Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 393259, 9 pages
doi:10.1155/2009/393259

Research Article

Positive Solutions for Some Beam Equation Boundary Value Problems

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Received 2 September 2009; Accepted 1 November 2009

Recommended by Wenming Zou

A new fixed point theorem in a cone is applied to obtain the existence of positive solutions of some fourth-order beam equation boundary value problems with dependence on the first-order derivative $u^{(iv)}(t) = f(t, u(t), u'(t)), 0 < t < 1, u(0) = u(1) = u''(0) = u''(1) = 0$, where $f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous.

1. Introduction

It is well known that beam is one of the basic structures in architecture. It is greatly used in the designing of bridge and construction. Recently, scientists bring forward the theory of combined beams. That is to say, we can bind up some stratified structure copings into one global combined beam with rock bolts. The deformations of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by following equation of deflection curve:

$$\frac{d^2}{dx^2}\left(EI_z\frac{d^2v}{dx^2}\right) = q(x), \tag{1.1}$$

where E is Yang's modulus constant, I_z is moment of inertia with respect to z axes, determined completely by the beam's shape cross-section. Specially, $I_z = bh^3/12$ if the cross-section is a rectangle with a height of h and a width of b. Also, $q(x)$ is loading at x. If the loading of beam considered is in relation to deflection and rate of change of deflection, we need to research the more general equation

$$u^{(4)}(x) = f\left(x, u(x), u'(x)\right). \tag{1.2}$$

According to the forms of supporting, various boundary conditions should be considered. Solving corresponding boundary value problems, one can obtain the expression of deflection curve. It is the key in design of constants of beams and rock bolts.

Owing to its importance in physics and engineering, the existence of solutions to this problem has been studied by many authors, see [1–10]. However, in practice, only its positive solution is significant. In [1, 9, 11, 12], Aftabizadeh, Del Pino and Manásevich, Gupta, and Pao showed the existence of positive solution for

$$u^{(iv)}(t) = f\left(t, u(t), u''(t)\right) \tag{1.3}$$

under some growth conditions of f and a nonresonance condition involving a two-parameter linear eigenvalue problem. All of these results are based on the Leray-Schauder continuation method and topological degree.

The lower and upper solution method has been studied for the fourth-order problem by several authors [2, 3, 7, 8, 13, 14]. However, all of these authors consider only an equation of the form

$$u^{(iv)}(t) = f(t, u(t)), \tag{1.4}$$

with diverse kind of boundary conditions. In [10], Ehme et al. gave some sufficient conditions for the existence of a solution of

$$u^{(iv)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right) \tag{1.5}$$

with some quite general nonlinear boundary conditions by using the lower and upper solution method. The conditions assume the existence of a strong upper and lower solution pair.

Recently, Krasnosel'skii's fixed point theorem in a cone has much application in studying the existence and multiplicity of positive solutions for differential equation boundary value problems, see [3, 6]. With this fixed point theorem, Bai and Wang [6] discussed the existence, uniqueness, multiplicity, and infinitely many positive solutions for the equation of the form

$$u^{(iv)}(t) = \lambda f(t, u(t)), \tag{1.6}$$

where $\lambda > 0$ is a constant.

In this paper, via a new fixed point theorem in a cone and concavity of function, we show the existence of positive solutions for the following problem:

$$u^{(iv)}(t) = f\left(t, u(t), u'(t)\right), \quad 0 < t < 1,$$
$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.7}$$

where $f : [0,1] \times [0,+\infty) \times R \to [0,+\infty)$ is continuous.

We point out that positive solutions of (1.7) are concave and this concavity provides lower bounds on positive concave functions of their maximum, which can be used in defining a cone on which a positive operator is defined, to which a new fixed point theorem in a cone due to Bai and Ge [5] can be applied to obtain positive solutions.

2. Fixed Point Theorem in a Cone

Let X be a Banach space and $P \subset X$ a cone. Suppose $\alpha, \beta : X \to R^+$ are two continuous nonnegative functionals satisfying

$$\alpha(\lambda x) \leq |\lambda|\alpha(x), \quad \beta(\lambda x) \leq |\lambda|\beta(x), \quad \text{for } x \in X, \lambda \in [0,1],$$
$$M_1 \max\{\alpha(x), \beta(x)\} \leq \|x\| \leq M_2 \max\{\alpha(x), \beta(x)\}, \quad \text{for } x \in X, \tag{2.1}$$

where M_1, M_2 are two positive constants.

