta, \\ \dfrac{t(1-s)}{1-\alpha\eta} - \dfrac{\alpha t(\eta-s)}{1-\alpha\eta}, & 0 \le t \le s \le 1,\ s \le \eta, \\ \dfrac{t(1-s)}{1-\alpha\eta}, & 0 \le t \le s \le 1,\ s \ge \eta, \\ \dfrac{t(1-s)}{1-\alpha\eta} - (t-s), & 0 \le s \le t \le 1,\ s \ge \eta, \end{cases} \tag{2.4}$$

is the Green's function corresponding the boundary value problem

$$\begin{gathered} -u''(t) = z(t), \quad t \in [0,1], \\ u(0) = 0, \qquad u(1) = \alpha u(\eta) \end{gathered} \tag{2.5}$$

whose integral representation is given by

$$u(t) = \int_0^1 H(t,s)\, z(s)\, ds. \tag{2.6}$$

Lemma 2.2 (see [9]). *Let $0 < \alpha < 1/\eta$. If $z \in C[0,1]$ and $z \ge 0$, then then unique solution u of the problem* (2.5) *satisfies*

$$\min_{t\in[\eta,1]} u(t) \ge \gamma \|u\|, \tag{2.7}$$

where $\gamma = \min\{\alpha\eta, \alpha(1-\eta)/(1-\alpha\eta), \eta\}$.

We need the following properties of the Green's function H_n in the sequel.

Lemma 2.3 (see [11]). *The function H_n can be written as*

$$H_n(t,s) = G_n(t,s) + \frac{\alpha(t-1/n)}{1-\ 2/n + \alpha/n - \alpha\eta} G_n(\eta,s), \tag{2.8}$$

where

$$G_n(t,s) = \frac{n}{n-2} \begin{cases} \left(s - \dfrac{1}{n}\right)\left(1 - \dfrac{1}{n} - t\right), & \dfrac{1}{n} \le s \le t \le 1 - \dfrac{1}{n}, \\ \left(t - \dfrac{1}{n}\right)\left(1 - \dfrac{1}{n} - s\right), & \dfrac{1}{n} \le t \le s \le 1 - \dfrac{1}{n}. \end{cases} \tag{2.9}$$

Following the idea in [10], we calculate upper bound for the Green's function H_n in the following lemma.

Lemma 2.4. *The function* H_n *satisfies*

$$H_n(t,s) \leq \mu_n \left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right), \quad (t,s) \in \left[\frac{1}{n},1-\frac{1}{n}\right] \times \left[\frac{1}{n},1-\frac{1}{n}\right], \tag{2.10}$$

where $\mu_n = \max\{1,\alpha\}/(1-2/n+\alpha/n-\alpha\eta)$.

Proof. For $(t,s) \in [1/n,1-1/n] \times [1/n,1-1/n]$, we discuss various cases.

Case 1. $s \leq \eta$, $s \leq t$; using (2.3), we obtain

$$H_n(t,s) = s-\frac{1}{n}+(\alpha-1)\frac{(t-1/n)(s-1/n)}{1-2/n+\alpha/n-\alpha\eta}. \tag{2.11}$$

If $\alpha > 1$, the maximum occurs at $t = 1-1/n$, hence

$$\begin{aligned} H_n(t,s) &\leq H_n\left(1-\frac{1}{n},s\right) = \alpha\frac{(s-1/n)(1-1/n-\eta)}{1-2/n+\alpha/n-\alpha\eta} \\ &\leq \alpha\frac{(s-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} \leq \mu_n\left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right), \end{aligned} \tag{2.12}$$

and if $\alpha \leq 1$, the maximum occurs at $t = s$, hence

$$\begin{aligned} H_n(t,s) &\leq H_n(s,s) = \frac{(s-1/n)(1-1/n-s+\alpha(s-\eta))}{1-2/n+\alpha/n-\alpha\eta} \\ &\leq \frac{(s-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} \leq \mu_n\left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right). \end{aligned} \tag{2.13}$$

Case 2. $s \leq \eta$, $s \geq t$; using (2.3), we have

$$\begin{aligned} H_n(t,s) &= \frac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} - \alpha\frac{(t-1/n)(\eta-s)}{1-2/n+\alpha/n-\alpha\eta} \leq \frac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} \\ &\leq \frac{(s-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} \leq \mu_n\left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right). \end{aligned} \tag{2.14}$$

Case 3. $s \geq \eta$, $t \leq s$; using (2.3), we have

$$H_n(t,s) = \frac{(t-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} \leq \frac{(s-1/n)(1-1/n-s)}{1-2/n+\alpha/n-\alpha\eta} \leq \mu_n\left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right). \tag{2.15}$$

