Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 491952, 20 pages
doi:10.1155/2009/491952

Research Article

Existence of Solutions for Fourth-Order Four-Point Boundary Value Problem on Time Scales

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Received 11 April 2009; Revised 8 July 2009; Accepted 28 July 2009

Recommended by Irena Rachůnková

We present an existence result for fourth-order four-point boundary value problem on time scales. Our analysis is based on a fixed point theorem due to Krasnoselskii and Zabreiko.

1. Introduction

Very recently, Karaca [1] investigated the following fourth-order four-point boundary value problem on time scales:

$$\begin{gathered}
y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = f\left(t, y(\sigma(t)), y^{\Delta^2}(t)\right), \\
y\left(\sigma^4(b)\right) = 0, \qquad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\
\gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \qquad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0,
\end{gathered} \tag{1.1}$$

for $t \in [a,b] \subset \mathbb{T}$, $a \leq \xi_1 \leq \xi_2 \leq \sigma(b)$, and $f \in C([a,b] \times \mathbb{R} \times \mathbb{R}) \times \mathbb{R})$. And the author made the following assumptions:

(A_1) $\alpha, \beta, \gamma, \delta, \zeta, \eta \geq 0$, and $a \leq \xi_1 \leq \xi_2 \leq \sigma(b)$,

(A_2) $q(t) \geq 0$. If $q(t) \equiv 0$, then $\gamma + \zeta > 0$.

The following key lemma is provided in [1].

Lemma 1.1 (see [1, Lemma 2.5]). *Assume that conditions (A_1) and (A_2) are satisfied. If $h \in C[a,b]$, then the boundary value problem*

$$
\begin{gathered}
y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = h(t), \quad t \in [a,b], \\
y\left(\sigma^4(b)\right) = 0, \qquad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\
\gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \qquad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0
\end{gathered}
\tag{1.2}
$$

has a unique solution

$$
y(t) = \int_a^{\sigma^4(b)} G_1(t,\xi) \int_{\xi_1}^{\xi_2} G_2(t,s)h(s)\Delta s\, \Delta\xi, \tag{1.3}
$$

where

$$
G_1(t,s) = \frac{1}{d}\begin{cases} \left(\sigma^4(b) - \sigma(s)\right)\left(\alpha(t-a)+\beta\right), & t \le s, \\ \left(\sigma^4(b) - t\right)\left(\alpha(\sigma(s)-a)+\beta\right), & t \ge \sigma(s), \end{cases} \tag{1.4}
$$

$$
G_2(t,s) = \frac{1}{D}\begin{cases} \psi(\sigma(s))\varphi(t), & t \le s, \\ \psi(t)\varphi(\sigma(s)), & t \ge \sigma(s). \end{cases} \tag{1.5}
$$

Here $D = \zeta\phi(\xi_1) - \eta\psi^{\Delta}(\xi_1) = \delta\varphi(\xi_2) + \gamma\varphi(\xi_2), d = \beta + \alpha(\sigma^4(b) - a)$, and $\varphi(t), \psi(t)$ are given as follows:

$$
\begin{gathered}
\varphi(t) = \eta + \zeta(t - \xi_1) + \int_{\xi_1}^{t}\int_{\xi_1}^{\tau} q(s)\varphi(\sigma(s))\Delta s\, \Delta\tau, \\
\psi(t) = \delta + \gamma(\xi_2 - t) + \int_{t}^{\xi_2}\int_{\tau}^{\xi_2} q(s)\psi(\sigma(s))\Delta s\, \Delta\tau.
\end{gathered}
\tag{1.6}
$$

Unfortunately, this lemma is wrong. Without considering the whole interval $[a, \sigma(b)]$, the author only considers $[\xi_1, \xi_2]$ in the Green's function $G_2(t,s)$. Thus, the expression of $y(t)$ (1.3) *which is a solution to BVP* (1.2) *is incorrect. In fact,if one takes $\mathbb{T} = \mathbb{R}, q(t) = 0, a = 0, \sigma^4(b) = 1,$*

$\alpha = 1, \beta = 0, f(t, y, y^{\Delta^2}) \equiv f(t, y)$, *then* (1.1) *reduces to the following boundary value problem:*

$$\begin{gathered}
y''''(t) = f\big(t, y(t)\big), \quad 0 < t < 1, \\
y(0) = y(1) = 0, \\
\gamma y''(\xi_1) - \delta y'''(\xi_1) = 0, \qquad \zeta y''(\xi_2) + \eta y'''(\xi_2) = 0.
\end{gathered} \tag{1.7}$$

The counterexample is given by [2], from which one can see clearly that [1, Lemma 2.5] is wrong. If one takes $q(t) = q$, here $q > 0$ is a constant, then (1.1) *reduces to the following fourth-order four-point boundary value problem on time scales:*

$$\begin{gathered}
y^{\Delta^4}(t) - q y^{\Delta^2}(\sigma(t)) = f\Big(t, y(\sigma(t)), y^{\Delta^2}(t)\Big), \quad t \in [a, b] \subset \mathbb{T}, \\
y\Big(\sigma^4(b)\Big) = 0, \qquad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\
\gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \qquad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0.
\end{gathered} \tag{1.8}$$

The purpose of this paper is to establish some existence criteria of solution for BVP (1.8) which is a special case of (1.1). The paper is organized as follows. In Section 2, some basic time-scale definitions are presented and several preliminary results are given. In Section 3, by employing a fixed point theorem due to Krasnoselskii and Zabreiko, we establish existence of solutions criteria for BVP (1.8). Section 4 is devoted to an example illustrating our main result.

2. Preliminaries

The study of dynamic equations on time scales goes back to its founder Hilger [3] and it is a new area of still fairly theoretical exploration in mathematics. In the recent years boundary value problem on time scales has received considerable attention [4–6]. And an increasing interest in studying the existence of solutions to dynamic equations on time scales is observed, for example, see [7–16].

For convenience, we first recall some definitions and calculus on time scales, so that the paper is self-contained. For the further details concerning the time scales, please see [17–19] which are excellent works for the calculus of time scales.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \qquad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \tag{2.1}$$

are called the forward jump operator and the backward jump operator, respectively.