Lemma 2.1 (see [5]). *Let $r_2 > r_1 > 0, L_2 > L_1 > 0$ are constants and*

$$\Omega_i = \left\{x \in X \mid \alpha(x) < r_i, \beta(x) < L_i\right\}, \quad i = 1, 2 \tag{2.2}$$

are two open subsets in X such that $\theta \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. In addition, let

$$C_i = \left\{x \in X \mid \alpha(x) = r_i, \beta(x) \leq L_i\right\}, \quad i = 1, 2;$$
$$D_i = \left\{x \in X \mid \alpha(x) \leq r_i, \beta(x) = L_i\right\}, \quad i = 1, 2. \tag{2.3}$$

Assume $T : P \to P$ is a completely continuous operator satisfying

(S_1) $\alpha(Tx) \leq r_1$, $x \in C_1 \cap P$; $\beta(Tx) \leq L_1, x \in D_1 \cap P$; $\alpha(Tx) \geq r_2, x \in C_2 \cap P; \beta(Tx) \geq L_2, x \in D_2 \cap P$;

or

(S_2) $\alpha(Tx) \geq r_1, x \in C_1 \cap P$; $\beta(Tx) \geq L_1, x \in D_1 \cap P \alpha(Tx) \leq r_2, x \in C_2 \cap P; \beta(Tx) \leq L_2, x \in D_2 \cap P$,

then T has at least one fixed point in $(\overline{\Omega_2} \setminus \Omega_1) \cap P$.

3. Existence of Positive Solutions

In this section, we are concerned with the existence of positive solutions for the fourth-order two-point boundary value problem (1.7).

Let $X = C^1[0,1]$ with $\|u\| = \max\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|\}$ be a Banach space, $P = \{u \in X \mid u(t) \ge 0, u \text{ is concave on } [0,1]\} \subset X$ a cone. Define functionals

$$\alpha(u) = \max_{0\le t\le 1}|u(t)|, \quad \beta(u) = \max_{0\le t\le 1}\left|u'(t)\right|, \quad \text{for } u \in X, \tag{3.1}$$

then $\alpha, \beta : X \to R^+$ are two continuous nonnegative functionals such that

$$\|u\| = \max\{\alpha(u), \beta(u)\} \tag{3.2}$$

and (2.1) hold.

Denote by $G(t,s)$ Green's function for boundary value problem

$$\begin{gathered} -y''(t) = 0, \quad 0 < t < 1, \\ y(0) = y(1) = 0. \end{gathered} \tag{3.3}$$

Then $G(t,s) \ge 0$, for $0 \le t, s \le 1$, and

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{3.4}$$

Let

$$\begin{aligned} M &= \max_{0\le t\le 1}\iint_0^1 G(t,s)G(s,x)dx\,ds, \\ N &= \max_{0\le t\le 1}\int_0^1\int_{1/4}^{3/4} G(t,s)G(s,x)dx\,ds, \\ A &= \max\left\{\iint_0^1 (1-s)G(s,x)\,dx\,ds, \iint_0^1 sG(s,x)\,dx\,ds\right\}, \\ \mathrm{B} &= \max\left\{\int_0^1\int_h^{1-h} (1-s)G(s,x)\,dx\,ds, \int_0^1\int_h^{1-h} sG(s,x)\,dx\,ds\right\}. \end{aligned} \tag{3.5}$$

However, (1.7) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = Tu(t) := \int_0^1\left[\int_0^1 G(t,s)G(s,x)f\left(x, u(x), u'(x)\right)dx\right]ds. \tag{3.6}$$

It is well know that $T : P \to P$ is completely continuous.