Case 4. $s \geq \eta$, $t \geq s$; using (2.3), we have

$$H_n(t,s) = s - \frac{1}{n} + \left(t - \frac{1}{n}\right) \frac{\alpha(\eta - 1/n) - (s - 1/n)}{1 - 2/n + \alpha/n - \alpha\eta}. \tag{2.16}$$

For $\alpha(\eta - 1/n) > s - 1/n$, the maximum occurs at $t = 1 - 1/n$, hence

$$\begin{aligned} H_n(t,s) &\leq H_n\left(1 - \frac{1}{n}, s\right) = \alpha \frac{(\eta - 1/n)(1 - 1/n - s)}{1 - 2/n + \alpha/n - \alpha\eta} \leq \alpha \frac{(s - 1/n)(1 - 1/n - s)}{1 - 2/n + \alpha/n - \alpha\eta} \\ &\leq \mu_n \left(s - \frac{1}{n}\right)\left(1 - \frac{1}{n} - s\right). \end{aligned} \tag{2.17}$$

For $\alpha(\eta - 1/n) \leq s - 1/n$, the maximum occurs at $t = s$, so

$$H_n(t,s) \leq H_n(s,s) = \frac{(s - 1/n)(1 - 1/n - s)}{1 - 2/n + \alpha/n - \alpha\eta} \leq \mu_n \left(s - \frac{1}{n}\right)\left(1 - \frac{1}{n} - s\right). \tag{2.18}$$

□

Now, we consider the nonlinear nonsingular system of BVPs

$$\begin{gathered} -x''(t) = f\left(t, \max\left\{x(t) + \frac{1}{n}, \frac{1}{n}\right\}, \max\left\{y(t) + \frac{1}{n}, \frac{1}{n}\right\}\right), \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\ -y''(t) = g\left(t, \max\left\{x(t) + \frac{1}{n}, \frac{1}{n}\right\}, \max\left\{y(t) + \frac{1}{n}, \frac{1}{n}\right\}\right), \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\ x\left(\frac{1}{n}\right) = 0, \qquad x\left(1 - \frac{1}{n}\right) = \alpha x(\eta), \\ y\left(\frac{1}{n}\right) = 0, \qquad y\left(1 - \frac{1}{n}\right) = \alpha y(\eta). \end{gathered} \tag{2.19}$$

We write (2.19) as an equivalent system of integral equations

$$\begin{aligned} x(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f\left(s, \max\left\{x(s) + \frac{1}{n}, \frac{1}{n}\right\}, \max\left\{y(s) + \frac{1}{n}, \frac{1}{n}\right\}\right) ds, \\ y(t) &= \int_{1/n}^{1-1/n} H_n(t,s) g\left(s, \max\left\{x(s) + \frac{1}{n}, \frac{1}{n}\right\}, \max\left\{y(s) + \frac{1}{n}, \frac{1}{n}\right\}\right) ds. \end{aligned} \tag{2.20}$$

By a solution of the system (2.19), we mean a solution of the corresponding system of integral equations (2.20). Define a retraction $\sigma_n : [0,1] \to [1/n, 1-1/n]$ by $\sigma_n(t) = \max\{1/n, \min\{t, 1-1/n\}\}$ and an operator $T_n : E \times E \to P \times P$ by

$$T_n(x,y) = (A_n(x,y), B_n(x,y)), \tag{2.21}$$

where operators $A_n, B_n : E \times E \to P$ are defined by

$$A_n(x,y)(t)=\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\,f\left(s,\max\left\{x(s)+\frac{1}{n},\frac{1}{n}\right\},\max\left\{y(s)+\frac{1}{n},\frac{1}{n}\right\}\right)ds,$$
$$B_n(x,y)(t)=\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\,g\left(s,\max\left\{x(s)+\frac{1}{n},\frac{1}{n}\right\},\max\left\{y(s)+\frac{1}{n},\frac{1}{n}\right\}\right)ds. \tag{2.22}$$

Clearly, if $(x_n,y_n)\in E\times E$ is a fixed point of T_n, then (x_n,y_n) is a solution of the system (2.19).