For all $t \in \mathbb{T}$, we assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. The notations $[a, b]$, $[a, b)$, and so on, will denote time-scale intervals

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}, \tag{2.2}$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

Definition 2.1. Fix $t \in \mathbb{T}$. Let $y : \mathbb{T} \to \mathbb{R}$. Then we define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighborhood U of t with

$$\left| [y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \quad \forall s \in U. \tag{2.3}$$

Then y^{Δ} is called derivative of $y(t)$.

Definition 2.2. If $F^{\Delta}(t) = f(t)$ then we define the integral by

$$\int_{a}^{t} f(\tau)\Delta\tau = F(t) - F(a). \tag{2.4}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}^n$ is regressive provided

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}, \tag{2.5}$$

where $\mu(t) = \sigma(t) - t$, which is called graininess function. If p is a regressive function, then the generalized exponential function e_p is defined by

$$e_p(t, s) = \exp\left(\int_{s}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \tag{2.6}$$

for $s, t \in \mathbb{T}, \xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases} \tag{2.7}$$

Let p, q be two regressive functions, then define

$$p \oplus q = p + q + \mu pq, \qquad \ominus q = -\frac{q}{1 + \mu q}, \qquad p \ominus q = p \oplus (\ominus q) = \frac{p - q}{1 + \mu q}. \tag{2.8}$$

The generalized function e_p has then the following properties.

Lemma 2.3 (see [18]). *Assume that p, q are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(iv) $1/e_p(t,s) = e_{\ominus p}(t,s)$;

(v) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$.

The following well-known fixed point theorem will play a very important role in proving our main result.

Theorem 2.4 (see [20]). *Let X be a Banach space, and let $F : X \to X$ be completely continuous. Assume that $A : X \to X$ is a bounded linear operator such that 1 is not an eigenvalue of A and*

$$\lim_{\|x\|\to\infty} \frac{\|F(x)-A(x)\|}{\|x\|} = 0. \tag{2.9}$$

Then F has a fixed point in X.

Throughout this paper, let $E = C^2[a,b]$ be endowed with the norm by

$$\|y\|_0 = \max\left\{\|y\|, \left\|y^{\Delta^2}\right\|\right\}, \tag{2.10}$$

where $\|y\| = \max_{t\in[a,b]}|y(t)|$. And we make the following assumptions:

(H_1) $\alpha,\beta,\gamma,\delta,\zeta,\eta \geq 0$, and $a \leq \xi_1 \leq \xi_2 \leq \sigma(b)$,

(H_2) $q > 0$, and $r_1 = \sqrt{q}, r_2 = -\sqrt{q}$,

(H_3) $d = \beta + \alpha(\sigma^4(b) - a) > 0$.

Set

$$-r_2 \ominus -r_1 := p_1, \qquad \ominus - r_2 = \ominus r_1 := p_2. \tag{2.11}$$

For convenience, we denote

$$
\begin{aligned}
&\int_a^t e_{p_1}(s,a)\Delta s = l(t,a),\\
&\qquad A = \left[\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\int_a^{\xi_1}e_{p_1}(s,a)\Delta s-\delta p_2(\sigma(\xi_1))e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\right]\\
&\qquad\qquad \times\left(\zeta+\eta p_2(\xi_2)\right)e_{p_2}(\xi_2,a)-\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\\
&\qquad\qquad \times\left[\left(\zeta+\eta p_2(\xi_2)\right)e_{p_2}(\xi_2,a)\int_a^{\xi_2}e_{p_1}(s,a)\Delta s+\eta p_2(\sigma(\xi_2))e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a)\right],\\
&\qquad A_{11} = \delta e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\left(\zeta+\eta p_2(\xi_2)\right)e_{p_2}(\xi_2,a)\\
&\qquad\qquad +\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\eta e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a),\\
&\qquad A_{12} = \left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\left(\zeta+\eta p_2(\xi_2)\right)e_{p_2}(\xi_2,a),\\
&\qquad A_{13} = \left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\eta e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a),\\
&\qquad B_{11} = \left[\left(\zeta+\eta p_2(\xi_2)\right)e_{p_2}(\xi_2,a)\int_a^{\xi_2}e_{p_1}(s,a)\Delta s+\eta p_2(\sigma(\xi_2))e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a)\right]\\
&\qquad\qquad \times\left(\delta p_2(\xi_1)-\gamma\right)e_{p_2}(\xi_1,a)\\
&\qquad\qquad +\left[\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\int_a^{\xi_1}e_{p_1}(s,a)\Delta s-\delta p_2(\sigma(\xi_1))e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\right]\\
&\qquad\qquad \times\left(\eta p_2(\xi_2)+\zeta\right)e_{p_2}(\xi_2,a),\\
&\qquad B_{12} = \left[\left(\zeta+\eta p_2(\xi_2)\right)e_{p_2}(\xi_2,a)\int_a^{\xi_2}e_{p_1}(s,a)\Delta s+\eta p_2(\sigma(\xi_2))e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a)\right]\\
&\qquad\qquad \times\delta e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\\
&\qquad\qquad +\left[\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\int_a^{\xi_1}e_{p_1}(s,a)\Delta s-\delta p_2(\sigma(\xi_1))e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\right]\\
&\qquad\qquad \times\eta e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a),\\
&\qquad B_{13} = \left[\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\int_a^{\xi_1}e_{p_1}(s,a)\Delta s-\delta p_2(\sigma(\xi_1))e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\right]\\
&\qquad\qquad \times\left(\eta p_2(\xi_2)+\zeta\right)e_{p_2}(\xi_2,a),\\
&\qquad B_{14} = \left[\left(\gamma-\delta p_2(\xi_1)\right)e_{p_2}(\xi_1,a)\int_a^{\xi_1}e_{p_1}(s,a)\Delta s-\delta p_2(\sigma(\xi_1))e_{p_2}(\sigma(\xi_1),a)e_{p_1}(\xi_1,a)\right]\\
&\qquad\qquad \times\eta e_{p_2}(\sigma(\xi_2),a)e_{p_1}(\xi_2,a).
\end{aligned}
\tag{2.12}
$$

First, we present two lemmas about the calculus on Green functions which are crucial in our main results.