Theorem 3.1. *Suppose there are four constants* $r_2 > r_1 > 0, L_2 > L_1 > 0$ *such that* $\max\{r_1, L_1\} \leq \min\{r_2, L_2\}$ *and the following assumptions hold:*

(A_1) $f(t, x_1, x_2) \geq \max\{r_1/M, L_1/A\}$, *for* $(t, x_1, x_2) \in [0,1] \times [0, r_1] \times [-L_1, L_1]$;

(A_2) $f(t, x_1, x_2) \leq \min\{r_2/M, L_2/A\}$, *for* $(t, x_1, x_2) \in [0,1] \times [0, r_2] \times [-L_2, L_2]$.

Then, (1.7) *has at least one positive solution* $u(t)$ *such that*

$$r_1 \leq \max_{0\leq t\leq 1} u(t) \leq r_2 \quad \text{or} \quad L_1 \leq \max_{0\leq t\leq 1}\left|u'(t)\right| \leq L_2. \tag{3.7}$$

Proof. Let

$$\Omega_i = \left\{u \in X \mid \alpha(u) < r_i,\ \beta(u) < L_i\right\}, \quad i = 1, 2, \tag{3.8}$$

be two bounded open subsets in X. In addition, let

$$\begin{aligned} C_i &= \left\{u \in X \mid \alpha(u) = r_i,\ \beta(u) \leq L_i\right\}, \quad i = 1, 2; \\ D_i &= \left\{u \in X \mid \alpha(u) \leq r_i,\ \beta(u) = L_i\right\}, \quad i = 1, 2. \end{aligned} \tag{3.9}$$

For $u \in C_1 \cap P$, by (A_1), there is

$$\begin{aligned} \alpha(Tu) &= \max_{t\in[0,1]}\left|\iint_0^1 G(t,s)G(s,x)f\left(x, u(x), u'(x)\right)dx\,ds\right| \\ &\geq \frac{r_1}{M}\cdot\max_{t\in[0,1]}\left|\iint_0^1 G(t,s)G(s,x)\,dx\,ds\right| = r_1. \end{aligned} \tag{3.10}$$

For $u \in P$, because $T : P \to P$, so $Tu \in P$, that is to say Tu concave on $[0,1]$, it follows that

$$\max_{t\in[0,1]}\left|(Tu)'(t)\right| = \max\left\{\left|(Tu)'(0)\right|, \left|(Tu)'(1)\right|\right\}. \tag{3.11}$$

Combined with (A_1) and $f \geq 0$, for $u \in D_1 \cap P$, there is

$$\begin{aligned}
\beta(Tu) &= \max_{t\in[0,1]} \left|(Tu)'(t)\right| \\
&= \max_{t\in[0,1]} \left| -\int_0^t s \int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds \right. \\
&\qquad \left. + \int_t^1 (1-s) \int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds \right| \\
&= \max\left\{ \int_0^1 (1-s) \int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds, \right. \\
&\qquad \left. \int_0^1 s \int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds \right\} \\
&\geq \frac{L_1}{A} \cdot \max\left\{ \iint_0^1 (1-s) G(s,x)\, dx\, ds, \iint_0^1 s G(s,x)\, dx\, ds \right\} \\
&= \frac{L_1}{A} \cdot A = L_1.
\end{aligned} \tag{3.12}$$

For $u \in C_2 \cap P$, by (A_2), there is

$$\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,1]} \left| \iint_0^1 G(t,s) G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds \right| \\
&\leq \max_{t\in[0,1]} \iint_0^1 G(t,s) G(s,x) \cdot \frac{r_2}{M}\, dx\, ds \\
&= \frac{r_2}{M} \cdot \max_{t\in[0,1]} \iint_0^1 G(t,s) G(s,x)\, dx\, ds = r_2.
\end{aligned} \tag{3.13}$$

For $u \in D_2 \cap P$, by (A_2), there is

$$\begin{aligned}
\beta(Tu) &= \max\left\{ \int_0^1 (1-s) \int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds, \right. \\
&\qquad \left. \int_0^1 s \int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds \right\} \\
&\leq \frac{L_2}{A} \cdot \max\left\{ \iint_0^1 (1-s) G(s,x)\, dx\, ds, \iint_0^1 s G(s,x)\, dx\, ds \right\} \\
&= \frac{L_2}{A} \cdot A = L_2.
\end{aligned} \tag{3.14}$$