Lemma 2.5. *Assume that* (A_1)–(A_3) *holds. Then* $T_n : P\times P\to P\times P$ *is completely continuous.*

Proof. Clearly, for any $(x,y)\in P\times P$, $A_n(x,y),B_n(x,y)\in P$. We show that the operator $A_n : P\times P\to P$ is uniformly bounded. Let $d>0$ be fixed and consider

$$D=\left\{(x,y)\in P\times P:\left\|(x,y)\right\|_1\le d\right\}. \tag{2.23}$$

Choose a constant $c\in(0,1]$ such that $c(x+1/3)\le 1$, $c(y+1/3)\le 1$, $(x,y)\in D$. Then, for every $(x,y)\in D$, using (2.22), Lemma 2.4, (A_1) and (A_2), we have

$$\begin{aligned}
A_n(x,y)(t)&=\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\,f\left(s,x(s)+\frac{1}{n},y(s)+\frac{1}{n}\right)ds\\
&=\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\,f\left(s,c\frac{x(s)+1/n}{c},c\frac{y(s)+1/n}{c}\right)ds\\
&\le\left(\frac{1}{c}\right)^{\beta_1}\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\,f\left(s,c\left(x(s)+\frac{1}{n}\right),c\frac{y(s)+1/n}{c}\right)ds\\
&\le\left(\frac{1}{c}\right)^{\beta_1}\left(\frac{1}{c}\right)^{\beta_2}\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\,f\left(s,c\left(x(s)+\frac{1}{n}\right),c\left(y(s)+\frac{1}{n}\right)\right)ds\\
&\le c^{\alpha_1-\beta_1-\beta_2}\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\left(x(s)+\frac{1}{n}\right)^{\alpha_1}f\left(s,1,c\left(y(s)+\frac{1}{n}\right)\right)ds\\
&\le c^{\alpha_1-\beta_1+\alpha_2-\beta_2}\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\left(x(s)+\frac{1}{n}\right)^{\alpha_1}\left(y(s)+\frac{1}{n}\right)^{\alpha_2}f(s,1,1)\,ds\\
&\le c^{\alpha_1-\beta_1+\alpha_2-\beta_2}\int_{1/n}^{1-1/n}H_n(\sigma_n(t),s)\left(\frac{1}{n}\right)^{\alpha_1}\left(\frac{1}{n}\right)^{\alpha_2}f(s,1,1)\,ds\\
&\le\mu_n c^{\alpha_1-\beta_1+\alpha_2-\beta_2}n^{-\alpha_1-\alpha 2}\int_{1/n}^{1-1/n}\left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right)f(s,1,1)\,ds\\
&\le\mu_n c^{\alpha_1-\beta_1+\alpha_2-\beta_2}n^{-\alpha_1-\alpha 2}\int_{1/n}^{1-1/n}s(1-s)\,f(s,1,1)\,ds\\
&\le\mu_n c^{\alpha_1-\beta_1+\alpha_2-\beta_2}n^{-\alpha_1-\alpha 2}\int_{0}^{1}s(1-s)\,f(s,1,1)\,ds=a\mu_n c^{\alpha_1-\beta_1+\alpha_2-\beta_2}n^{-\alpha_1-\alpha 2},
\end{aligned} \tag{2.24}$$

which implies that

$$\|A_n(x,y)\| \leq a\mu_n c^{\alpha_1-\beta_1+\alpha_2-\beta_2} n^{-\alpha_1-\alpha 2}, \tag{2.25}$$

that is, $A_n(D)$ is uniformly bounded. Similarly, using (2.22), Lemma 2.4, (A_1) and (A_3), we can show that $B_n(D)$ is also uniformly bounded. Thus, $T_n(D)$ is uniformly bounded. Now we show that $A_n(D)$ is equicontinuous. Define

$$\begin{aligned} \omega = \max\Bigg\{ & \max_{(t,x,y)\in[1/n,1-1/n]\times[0,d]\times[0,d]} f\left(t, x+\frac{1}{n}, y+\frac{1}{n}\right), \\ & \max_{(t,x,y)\in[1/n,1-1/n]\times[0,d]\times[0,d]} g\left(t, x+\frac{1}{n}, y+\frac{1}{n}\right)\Bigg\}. \end{aligned} \tag{2.26}$$

Let $t_1, t_2 \in [0,1]$ such that $t_1 \leq t_2$. Since $H_n(t,s)$ is uniformly continuous on $[1/n, 1-1/n] \times [1/n, 1-1/n]$, for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies

$$|H_n(\sigma_n(t_1), s) - H_n(\sigma_n(t_2), s)| < \frac{\varepsilon}{\omega(1-2/n)} \quad \text{for } s \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]. \tag{2.27}$$