Lemma 2.5. *Assume that* (H_1) *and* (H_2) *are satisfied. If* $h \in C[a,b]$, *then* $u \in C^2[a,b]$ *is a solution of the following boundary value problem (BVP):*

$$\begin{gathered} y^{\Delta^2}t - qy(\sigma(t)) = h(t), \quad t \in [a,b], \\ \gamma y(\xi_1) - \delta y^{\Delta}(\xi_1) = 0, \qquad \zeta y(\xi_2) + \eta y^{\Delta}(\xi_2) = 0, \end{gathered} \tag{2.13}$$

if and only if

$$y(t) = \int_a^{\sigma(b)} G(t,s)h(s)\Delta s, \quad t \in [a,b], \tag{2.14}$$

where the Green's function of (2.13) *is as follows:*

$$G(t,s) = e_{p_2}(t,a)\begin{cases} \dfrac{-B_{11}}{A}\displaystyle\int_s^{\xi_1} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi & \\ \quad + \left[\dfrac{A_{12}l(t,a) - B_{13}}{A}\right]\displaystyle\int_{\xi_1}^{\xi_2} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi & \\ \quad + \left[\dfrac{A_{11}l(t,a) - B_{12}}{A}\right]e_{-r_1}(s,a) & \\ \quad + \displaystyle\int_s^{t} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi, & a \le \sigma(s) \le \min\{t,\xi_1\}, \\ \dfrac{-B_{11}}{A}\displaystyle\int_s^{\xi_1} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi & \\ \quad + \left[\dfrac{A_{12}l(t,a) - B_{13}}{A}\right]\displaystyle\int_{\xi_1}^{\xi_2} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi & \\ \quad + \left[\dfrac{A_{11}l(t,a) - B_{12}}{A}\right]e_{-r_1}(s,a), & a \le t \le s \le \xi_1, \\ \left[\dfrac{A_{12}l(t,a) - B_{13}}{A}\right]\displaystyle\int_s^{\xi_2} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi & \\ \quad + \dfrac{-B_{14}}{A}e_{-r_1}(s,a) + \dfrac{A_{13}}{A}l(t,a)e_{-r_2}(s,a) & \\ \quad + \displaystyle\int_s^{t} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi, & \xi_1 \le \sigma(s) \le \min\{t,\xi_2\}, \\ \left[\dfrac{A_{12}l(t,a) - B_{13}}{A}\right]\displaystyle\int_s^{\xi_2} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi & \\ \quad + \dfrac{-B_{14}}{A}e_{-r_1}(s,a) + \dfrac{A_{13}}{A}l(t,a)e_{-r_2}(s,a), & \max\{\xi_1,t\} \le s \le \xi_2, \\ \displaystyle\int_s^{t} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi, & \xi_2 \le \sigma(s) \le t \le b, \\ 0, & \max\{t,\xi_2\} \le s \le b, \end{cases} \tag{2.15}$$

where $l(t,a)$, A, A_{11}, A_{12}, A_{13}, B_{11}, B_{12}, B_{13}, B_{14} *are given as* (2.12), *respectively.*

Proof. If $y \in C^2[a,b]$ is a solution of (2.13), setting

$$u(s) = y^{\Delta}(s) - r_2 y(\sigma(s)), \quad t \in [a,b], \tag{2.16}$$

then it follows from the first equation of (2.13) that

$$u^{\Delta}(s) - r_1 u(\sigma(s)) = h(s), \quad t \in [a,b]. \tag{2.17}$$

Multiplying (2.17) by $e_{-r_1}(s,a)$ and integrating from a to t, we get

$$u(t) = e_{\ominus -r_1}(t,a)\left[u(a) + \int_a^t e_{-r_1}(s,a)h(s)\Delta s\right], \quad t \in [a,b]. \tag{2.18}$$

Similarly, by (2.18), we have

$$y(t) = e_{\ominus -r_2}(t,a)\left[y(a) + \int_a^t e_{-r_2}(s,a)u(s)\Delta s\right], \quad t \in [a,b]. \tag{2.19}$$

Then substituting (2.18) into (2.19), we get for each $t \in [a,b]$ that

$$\begin{aligned} y(t) &= e_{p_2}(t,a)y(a) + e_{p_2}(t,a)u(a)\int_a^t e_{p_1}(s,a)\Delta s \\ &\quad + e_{p_2}(t,a)\int_a^t e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s. \end{aligned} \tag{2.20}$$

Substituting this expression for $y(t)$ into the boundary conditions of (2.13). By some calculations, we get

$$\begin{aligned} u(a) &= \frac{1}{A}\left[A_{11}\int_a^{\xi_1} e_{-r_1}(s,a)h(s)\Delta s + A_{12}\int_{\xi_1}^{\xi_2} e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s \right. \\ &\quad \left. + A_{13}\int_{\xi_1}^{\xi_2} e_{-r_2}(s,a)h(s)\Delta s\right], \\ y(a) &= -\frac{1}{A}\left[B_{11}\int_a^{\xi_1} e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s + B_{12}\int_a^{\xi_1} e_{-r_1}(s,a)h(s)\Delta s \right. \\ &\quad \left. + B_{13}\int_{\xi_1}^{\xi_2} e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s + B_{14}\int_{\xi_1}^{\xi_2} e_{-r_1}(s,a)h(s)\Delta s\right]. \end{aligned} \tag{2.21}$$

Then substituting (2.21) into (2.20), we get

$$
\begin{aligned}
y(t) = -\frac{e_{p_2}(t,a)}{A}&\left[B_{11}\int_a^{\xi_1} e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s + B_{12}\int_a^{\xi_1} e_{-r_1}(s,a)h(s)\Delta s\right.\\
&\quad\left.+B_{13}\int_{\xi_1}^{\xi_2} e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s + B_{14}\int_{\xi_1}^{\xi_2} e_{-r_1}(s,a)h(s)\Delta s\right]\\
&+\frac{e_{p_2}(t,a)}{A}\int_a^t e_{p_1}(s,a)\Delta s\left[A_{11}\int_a^{\xi_1} e_{-r_1}(s,a)h(s)\Delta s + A_{12}\int_{\xi_1}^{\xi_2} e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s\right.\\
&\qquad\left.+A_{13}\int_{\xi_1}^{\xi_2} e_{-r_2}(s,a)h(s)\Delta s\right]\\
&+e_{p_2}(t,a)\int_a^t e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s.
\end{aligned}
\tag{2.22}
$$