Now, Lemma 2.1 implies there exists $u \in (\overline{\Omega}_2 \setminus \Omega_1) \cap P$ such that $u = Tu$, namely, (1.7) has at least one positive solution $u(t)$ such that

$$r_1 \leq \alpha(u) \leq r_2 \qquad \text{or} \qquad L_1 \leq \beta(u) \leq L_2, \tag{3.15}$$

that is,

$$r_1 \leq \max_{0 \leq t \leq 1} u(t) \leq r_2 \qquad \text{or} \qquad L_1 \leq \max_{0 \leq t \leq 1} \left| u'(t) \right| \leq L_2. \tag{3.16}$$

The proof is complete. □

Theorem 3.2. *Suppose there are five constants* $0 < r_1 < r_2, 0 < L_1 < L_2, 0 \leq h < 1/2$ *such that* $\max\{r_1/N, L_1/B\} \leq \min\{r_2/M, L_2/A\}$, *and the following assumptions hold*

(A_3) $f(t, x_1, x_2) \geq r_1/N$, *for* $(t, x_1, x_2) \in [1/4, 3/4] \times [r_1/4, r_1] \times [-L_1, L_1]$;

(A_4) $f(t, x_1, x_2) \geq L_1/B$, *for* $(t, x_1, x_2) \in [h, 1-h] \times [0, r_1] \times [-L_1, L_1]$;

(A_5) $f(t, x_1, x_2) \leq \min\{r_2/M, L_2/A\}$, *for* $(t, x_1, x_2) \in [0, 1] \times [0, r_2] \times [-L_2, L_2]$.

Then, (1.7) *has at least one positive solution* $u(t)$ *such that*

$$r_1 \leq \max_{0 \leq t \leq 1} u(t) \leq r_2 \quad \text{or} \quad L_1 \leq \max_{0 \leq t \leq 1} \left| u'(t) \right| \leq L_2. \tag{3.17}$$

Proof. We just need notice the following difference to the proof of Theorem 3.1.

For $u \in C_1 \cap P$, the concavity of u implies that $u(t) \geq (1/4)\alpha(u) = r_1/4$ for $t \in [1/4, 3/4]$. By (A_3), there is

$$\begin{aligned}
\alpha(Tu) &= \max_{t \in [0,1]} \left| \int\int_0^1 G(t,s) G(s,x) f(x, u(x), u'(x)) dx\, ds \right| \\
&\geq \max_{t \in [0,1]} \left| \int_0^1 \int_{1/4}^{3/4} G(t,s) G(s,x) f(x, u(x), u'(x)) dx\, ds \right| \\
&\geq \max_{t \in [0,1]} \left| \int_0^1 \int_{1/4}^{3/4} G(t,s) G(s,x) \cdot \frac{r_1}{N} dx\, ds \right| \\
&= \frac{r_1}{N} \cdot \max_{t \in [0,1]} \left| \int_0^1 \int_{1/4}^{3/4} G(t,s) G(s,x) dx\, ds \right| = r_1.
\end{aligned} \tag{3.18}$$

For $u \in D_1 \cap P$, by (A_4), there is

$$\begin{aligned}
\beta(Tu) &= \max\left\{\int_0^1 (1-s)\int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds, \right.\\
&\qquad\qquad \left. \int_0^1 s\int_0^1 G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds\right\}\\
&\geq \max\left\{\int_0^1 (1-s)\int_h^{1-h} G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds, \right.\\
&\qquad\qquad \left. \int_0^1 s\int_h^{1-h} G(s,x) f\left(x, u(x), u'(x)\right) dx\, ds\right\}\\
&\geq \frac{L_1}{B}\cdot \max\left\{\int_0^1\int_h^{1-h} (1-s) G(s,x) dx\, ds, \int_0^1\int_h^{1-h} sG(s,x) dx\, ds\right\}\\
&= \frac{L_1}{B}\cdot B = L_1
\end{aligned} \tag{3.19}$$

The rest of the proof is similar to Theorem 3.1 and the proof is complete. □

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