For $(x,y) \in D$, using (2.22)–(2.27), we have

$$\begin{aligned} & \left|A_n(x,y)(t_1) - A_n(x,y)(t_2)\right| \\ & = \left|\int_{1/n}^{1-1/n} \left(H_n(\sigma_n(t_1), s) - H_n(\sigma_n(t_2), s) f\left(s, x(s)+\frac{1}{n}, y(s)+\frac{1}{n}\right)\right) ds\right| \\ & \leq \int_{1/n}^{1-1/n} |H_n(\sigma_n(t_1), s) - H_n(\sigma_n(t_2), s)| f\left(s, x(s)+\frac{1}{n}, y(s)+\frac{1}{n}\right) ds \\ & \leq \omega \int_{1/n}^{1-1/n} |H_n(\sigma_n(t_1), s) - H_n(\sigma_n(t_2), s)| ds \\ & < \omega \frac{\varepsilon}{\omega(1-2/n)} \int_{1/n}^{1-1/n} ds = \frac{\varepsilon}{(1-2/n)}\left(1-\frac{2}{n}\right) = \varepsilon. \end{aligned} \tag{2.28}$$

Hence,

$$\left|A_n(x,y)(t_1) - A_n(x,y)(t_2)\right| < \varepsilon, \quad \forall (x,y) \in D, \ |t_1 - t_2| < \delta, \tag{2.29}$$

which implies that $A_n(D)$ is equicontinuous. Similarly, using (2.22)–(2.27), we can show that $B_n(D)$ is also equicontinuous. Thus, $T_n(D)$ is equicontinuous. By Arzelà-Ascoli theorem, $T_n(D)$ is relatively compact. Hence, T_n is a compact operator.

Now we show that T_n is continuous. Let $(x_m, y_m), (x, y) \in P \times P$ such that $\|(x_m, y_m) - (x, y)\|_1 \to 0$ as $m \to +\infty$. Then by using (2.22) and Lemma 2.4, we have

$$\begin{aligned}
&\left|A_n(x_m, y_m)(t) - A_n(x, y)(t)\right| \\
&= \left|\int_{1/n}^{1-1/n} H_n(\sigma_n(t), s)\left(f\left(s, x_m(s) + \frac{1}{n}, y_m(s) + \frac{1}{n}\right) - f\left(s, x(s) + \frac{1}{n}, y(s) + \frac{1}{n}\right)\right) ds\right| \\
&\leq \int_{1/n}^{1-1/n} H_n(\sigma_n(t), s)\left|f\left(s, x_m(s) + \frac{1}{n}, y_m(s) + \frac{1}{n}\right) - f\left(s, x(s) + \frac{1}{n}, y(s) + \frac{1}{n}\right)\right| ds \\
&\leq \mu_n \int_{1/n}^{1-1/n}\left(s - \frac{1}{n}\right)\left(1 - \frac{1}{n} - s\right)\left|f\left(s, x_m(s) + \frac{1}{n}, y_m(s) + \frac{1}{n}\right) - f\left(s, x(s) + \frac{1}{n}, y(s) + \frac{1}{n}\right)\right| ds.
\end{aligned} \tag{2.30}$$

Consequently,

$$\begin{aligned}
&\|A_n(x_m, y_m) - A_n(x, y)\| \\
&\quad\leq \mu_n \int_{1/n}^{1-1/n}\left(s - \frac{1}{n}\right)\left(1 - \frac{1}{n} - s\right) \\
&\qquad\times\left|f\left(s, x_m(s) + \frac{1}{n}, y_m(s) + \frac{1}{n}\right) - f\left(s, x(s) + \frac{1}{n}, y(s) + \frac{1}{n}\right)\right| ds.
\end{aligned} \tag{2.31}$$

By Lebesgue dominated convergence theorem, it follows that

$$\|A_n(x_m, y_m) - A_n(x, y)\| \longrightarrow 0 \quad \text{as } m \longrightarrow +\infty. \tag{2.32}$$

Similarly, by using (2.22) and Lemma 2.4, we have

$$\|B_n(x_m, y_m) - B_n(x, y)\| \longrightarrow 0 \quad \text{as } m \longrightarrow +\infty. \tag{2.33}$$

From (2.32) and (2.33), it follows that

$$\|T_n(x_m, y_m) - T_n(x, y)\|_1 \longrightarrow 0 \quad \text{as } m \longrightarrow +\infty, \tag{2.34}$$

that is, $T_n : P \times P \to P \times P$ is continuous. Hence, $T_n : P \times P \to P \times P$ is completely continuous. □

3. Main Results

Proof of Theorem 1.1. Let $M = \max\{\mu_{n_0}, \max\{1, \alpha\}/(1 - \alpha\eta)\}$. Choose a constant $R > 0$ such that

$$R \geq \max\left\{(2aM)^{1/(1-\alpha_1-\alpha_2)}, (2bM)^{1/(1-\gamma_1-\gamma_2)}\right\}. \tag{3.1}$$