By interchanging the order of integration and some rearrangement of (2.22), we obtain

$$
\begin{aligned}
y(t) &= e_{p_2}(t,a)\\
&\times\left(\int_a^{\xi_1}\left(\frac{-B_{11}}{A}\int_s^{\xi_1} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi + \left[\frac{A_{12}l(t,a)-B_{13}}{A}\right]\int_{\xi_1}^{\xi_2} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi\right.\right.\\
&\qquad\left.+e_{-r_1}(s,a)\left[\frac{A_{11}l(t,a)-B_{12}}{A}\right]\right)h(s)\Delta s\\
&\quad+\int_{\xi_1}^{\xi_2}\left(\left[\frac{A_{12}l(t,a)-B_{13}}{A}\right]\int_s^{\xi_2} e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi + \frac{-B_{14}}{A}e_{-r_1}(s,a)\right.\\
&\qquad\left.\left.+\frac{A_{13}}{A}l(t,a)e_{-r_2}(s,a)\right)h(s)\Delta s + \int_a^t\left(\int_s^t e_{p_1}(\xi,a)e_{-r_1}(s,a)\Delta\xi\right)h(s)\Delta s\right).
\end{aligned}
\tag{2.23}
$$

Thus, we obtain (2.14) consequently.

On the other hand, if y satisfies (2.14), then direct differentiation of (2.14) yields

$$
y^{\Delta^2}(t) - qy^{\Delta}(\sigma(t)) = h(t), \quad t\in[a,b]. \tag{2.24}
$$

And it is easy to know that $y\in C^2[a,b]$ and y satisfies (2.13). □

Corollary 2.6. *If* $\mathbb{T} = \mathbb{R}$, *then BVP* (2.13) *reduces to the following problem:*

$$\begin{aligned} &y''(t) - qy(t) = h(t), \quad t \in [a,b], \\ &\gamma y(\xi_1) - \delta y'(\xi_1) = 0, \qquad \zeta y(\xi_2) + \eta y'(\xi_2) = 0. \end{aligned} \tag{2.25}$$

From Lemma 2.5, BVP (2.25) *has a unique solution*

$$y(t) = \int_a^b G(t,s)h(s)ds, \tag{2.26}$$

where the Green's function of (2.25) *is as follows:*

$$G(t,s) = \frac{1}{r_1 - r_2}\begin{cases} \dfrac{1}{\Delta_1}\left(e^{r_1(t-a)}M_1(s) - e^{r_2(t-a)}M_2(s)\right) + e^{r_1(t-s)} - e^{r_2(t-s)}, & a \le s \le \min\{t,\xi_1\}, \\ \dfrac{1}{\Delta_1}\left(e^{r_1(t-a)}M_1(s) - e^{r_2(t-a)}M_2(s)\right), & a \le t \le s \le \xi_1, \\ \dfrac{1}{\Delta_1}\left(e^{r_1(t-a)}M_3(s) - e^{r_2(t-a)}M_4(s)\right) + e^{r_1(t-s)} - e^{r_2(t-s)}, & \xi_1 \le s \le \min\{t,\xi_2\}, \\ \dfrac{1}{\Delta_1}\left(e^{r_1(t-a)}M_3(s) - e^{r_2(t-a)}M_4(s)\right), & \max\{\xi_1,t\} \le s \le \xi_2, \\ e^{r_1(t-s)} - e^{r_2(t-s)}, & \xi_2 \le s \le t \le b, \\ 0, & \max\{t,\xi_2\} \le s \le b, \end{cases} \tag{2.27}$$

where

$$\Delta_1 = (\gamma - \delta r_2)(\zeta + \eta r_1)e^{r_1(\xi_2-a)+r_2(\xi_1-a)} - (\gamma - \delta r_1)(\zeta + \eta r_2)e^{r_1(\xi_1-a)+r_2(\xi_2-a)}, \tag{2.28}$$

$$\begin{aligned} M_1(s) &= (\delta r_2 - \gamma)(\eta r_1 + \zeta)e^{r_1(\xi_2-s)+r_2(\xi_1-a)} - (\delta r_1 - \gamma)(\eta r_2 + \zeta)e^{r_2(\xi_2-a)+r_1(\xi_1-s)}, \\ M_2(s) &= (\gamma - \delta r_1)(\eta r_2 + \zeta)e^{r_2(\xi_2-s)+r_1(\xi_1-a)} + (\delta r_2 - \gamma)(\eta r_1 + \zeta)e^{r_1(\xi_2-a)+r_2(\xi_1-s)}, \\ M_3(s) &= (\gamma - \delta r_2)(\eta r_2 + \zeta)e^{r_2(\xi_2-s)+r_2(\xi_1-a)} - (\gamma - \delta r_2)(\eta r_1 + \zeta)e^{r_2(\xi_1-a)+r_1(\xi_2-s)}, \\ M_4(s) &= (\gamma - \delta r_1)(\eta r_2 + \zeta)e^{r_2(\xi_2-s)+r_1(\xi_1-a))} - (\gamma - \delta r_1)(\eta r_1 + \zeta)e^{r_1(\xi_1-a)+r_1(\xi_2-s)}. \end{aligned} \tag{2.29}$$

Proof. If $y \in C^2[a,b]$ is a solution of (2.25), take $\mathbb{T} = \mathbb{R}$, then $p_1 = r_1 - r_2, p_2 = r_2$. Hence, from (2.20) we have

$$\begin{aligned}
y(t) &= e_{p_2}(t,a)y(a) + e_{p_2}(t,a)u(a)\int_a^t e_{p_1}(s,a)\Delta s \\
&\quad + e_{p_2}(t,a)\int_a^t e_{p_1}(s,a)\int_a^s e_{-r_1}(\xi,a)h(\xi)\Delta\xi\,\Delta s \\
&= e^{r_2(t-a)}y(a) + e^{r_2(t-a)}u(a)\int_a^t e^{(r_1-r_2)(s-a)}ds \\
&\quad + e^{r_2(t-a)}\int_a^t\int_a^s e^{(r_1-r_2)(s-a)}e^{-r_1(\xi-a)}h(\xi)d\xi\,ds \\
&= e^{r_2(t-a)}y(a) + \frac{u(a)}{r_1-r_2}\left(e^{r_1(t-a)} - e^{r_2(t-a)}\right) \\
&\quad + \frac{1}{r_1-r_2}\int_a^t \left(e^{r_1(t-s)} - e^{r_2(t-s)}\right)h(s)ds.
\end{aligned} \tag{2.30}$$