Choose a constant $c_1 \in (0,1]$ such that $c_1\,(x(t)+1/n_0) \le 1$, $c_1(y(t)+1/n_0) \le 1$, $(x,y) \in \partial\Omega_R \cap (P\times P)$, $t \in (0,1)$. For any $(x,y) \in \partial\Omega_R \cap (P\times P)$, using (2.22), (3.1), (A_1), and (A_2), we have

$$\begin{aligned}
A_n(x,y)(t) &= \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, x(s)+\frac{1}{n}, y(s)+\frac{1}{n}\right) ds \\
&= \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, c_1\frac{x(s)+1/n}{c_1}, c_1\frac{y(s)+1/n}{c_1}\right) ds \\
&\le \left(\frac{1}{c_1}\right)^{\beta_1} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, c_1\left(x(s)+\frac{1}{n}\right), c_1\frac{y(s)+1/n}{c_1}\right) ds \\
&\le \left(\frac{1}{c_1}\right)^{\beta_1}\left(\frac{1}{c_1}\right)^{\beta_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, c_1\left(x(s)+\frac{1}{n}\right), c_1\left(y(s)+\frac{1}{n}\right)\right) ds \\
&\le c_1^{\alpha_1-\beta_1-\beta_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\left(x(s)+\frac{1}{n}\right)^{\alpha_1} f\left(s, 1, c_1\left(y(s)+\frac{1}{n}\right)\right) ds \\
&\le c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\left(x(s)+\frac{1}{n}\right)^{\alpha_1}\left(y(s)+\frac{1}{n}\right)^{\alpha_2} f(s,1,1)\, ds \\
&\le c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\,(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds \\
&\le \mu_n c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{1/n}^{1-1/n} \left(s-\frac{1}{n}\right)\left(1-\frac{1}{n}-s\right)(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds \\
&\le \mu_n c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{1/n}^{1-1/n} s(1-s)\,(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds \\
&\le \mu_n c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{0}^{1} s(1-s)\,(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds \\
&\le M c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{0}^{1} s(1-s)\,(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds.
\end{aligned} \tag{3.2}$$

Since,

$$\begin{aligned}
&M c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2} \int_{0}^{1} s(1-s)\,(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds \\
&\quad\le M \int_{0}^{1} s(1-s)\,(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)\, ds \\
&\quad\le aMR^{\alpha_1+\alpha_2} \le \frac{R}{2},
\end{aligned} \tag{3.3}$$

it follows that

$$\left\|A_n(x,y)\right\| \le \frac{R}{2} = \frac{\left\|(x,y)\right\|_1}{2}, \quad \forall (x,y) \in \partial\Omega_R \cap (P \times P). \tag{3.4}$$

Similarly, using (2.22), (3.1), (A_1), and (A_3), we have

$$\left\|B_n(x,y)\right\| \le \frac{\left\|(x,y)\right\|_1}{2}, \quad \forall (x,y) \in \partial\Omega_R \cap (P \times P). \tag{3.5}$$

From (3.4), and (3.5), it follows that

$$\left\|T_n(x,y)\right\|_1 \le \left\|(x,y)\right\|_1, \quad \forall (x,y) \in \partial\Omega_R \cap (P \times P). \tag{3.6}$$

Choose a real constant $r \in (0,R)$ such that

$$r \le \min\left\{(2aM)^{1/(1-\beta_1-\beta_2)}, (2bM)^{1/(1-\rho_1-\rho_2)}\right\}. \tag{3.7}$$

Choose a constant $c_2 \in (0,1]$ such that $c_2(x(t)+1/n_0) \le 1$, $c_2(y(t)+1/n_0) \le 1$, $(x,y) \in \partial\Omega_r \cap (P \times P)$, $t \in (0,1)$. For any $(x,y) \in \partial\Omega_r \cap (P \times P)$, using (2.22), (3.7), (A_1), and (A_2), we have