Substituting this expression for $y(t)$ into the boundary conditions of (2.25). By some calculations, we obtain

$$\begin{aligned}
u(a) &= \frac{1}{\Delta_1}\Bigg[\int_a^{\xi_1}\Big((\delta r_2-\gamma)(\eta r_1+\zeta)e^{r_1(\xi_2-s)+r_2(\xi_1-a)} - (\delta r_1-\gamma)(\eta r_2+\zeta)e^{r_2(\xi_2-a)+r_1(\xi_1-s)}\Big)h(s)ds \\
&\quad + \int_{\xi_1}^{\xi_2}\Big((\gamma-\delta r_2)(\eta r_2+\zeta)e^{r_2(\xi_2-s)+r_2(\xi_1-a)} - (\gamma-\delta r_2)(\eta r_1+\zeta)e^{r_2(\xi_1-a)+r_1(\xi_2-s)}\Big)h(s)ds\Bigg], \\
y(a) &= -\frac{1}{(r_1-r_2)\Delta_1} \\
&\quad \times\Bigg[\int_a^{\xi_1}\Big((\gamma-\delta r_1)(\eta r_2+\zeta)e^{r_1(\xi_1-a)+r_2(\xi_2-s)} + (\zeta+\eta r_1)(\delta r_2-\gamma)e^{r_1(\xi_2-a)+r_2(\xi_1-s)} \\
&\qquad + (\gamma-\delta r_2)(\eta r_1+\zeta)e^{r_1(\xi_2-s)+r_2(\xi_1-a)} + (\delta r_1-\gamma)(\eta r_2+\zeta)e^{r_2(\xi_2-a)+r_1(\xi_1-s)}\Big)h(s)ds \\
&\quad + \int_{\xi_1}^{\xi_2}\Big((\gamma-\delta r_1)(\eta r_2+\zeta)e^{r_2(\xi_2-s)+r_1(\xi_1-a))} - (\gamma-\delta r_1)(\eta r_1+\zeta)e^{r_1(\xi_1-a)+r_1(\xi_2-s)} \\
&\qquad - (\gamma-r_2\delta)(\eta r_2+\zeta)e^{r_2(\xi_1-a)+r_2(\xi_2-s)} + (\gamma-r_2\delta)(\eta r_1+\zeta)e^{r_2(\xi_1-a)+r_1(\xi_2-s)}\Big)h(s)ds\Bigg],
\end{aligned} \tag{2.31}$$

where Δ_1 is given as (2.28). Then substituting (2.31) into (2.30), we get

$$y(t) = \frac{1}{r_1 - r_2}\left[\int_a^{\xi_1}\left(\frac{e^{r_1(t-a)}}{\Delta_1}M_1(s) - \frac{e^{r_2(t-a)}}{\Delta_1}M_2(s)\right)h(s)ds\right.$$
$$\left.+\int_{\xi_1}^{\xi_2}\left(\frac{e^{r_1(t-a)}}{\Delta_1}M_3(s) - \frac{e^{r_2(t-a)}}{\Delta_1}M_4(s)\right)h(s)ds + \int_a^t\left(e^{r_1(t-s)} - e^{r_2(t-s)}\right)h(s)ds\right], \tag{2.32}$$

where $M_i(s)$ $(i = \{1,2,3,4\})$ are as in (2.29), respectively. By some rearrangement of (2.32), we obtain (2.26) consequently. □

From the proof of Corollary 2.6, if $\mathbb{T} = \mathbb{R}$, take $\gamma = \zeta = 1, \delta = \eta = 0, a = \xi_1 = 0, b = \xi_2 = 1$, we get the following result.

Corollary 2.7. *The following boundary value problem:*

$$\begin{gathered} -y''(t) + qy(t) = h(t), \quad t \in [0,1], \\ y(0) = y(1) = 0 \end{gathered} \tag{2.33}$$

has a unique solution

$$y(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.34}$$

where the Green's function of (2.33) *is as follows:*

$$G(t,s) = \frac{-1}{r_1 - r_2}\begin{cases} \dfrac{1}{\Delta_1}\left(e^{r_1 t}M_3(s) - e^{r_2 t}M_4(s)\right) + e^{r_1(t-s)} - e^{r_2(t-s)}, & 0 \le t \le s \le 1, \\ \dfrac{1}{\Delta_1}\left(e^{r_1 t}M_3(s) - e^{r_2 t}M_4(s)\right), & 0 \le s \le t \le 1, \end{cases} \tag{2.35}$$

where

$$\Delta_1 = e^{r_1} - e^{r_2}, \qquad M_3(s) = M_4(s) = e^{r_2(1-s)} - e^{r_1(1-s)}. \tag{2.36}$$

After some rearrangement of (2.35), *one obtains*

$$G(t,s) = \begin{cases} \dfrac{\sinh r_1 t \sinh r_1(1-s)}{r_1 \sinh r_1}, & 0 \le t \le s \le 1, \\ \dfrac{\sinh r_1 s \sinh r_1(1-t)}{r_1 \sinh r_1}, & 0 \le s \le t \le 1. \end{cases} \tag{2.37}$$

Remark 2.8. Green function (2.37) associated with BVP (2.33) which is a special case of (2.13) is coincident with part of [21, Lemma 1].

Lemma 2.9. *Assume that conditions* (H_1)*–*(H_3) *are satisfied. If* $h \in C[a,b]$, *then boundary value problem*

$$\begin{gathered} y^{\Delta^4}(t) - q y^{\Delta^2}(\sigma(t)) = h(t), \quad t \in [a,b], \\ y\left(\sigma^4(b)\right) = 0, \qquad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \qquad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0 \end{gathered} \tag{2.38}$$

has a unique solution

$$y(t) = \int_a^{\sigma^4(b)} G_1(t,\xi) \int_a^{\sigma(b)} G(\xi,s) h(s) \Delta s \, \Delta\xi, \tag{2.39}$$

where

$$G_1(t,s) = \frac{1}{d} \begin{cases} \left(\sigma^4(b) - \sigma(s)\right)\left(\alpha(t-a) + \beta\right), & t \le s, \\ \left(\sigma^4(b) - t\right)\left(\alpha(\sigma(s) - a) + \beta\right), & t \ge \sigma(s), \end{cases} \tag{2.40}$$

and $G(t,s)$ *is given in Lemma 2.5.*

Proof. Consider the following boundary value problem:

$$\begin{gathered} y^{\Delta^2}(t) - q y^{\Delta}(\sigma(t)) = \int_a^{\sigma(b)} G(t,s) h(s) \Delta s, \quad t \in \left[a, \sigma^2(b)\right], \\ y\left(\sigma^4(b)\right) = 0, \qquad \alpha y(a) - \beta y^{\Delta}(a) = 0. \end{gathered} \tag{2.41}$$

The Green's function associated with the BVP (2.41) is $G_1(t,s)$. This completes the proof. □

Remark 2.10. In [1, Lemma 2.5], the solution of (1.2) is defined as

$$y(t) = \int_a^{\sigma^4(b)} G_1(t,\xi) \int_{\xi_1}^{\xi_2} G_2(\xi,s) h(s) \Delta s \, \Delta\xi, \tag{2.42}$$

where $G_1(t,s)$ and $G_2(t,s)$ are given as (1.4) and (1.5), respectively. In fact, $y(t)$ is incorrect. Thus, we give the right form of $y(t)$ as the special case $q(t) = q$ in our Lemma 2.9.

3. Main Results

Theorem 3.1. *Assume* (H_1)–(H_3) *are satisfied. Moreover, suppose that the following condition is satisfied:*

(H_4) $f(t, y(\sigma(t)), y^{\Delta^2}(t)) = m(t)g(u) + n(t)h(v)$, *where* $g, h : \mathbb{R} \to \mathbb{R}$ *are continuous,* $m(t), n(t) \in C[a,b]$, *with*

$$\lim_{u\to\infty}\frac{g(u)}{u} = \lambda, \qquad \lim_{u\to\infty}\frac{h(v)}{v} = \mu, \tag{3.1}$$

and there exists a continuous nonnegative function $w : [a,b] \to \mathbb{R}^+$ *such that* $|m(s)| + |n(s)| \le w(s)$, $s \in [a,b]$. *If*

$$\max\{|\lambda|, |\mu|\} < \min\left\{\frac{1}{L_1}, \frac{1}{L_2}\right\}, \tag{3.2}$$

where

$$\begin{gathered} L_1 = \max_{a\le t\le b}\left(\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} |G(\xi,s)|w(s)\Delta s\,\Delta\xi\right), \\ L_2 = \max_{a\le t\le b}\int_a^{\sigma(b)} |G(t,s)|w(s)\Delta s, \end{gathered} \tag{3.3}$$

then BVP (1.8) *has a solution* $y \in C^2[a,b]$.

Proof. Define an operator $F : C^2[a,b] \to C^2[a,b]$ by

$$Fy(t) = \int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y(s)) + n(s)h\left(y^{\Delta^2}(s)\right)\right]\Delta s\,\Delta\xi, \tag{3.4}$$

where $G_1(t,s)$ is given by (2.40). Then by Lemmas 2.5 and 2.9, it is clear that the fixed points of F are the solutions to the boundary value problem (1.8). First of all, we claim that F is a completely continuous operator, which is divided into 3 steps.

Step 1 ($F : C^2[a,b] \to C^2[a,b]$ is continuous). Let $\{y_n\}_{n=1}^{\infty}$ be a sequence such that $y_n \to y(n \to \infty)$, then we have

$$\begin{aligned}
&\left|(Fy_n)(t) - (Fy)(t)\right| \\
&\quad= \left|\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y_n(s)) + n(s)h\left(y_n^{\Delta^2}(s)\right)\right]\Delta s\,\Delta\xi\right. \\
&\qquad\left.-\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y(s)) + n(s)h\left(y^{\Delta^2}(s)\right)\right]\Delta s\,\Delta\xi\right| \\
&\quad= \left|\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} G(\xi,s)\left[m(s)(g(y_n(s)) - g(y(s))) + n(s)\left(h\left(y_n^{\Delta^2}(s)\right) - h\left(y^{\Delta^2}(s)\right)\right)\right]\Delta s\,\Delta\xi\right| \\
&\quad\le \int_a^{\sigma^4(b)} |G_1(t,\xi)|\int_a^{\sigma(b)} |G(\xi,s)m(s)|\left|g(y_n(s)) - g(y(s))\right|\Delta\xi \\
&\qquad+ \int_a^{\sigma^4(b)} |G_1(t,\xi)|\int_a^{\sigma(b)} |G(\xi,s)m(s)|\left|h\left(y_n^{\Delta^2}(s)\right) - h\left(y^{\Delta^2}(s)\right)\right|\Delta\xi, \\
&\left|(Fy_n)^{\Delta^2}(t) - (Fy)^{\Delta^2}(t)\right| \\
&\quad= \left|\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y_n(s)) + n(s)h\left(y_n^{\Delta^2}(s)\right)\right]\Delta s\right. \\
&\qquad\left.-\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y(s)) + n(s)h\left(y^{\Delta^2}(s)\right)\right]\Delta s\right| \\
&\quad= \left|\int_a^{\sigma(b)} G(\xi,s)\left[m(s)(g(y_n(s)) - g(y(s))) + n(s)\left(h\left(y_n^{\Delta^2}(s)\right) - h\left(y^{\Delta^2}(s)\right)\right)\right]\Delta s\right| \\
&\quad\le \int_a^{\sigma(b)} |G(\xi,s)m(s)|\left|g(y_n(s)) - g(y(s))\right|\Delta s + \int_a^{\sigma(b)} |G(\xi,s)m(s)|\left|h\left(y_n^{\Delta^2}(s)\right) - h\left(y^{\Delta^2}(s)\right)\right|\Delta s.
\end{aligned} \tag{3.5}$$

Since g, h are continuous, we have $|(Fy_n)(t) - (Fy)(t)| \to 0$, which yields $\|Fy_n - Fy\| \to 0(n \to \infty)$. That is,$F : C^2[a,b] \to C^2[a,b]$ is continuous.