$$\begin{aligned}
A_n(x,y)(t) &= \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, x(s)+\frac{1}{n}, y(s)+\frac{1}{n}\right) ds \\
&= \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, c_2\frac{x(s)+1/n}{c_2}, c_2\frac{y(s)+1/n}{c_2}\right) ds \\
&\ge \left(\frac{1}{c_2}\right)^{\alpha_1} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, c_2\left(x(s)+\frac{1}{n}\right), c_2\frac{y(s)+1/n}{c_2}\right) ds \\
&\ge \left(\frac{1}{c_2}\right)^{\alpha_1}\left(\frac{1}{c_2}\right)^{\alpha_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\, f\left(s, c_2\left(x(s)+\frac{1}{n}\right), c_2\left(y(s)+\frac{1}{n}\right)\right) ds \\
&\ge c_2^{\beta_1-\alpha_1-\alpha_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\left(x(s)+\frac{1}{n}\right)^{\beta_1} f\left(s, 1, c_2\left(y(s)+\frac{1}{n}\right)\right) ds \\
&\ge c_2^{\beta_1-\alpha_1+\beta_2-\alpha_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)\left(x(s)+\frac{1}{n}\right)^{\beta_1}\left(y(s)+\frac{1}{n}\right)^{\beta_2} f(s,1,1)\, ds \\
&\ge c_2^{\beta_1-\alpha_1+\beta_2-\alpha_2} \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)(x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1)\, ds \ge \frac{r}{2}.
\end{aligned} \tag{3.8}$$

We used the fact that

$$
\begin{aligned}
c_2^{\beta_1-\alpha_1+\beta_2-\alpha_2}&\int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)(x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1)\,ds \\
&\geq \int_{1/n}^{1-1/n} H_n(\sigma_n(t),s)(x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1)\,ds \\
&\geq aMr^{\beta_1+\beta_2}.
\end{aligned}
\tag{3.9}
$$

Thus,

$$
\|A_n(x,y)\| \geq \frac{\|(x,y)\|_1}{2}, \quad \forall (x,y) \in \partial\Omega_r \cap (P \times P). \tag{3.10}
$$

Similarly, using (2.22), (3.7), (A_1) and (A_3), we have,

$$
\|B_n(x,y)\| \geq \frac{\|(x,y)\|_1}{2}, \quad \forall (x,y) \in \partial\Omega_r \cap (P \times P). \tag{3.11}
$$

From (3.10) and (3.11), it follows that

$$
\|T_n(x,y)\|_1 \geq \|(x,y)\|_1, \quad \forall (x,y) \in \partial\Omega_r \cap (P \times P). \tag{3.12}
$$

Hence by Lemma 2.1, T_n has a fixed point $(x_n, y_n) \in (P \times P) \cap (\overline{\Omega}_R \setminus \Omega_r)$, that is,

$$
x_n = A_n(x_n, y_n), \qquad y_n = B_n(x_n, y_n). \tag{3.13}
$$

Moreover, by (3.4), (3.5), (3.10) and (3.11), we have

$$
\begin{aligned}
\frac{r}{2} &\leq \|x_n\| \leq \frac{R}{2}, \\
\frac{r}{2} &\leq \|y_n\| \leq \frac{R}{2}.
\end{aligned}
\tag{3.14}
$$

Since (x_n, y_n) is a solution of the system (2.19), hence x_n and y_n are concave on $[1/n, 1-1/n]$. Moreover, $\max_{t\in[1/n,1-1/n]} x_n(t) = \|x_n\|$ and $\max_{t\in[1/n,1-1/n]} y_n(t) = \|y_n\|$. For $h \in (1/n, 1/2)$, using result (2.2) and (3.14), we have

$$
\begin{aligned}
\frac{rh}{2} &\leq x_n(t) \leq \frac{R}{2}, \quad \forall t \in [h, 1-h], \\
\frac{rh}{2} &\leq y_n(t) \leq \frac{R}{2}, \quad \forall t \in [h, 1-h],
\end{aligned}
\tag{3.15}
$$

which implies that $\{(x_n, y_n)\}$ is uniformly bounded on $[h, 1-h]$. Now we show that $\{(x_n, y_n)\}$ is equicontinuous on $[h, 1-h]$. Choose $\eta \in (h, 1-h)$ and $0 < \alpha < (1-2h)/(\eta-h)$ and consider the integral equation

$$x_n(t) = \frac{x_n(1-h) - \alpha\, x_n(\eta) - (1-\alpha)\, x_n(h)}{1-2h+\alpha h-\alpha\eta}(t-h) + x_n(h) + \int_h^{1-h} H_{h^{-1}}(t,s)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds, \quad t \in [h, 1-h]. \tag{3.16}$$