Step 2 (F maps bounded sets into bounded sets in $C^2[a,b]$). Let $\Omega \subset C^2[a,b]$ be a bounded set. Then, for $t \in [a,b]$ and any $y \in \Omega$, we have

$$\begin{aligned} \left|Fy(t)\right| &= \left|\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y(s)) + n(s)h\left(y^{\Delta^2}(s)\right)\right]\Delta s\,\Delta\xi\right| \\ &\le \int_a^{\sigma^4(b)} |G_1(t,\xi)|\int_a^{\sigma(b)} |G(\xi,s)|\left(\left|m(s)g(y(s))\right| + \left|n(s)h\left(y^{\Delta^2}(s)\right)\right|\right)\Delta s\,\Delta\xi. \end{aligned} \tag{3.6}$$

By virtue of the continuity of g and h, we conclude that Fu is bounded uniformly, and so $F(\Omega)$ is a bounded set.

Step 3 (F maps bounded sets into equicontinuous sets of $C^2[a,b]$). Let $t_1, t_2 \in [a,b], y \in \Omega$, then

$$\begin{aligned} &\left|(Fy)(t_1) - (Fy)(t_2)\right| \\ &\quad = \left|\int_a^{\sigma^4(b)} (G_1(t_1,\xi) - G_1(t_2,\xi))\int_a^{\sigma(b)} G(\xi,s)\left[m(s)g(y(s)) + n(s)h\left(y^{\Delta^2}(s)\right)\right]\Delta s\,\Delta\xi\right| \\ &\quad \le \int_a^{\sigma^4(b)} |G_1(t_1,\xi) - G_1(t_2,\xi)|\int_a^{\sigma(b)} \left|G(\xi,s)\left[m(s)g(y(s)) + n(s)h\left(y^{\Delta^2}(s)\right)\right]\right|\Delta s\,\Delta\xi. \end{aligned} \tag{3.7}$$

The right hand side tends to uniformly zero as $t_1 - t_2 \to 0$. Consequently, Steps 1–3 together with the Arzela-Ascoli theorem show that F is completely continuous.

Now we consider the following boundary value problem:

$$\begin{gathered} y^{\Delta^4}(t) - qy^{\Delta^2}(\sigma(t)) = \lambda m(t)y(t) + \mu n(t)y^{\Delta^2}(t), \quad t \in [a,b], \\ y\left(\sigma^4(b)\right) = 0, \qquad \alpha y(a) - \beta y^{\Delta}(a) = 0, \\ \gamma y^{\Delta^2}(\xi_1) - \delta y^{\Delta^3}(\xi_1) = 0, \qquad \zeta y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0. \end{gathered} \tag{3.8}$$

Define

$$Ay(t) = \int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} G(\xi,s)\left[\lambda m(s)y(s) + \mu n(s)y^{\Delta^2}(s)\right]\Delta s\,\Delta\xi. \tag{3.9}$$

Obviously, A is a completely continuous bounded linear operator. Moreover, the fixed point of A is a solution of the BVP (3.8) and conversely.

We are now in the position to claim that 1 is not an eigenvalue of A.

If $\lambda = 0$ and $\mu = 0$, then (3.8) has no nontrivial solution.

If $\lambda \neq 0$ or $\mu \neq 0$, suppose that the BVP (3.8) has a nontrivial solution y and $\|y\|_0 > 0$, then we have

$$\begin{aligned}
\left|Ay(t)\right| &\leq \int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} \left|G(\xi,s)\left[\lambda m(s)y(s)+\mu n(s)y^{\Delta^2}(s)\right]\right|\Delta s\,\Delta\xi \\
&\leq \int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} |G(\xi,s)|\left[|\lambda||m(s)|\left|y(s)\right|+\left|\mu\right||n(s)|\left|y^{\Delta^2}(s)\right|\right]\Delta s\,\Delta\xi \\
&\leq \max_{a\leq t\leq b}\left(\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{\sigma(b)} |G(\xi,s)|\left[|\lambda||m(s)|+\left|\mu\right||n(s)|\right]\|y\|_0\Delta s\,\Delta\xi\right) \\
&\leq \max\{|\lambda|,\left|\mu\right|\}\max_{a\leq t\leq b}\left(\int_a^{\sigma^4(b)} G_1(t,\xi)\int_a^{(\sigma(b))} |G(\xi,s)|w(s)\Delta s\,\Delta\xi\right)\|y\|_0,
\end{aligned} \tag{3.10}$$

which yields

$$\left|Ay(t)\right| \leq \max\{|\lambda|,\left|\mu\right|\}L_1\|y\|_0 = \|y\|_0. \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}
\left|(Ay)^{\Delta^2}(t)\right| &\leq \int_a^{\sigma(b)} \left|G(t,s)\lambda m(s)y(s)+\mu n(s)y^{\Delta^2}(s)\right|\Delta s \\
&\leq \max_{a\leq t\leq b}\int_a^{\sigma(b)} |G(t,s)|\left[|\lambda||m(s)|+\left|\mu\right||n(s)|\right]\|y\|_0\Delta s \\
&\leq \max\{|\lambda|,\left|\mu\right|\}\max_{a\leq t\leq b}\int_a^{\sigma(b)} |G(t,s)|w(s)\Delta s\|y\|_0 \\
&\leq \max\{|\lambda|,\left|\mu\right|\}L_2\|y\|_0 < \frac{1}{L_2}L_2\|y\|_0 = \|y\|_0.
\end{aligned} \tag{3.12}$$

From the above discussion (3.11) and (3.12), we have $\|Ay\|_0 < \|y\|_0$. This contradiction implies that boundary value problem (3.8) has no trivial solution. Hence, 1 is not an eigenvalue of A.

At last, we show that

$$\lim_{\|x\|_0\to\infty}\frac{\|F(x)-A(x)\|_0}{\|x\|_0} = 0. \tag{3.13}$$

By $\lim_{u\to\infty}(g(u)/u) = \lambda, \lim_{u\to\infty}(h(v)/v) = \mu$, then for any $\varepsilon > 0$, there exist a $R > 0$ such that

$$\left|g(u)-\lambda u\right| < \varepsilon|u|, \quad \left|h(v)-\mu v\right| < \varepsilon|v|, \quad |u|,|v| > R. \tag{3.14}$$

Set $R^* = \max\{\max_{|u|\leq R}|g(u)|, \max_{|v|\leq R}|h(v)|\}$ and select $M > 0$ such that $R^* + \max\{|\lambda|,|\mu|\}R < \varepsilon M$.