Using Lemma 2.3, we have

$$\begin{aligned} x_n(t) &= \frac{x_n(1-h) - \alpha\, x_n(\eta) - (1-\alpha)\, x_n(h)}{1-2h+\alpha h-\alpha\eta}(t-h) + x_n(h) \\ &+ \frac{1-h-t}{1-2h}\int_h^t (s-h)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds \\ &+ \frac{t-h}{1-2h}\int_t^{1-h} (1-h-s)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds \\ &+ \frac{\alpha(t-h)}{1-2h+\alpha h-\alpha\eta}\int_h^{1-h} G_{h^{-1}}(\eta,s)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds, \quad t \in [h, 1-h]. \end{aligned} \tag{3.17}$$

Differentiating with respect to t, we obtain

$$\begin{aligned} x_n'(t) &= \frac{x_n(1-h) - \alpha\, x_n(\eta) - (1-\alpha)\, x_n(h)}{1-2h+\alpha h-\alpha\eta} \\ &- \frac{1}{1-2h}\int_h^t (s-h)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds \\ &+ \frac{1}{1-2h}\int_t^{1-h} (1-h-s)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds \\ &+ \frac{\alpha}{1-2h+\alpha h-\alpha\eta}\int_h^{1-h} G_{h^{-1}}(\eta,s)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds, \quad t \in [h, 1-h], \end{aligned} \tag{3.18}$$

which implies that

$$\begin{aligned} \left|x_n'(t)\right| &\le \frac{(1+\alpha)R}{1-2h+\alpha h-\alpha\eta} + \int_h^{1-h} f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds \\ &+ \frac{\alpha}{1-2h+\alpha h-\alpha\eta}\int_h^{1-h} G_{h^{-1}}(\eta,s)\, f\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right) ds, \quad t \in [h, 1-h], \end{aligned} \tag{3.19}$$

In view of (A_2) and (3.15), we have

$$\left|x_n'(t)\right| \leq \frac{(1+\alpha)R}{1-2h+\alpha h-\alpha\eta} + c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2}\left(\frac{hr}{2}\right)^{\alpha_1+\alpha_2}\int_h^{1-h} f(s,1,1)\,ds$$
$$+\frac{\alpha}{1-2h+\alpha h-\alpha\eta}c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2}\left(\frac{hr}{2}\right)^{\alpha_1+\alpha_2}\int_h^{1-h} G_{h^{-1}}(\eta,s)\, f(s,1,1)\,ds, \quad t\in[h,1-h], \tag{3.20}$$

which implies that

$$\|x_n'\| \leq \frac{(1+\alpha)R}{1-2h+\alpha h-\alpha\eta} + c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2}\left(\frac{hr}{2}\right)^{\alpha_1+\alpha_2}\int_h^{1-h} f(s,1,1)\,ds$$
$$+\frac{\alpha}{1-2h+\alpha h-\alpha\eta}c_1^{\alpha_1-\beta_1+\alpha_2-\beta_2}\left(\frac{hr}{2}\right)^{\alpha_1+\alpha_2}\int_h^{1-h} G_{h^{-1}}(\eta,s)\, f(s,1,1)\,ds, \quad t\in[h,1-h]. \tag{3.21}$$

Similarly, consider the integral equation

$$y_n(t) = \frac{y_n(1-h)-\alpha\, y_n(\eta)-(1-\alpha)\,y_n(h)}{1-2h+\alpha h-\alpha\eta}(t-h)+y_n(h)$$
$$+\int_h^{1-h} H_{h^{-1}}(t,s)\, g\left(s, x_n(s)+\frac{1}{n}, y_n(s)+\frac{1}{n}\right)ds, \quad t\in[h,1-h], \tag{3.22}$$

using (A_3) and (3.15), we can show that

$$\|y_n'\| \leq \frac{(1+\alpha)R}{1-2h+\alpha h-\alpha\eta} + c_1^{\gamma_1-\rho_1+\gamma_2-\rho_2}\left(\frac{hr}{2}\right)^{\gamma_1+\gamma_2}\int_h^{1-h} g(s,1,1)\,ds$$
$$+\frac{\alpha}{1-2h+\alpha h-\alpha\eta}c_1^{\gamma_1-\rho_1+\gamma_2-\rho_2}\left(\frac{hr}{2}\right)^{\gamma_1+\gamma_2}\int_h^{1-h} G_{h^{-1}}(\eta,s)\, g(s,1,1)\,ds, \quad t\in[h,1-h]. \tag{3.23}$$