Denote

$$E_1=\{t\in[a,b]:|u(t)|\le R,|v(t)|>R\},\qquad E_2=\{t\in[a,b]:|u(t)|\le R,|v(t)|>R\},$$
$$E_3=\{t\in[a,b]:|u(t)|,|v(t)|\le R\},\qquad E_4=\{t\in[a,b]:|u(t)|,|v(t)|>R\}. \tag{3.15}$$

Thus for any $y\in E$ and $\|y\|_0>M$, when $t\in E_1$, it follows that

$$\begin{gathered}\left|g(u(t))-\lambda u(t)\right|\le\left|g(u(t))\right|+|\lambda||u(t)|\le R^*+|\lambda|R<\varepsilon M<\varepsilon\|u\|_0,\\ \left|h(v(t))-\mu v(t)\right|<\varepsilon|v(t)|\le\varepsilon\|v\|_0.\end{gathered} \tag{3.16}$$

In a similar way, we also conclude that for any $t\in E_i$, $(i=2,3,4)$,

$$\left|g(u(t))-\lambda u(t)\right|<\varepsilon\|u\|_0,\qquad \left|h(v(t))-\mu v(t)\right|<\varepsilon\|v\|_0. \tag{3.17}$$

Therefore,

$$\begin{aligned}&\left|Fy(t)-Ay(t)\right|\\ &\quad=\left|\int_a^{\sigma^4(b)}G_1(t,\xi)\int_a^{\sigma(b)}G(\xi,s)\left(m(s)\left[g(y(s))-\lambda y(s)\right]+n(s)\left[h\left(y^{\Delta^2}(s)\right)-\mu y^{\Delta^2}(s)\right]\right)\Delta s\,\Delta\xi\right|\\ &\quad\le\max_{a\le t\le b}\left(\int_a^{\sigma^4(b)}G_1(t,\xi)\int_a^{\sigma(b)}|G(\xi,s)|w(s)\Delta s\,\Delta\xi\right)\varepsilon\|y\|_0=\varepsilon L_1\|y\|_0.\end{aligned} \tag{3.18}$$

On the other hand, we get

$$\begin{aligned}\left|(Fy-Ay)^{\Delta^2}(t)\right|&\le\int_a^{\sigma(b)}\left|G(t,s)\left(m(s)\left[g(y(s))-\lambda y(s)\right]+n(s)\left[h\left(y^{\Delta^2}(s)\right)-\mu y^{\Delta^2}(s)\right]\right)\right|\Delta s\\ &\le\max_{a\le t\le b}\left(\int_a^{\sigma(b)}|G(t,s)|w(s)\Delta s\right)\varepsilon\|y\|_0\\ &=\varepsilon L_2\|y\|_0.\end{aligned} \tag{3.19}$$

Combining (3.18) with (3.19), we have

$$\lim_{\|x\|_0\to\infty}\frac{\|F(x)-A(x)\|_0}{\|x\|_0}=0. \tag{3.20}$$

Theorem 2.4 guarantees that boundary value problem (1.8) has a solution $y^*\in C^2[a,b]$. It is obvious that $y^*\neq 0$ when $m(t_0)g(0)+n(t_0)h(0)\neq 0$ for some $t_0\in[a,b]$. In fact, if $m(t_0)g(0) + n(t_0)h(0) \neq 0$, then $(0)^{\Delta 4} - q(0)^{\Delta^2} = m(t_0)g(0) + n(t_0)h(0) \neq 0$ will lead to a contradiction, which completes the proof. □

4. Application

We give an example to illustrate our result.

Example 4.1. Consider the fourth-order four-pint boundary value problem

$$\begin{gathered} y^4(t) - \frac{1}{4}y''(t) = \frac{t\sin 2\pi t}{t^2+1}y(t) - \frac{1}{2}te^{\cos t}\cos y''(t), \quad 0 < t < 1, \\ y(0) = y(1) = 0, \\ y''\left(\frac{1}{3}\right) - y'''\left(\frac{1}{3}\right) = 0, \qquad y''\left(\frac{2}{3}\right) + y'''\left(\frac{2}{3}\right) = 0. \end{gathered} \tag{4.1}$$

Notice that $\mathbb{T} = \mathbb{R}$. To show that (4.1) has at least one nontrivial solution we apply Theorem 3.1 with $m(t) = t\sin 2\pi t/(t^2+1), n(t) = (1/2)te^{\cos t}, g(u) = u, h(u) = \cos u, \alpha = \gamma = \delta = \eta = \zeta = 1, \beta = 0, q = 1/4, \xi_1 = 1/3$, and $\xi_2 = 2/3$. Obviously, (H_1)–(H_3) are satisfied. And

$$m(t_0)g(0) + n(t_0)h(0) = \frac{1}{2}t_0e^{\cos t_0} \neq 0, \quad t_0 \in (0,1]. \tag{4.2}$$

Since $|m(s)| + |n(s)| \leq ((1/2)e+1)s := w(s)$, for each $s \in [0,1]$, we have the following.

By simple calculation we have

$$\begin{aligned} L_1 &= \max_{0\leq t\leq 1}\left(\int_0^1 G_1(t,\xi)\int_0^1 |G(\xi,s)|w(s)ds\,d\xi\right) \approx 0.05 < 1, \\ L_2 &= \max_{0\leq t\leq 1}\int_0^1 |G(t,s)|w(s)ds \approx 0.82 < 1. \end{aligned} \tag{4.3}$$

On the other hand, we notice that

$$\lambda = \lim_{u\to\infty}\frac{g(u)}{u} = 1, \qquad \mu = \lim_{u\to\infty}\frac{h(u)}{u} = 0. \tag{4.4}$$

Hence,

$$\max\{\lambda,\mu\} < 1 < \min\left\{\frac{1}{L_1},\frac{1}{L_2}\right\}. \tag{4.5}$$

That is, (H_4) is satisfied. Thus, Theorem 3.1 guarantees that (4.1) has at least one nontrivial solution $u \in C^2[0,1]$.

Acknowledgments

The authors would like to thank the referees for their valuable suggestions and comments. This work is supported by the National Natural Science Foundation of China (10771212) and the Natural Science Foundation of Jiangsu Education Office (06KJB110010).

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