In view of (3.21) and (3.23), $\{(x_n, y_n)\}$ is equicontinuous on $[h, 1-h]$. Hence by Arzelà-Ascoli theorem, the sequence $\{(x_n, y_n)\}$ has a subsequence $\{(x_{n_k}, y_{n_k})\}$ converging uniformly on $[h, 1-h]$ to $(x, y) \in (P\times P)\cap(\overline{\Omega}_R \setminus \Omega_r)$. Let us consider the integral equation

$$x_{n_k}(t) = \frac{x_{n_k}(1-h)-\alpha\, x_{n_k}(\eta)-(1-\alpha)\,x_{n_k}(h)}{1-2h+\alpha h-\alpha\eta}(t-h)+x_{n_k}(h)$$
$$+\int_h^{1-h} H_{h^{-1}}(t,s)\, f\left(s, x_{n_k}(s)+\frac{1}{n_k}, y_{n_k}(s)+\frac{1}{n_k}\right)ds, \quad t\in[h,1-h]. \tag{3.24}$$

Letting $n_k \to \infty$, we have

$$x(t) = \frac{x(1-h) - \alpha\, x(\eta) - (1-\alpha)\, x(h)}{1 - 2h + \alpha h - \alpha\eta}(t-h) + x(h) + \int_h^{1-h} H_{h^{-1}}(t,s) f(s, x(s), y(s))\, ds, \quad t \in [h, 1-h]. \tag{3.25}$$

Differentiating twice with respect to t, we have

$$-x''(t) = f(t, x(t), y(t)), \quad t \in [h, 1-h]. \tag{3.26}$$

Letting $h \to 0$, we have

$$-x''(t) = f(t, x(t), y(t)), \quad t \in (0,1). \tag{3.27}$$

Similarly, consider the integral equation

$$y_{n_k}(t) = \frac{y_{n_k}(1-h) - \alpha\, y_{n_k}(\eta) - (1-\alpha)\, y_{n_k}(h)}{1 - 2h + \alpha h - \alpha\eta}(t-h) + y_{n_k}(h) + \int_h^{1-h} H_{h^{-1}}(t,s)\, g\left(s, x_{n_k}(s) + \frac{1}{n_k}, y_{n_k}(s) + \frac{1}{n_k}\right) ds, \quad t \in [h, 1-h], \tag{3.28}$$

we can show that

$$-y''(t) = g(t, x(t), y(t)), \quad t \in (0,1). \tag{3.29}$$

Now, we show that (x, y) also satisfies the boundary conditions. Since,

$$\begin{aligned} x(0) &= \lim_{n_k \to \infty} x\left(\frac{1}{n_k}\right) = \lim_{n_k \to \infty} x_{n_k}\left(\frac{1}{n_k}\right) = 0, \\ x(1) &= \lim_{n_k \to \infty} x\left(1 - \frac{1}{n_k}\right) = \lim_{n_k \to \infty} x_{n_k}\left(1 - \frac{1}{n_k}\right) = \lim_{n_k \to \infty} \alpha x_{n_k}(\eta) = \alpha x(\eta). \end{aligned} \tag{3.30}$$

Similarly, we can show that

$$y(t) = 0, \qquad y(1) = \alpha y(\eta). \tag{3.31}$$

Equations (3.27)–(3.31) imply that (x, y) is a solution of the system (1.3). Moreover, (x, y) is positive. In fact, by (3.27) x is concave and by Lemma 2.2

$$x(1) \geq \min_{t \in [\eta, 1]} x(t) \geq \gamma \|x\| > 0, \tag{3.32}$$

implies that $x(t) > 0$ for all $t \in (0,1)$. Similarly, $y(t) > 0$ for all $t \in (0,1)$. The proof of Theorem 1.1 is complete. □

Example 3.1. Let

$$f(t,x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} t^{p_i}(1-t)^{q_j} x^{r_i} y^{s_j},$$
$$g(t,x,y) = \sum_{k=1}^{m'}\sum_{l=1}^{n'} t^{p'_k}(1-t)^{q'_l} x^{r'_k} y^{s'_l}, \tag{3.33}$$

where the real constants p_i, q_j, r_i, s_j satisfy $p_i, q_j > -2$, $r_i, s_j < 1$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, with $\max_{1\le i\le m} r_i + \max_{1\le j\le n} s_j < 1$ and the real constants p'_k, q'_l, r'_k, s'_l satisfy $p'_k, q'_l > -2$, $r'_k, s'_l < 1$, $k = 1,2,\ldots,m'$; $l = 1,2,\ldots,n'$, with $\max_{1\le k\le m'} r'_k + \max_{1\le l\le n'} s'_l < 1$. Clearly, f and g satisfy the assumptions (A_1)–(A_3). Hence, by Theorem 1.1, the system (1.3) has a positive solution.

Acknowledgement

Research of R. A. Khan is supported by HEC, Pakistan, Project 2- 3(50)/PDFP/HEC/2008/1.

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