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Research Article

Blowup Analysis for a Semilinear Parabolic System with Nonlocal Boundary Condition

Yulan Wang[1] and Zhaoyin Xiang[2]

[1] *School of Mathematics and Computer Engineering, Xihua University, Chengdu 610039, China*
[2] *School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu 610054, China*

Correspondence should be addressed to Zhaoyin Xiang, zxiangmath@gmail.com

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This paper deals with the properties of positive solutions to a semilinear parabolic system with nonlocal boundary condition. We first give the criteria for finite time blowup or global existence, which shows the important influence of nonlocal boundary. And then we establish the precise blowup rate estimate for small weighted nonlocal boundary.

1. Introduction

In this paper, we devote our attention to the singularity analysis of the following semilinear parabolic system:

$$u_t - \Delta u = v^p, \quad v_t - \Delta v = u^q, \quad x \in \Omega,\ t > 0 \tag{1.1}$$

with nonlocal boundary condition

$$u(x,t) = \int_\Omega f(x,y)u(y,t)dy, \quad v(x,t) = \int_\Omega g(x,y)v(y,t)dy, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded connected domain with smooth boundary $\partial\Omega$, p and q are positive parameters. Most physical settings lead to the default assumption that the functions $f(x,y)$, $g(x,y)$ defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$ are nonnegative and continuous, and that the initial data $u_0(x)$, $v_0(x) \in C^1(\overline{\Omega})$ are nonnegative, which are mathematically convenient and currently followed throughout this paper. We also assume that (u_0, v_0) satisfies the compatibility condition on $\partial\Omega$, and that $f(x,\cdot)\not\equiv 0$ and $g(x,\cdot)\not\equiv 0$ for any $x \in \partial\Omega$ for the sake of the meaning of *nonlocal boundary*.

Over the past few years, a considerable effort has been devoted to studying the blowup properties of solutions to parabolic equations with *local boundary conditions*, say Dirichlet, Neumann, or Robin boundary condition, which can be used to describe heat propagation on the boundary of container (see the survey papers [1, 2]). For example, the system (1.1) and (1.3) with homogeneous Dirichlet boundary condition

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0 \tag{1.4}$$

has been studied extensively (see [3–5] and references therein), and the following proposition was proved.

Proposition 1.1. (i) *All solutions are global if $pq \le 1$, while there exist both global solutions and finite time blowup solutions depending on the size of initial data when $pq > 1$ (See [4]).* (ii) *The asymptotic behavior near the blowup time is characterized by*

$$C_1^{-1} \le \max_{x\in\overline{\Omega}} u(x,t)(T-t)^{p+1/pq-1} \le C_1, \qquad C_2^{-1} \le \max_{x\in\overline{\Omega}} v(x,t)(T-t)^{(q+1)/(pq-1)} \le C_2 \tag{1.5}$$

for some $C_1, C_2 > 0$ (See [3, 5]).

For the more parabolic problems related to the local boundary, we refer to the recent works [6–9] and references therein.

On the other hand, there are a number of important phenomena modeled by parabolic equations coupled with nonlocal boundary condition of form (1.2). In this case, the solution could be used to describe the entropy per volume of the material [10–12]. Over the past decades, some basic results such as the global existence and decay property have been obtained for the nonlocal boundary problem (1.1)–(1.3) in the case of scalar equation (see [13–16]). In particular, for the blowup solution u of the single equation

$$\begin{gathered} u_t - \Delta u = u^p, \quad x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega f(x,y)u(y,t)dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.6}$$

under the assumption that $\int_\Omega f(x,y)dy = 1$, Seo [15] established the following blowup rate estimate

$$(p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)} \le \max_{x\in\overline{\Omega}} u(x,t) \le C_1(T-t)^{-1/(\gamma-1)} \tag{1.7}$$

for any $\gamma \in (1, p)$. For the more nonlocal boundary problems, we also mention the recent works [17–22]. In particular, Kong and Wang in [17], by using some ideas of Souplet [23], obtained the blowup conditions and blowup profile of the following system:

$$u_t = \Delta u + \int_\Omega u^m(x,t)v^n(x,t)dx, \quad v_t = \Delta v + \int_\Omega u^p(x,t)v^q(x,t)dx, \quad x \in \Omega,\ t > 0 \tag{1.8}$$

subject to nonlocal boundary (1.2), and Zheng and Kong in [22] gave the condition for global existence or nonexistence of solutions to the following similar system:

$$u_t = \Delta u + u^m \int_\Omega v^n(y,t)dy, \quad v_t = \Delta v + v^q \int_\Omega u^p(y,t)dy, \quad x \in \Omega,\ t > 0 \tag{1.9}$$

with nonlocal boundary condition (1.2). The typical characterization of systems (1.8) and (1.9) is the complete couple of the nonlocal sources, which leads to the analysis of simultaneous blowup.

To our surprise, however, it seems that there is no work dealing with singularity analysis of the parabolic system (1.1) with nonlocal boundary condition (1.2) except for the single equation case, although this is a very classical model. Therefore, the basic motivation for the work under consideration was our desire to understand the role of weight function in the blowup properties of that nonlinear system. We first remark by the standard theory [4, 13] that there exist local nonnegative classical solutions to this system.

Our main results read as follows.

Theorem 1.2. *Suppose that* $0 < pq \leq 1$. *All solutions to* (1.1)–(1.3) *exist globally.*

It follows from Theorem 1.2 and Proposition 1.1(i) that any weight perturbation on the boundary has no influence on the global existence when $pq \leq 1$, while the following theorem shows that it plays an important role when $pq > 1$. In particular, Theorem 1.3(ii) is completely different from the case of the local boundary (1.4) (by comparing with Proposition 1.1(i)).

Theorem 1.3. *Suppose that* $pq > 1$.

(i) *For any nonnegative* $f(x,y)$ *and* $g(x,y)$, *solutions to* (1.1)–(1.3) *blow up in finite time provided that the initial data are large enough.*

(ii) *If* $\int_\Omega f(x,y)dy \geq 1$, $\int_\Omega g(x,y)dy \geq 1$ *for any* $x \in \partial\Omega$, *then any solutions to* (1.1)–(1.3) *with positive initial data blow up in finite time.*

(iii) *If* $\int_\Omega f(x,y)dy < 1$, $\int_\Omega g(x,y)dy < 1$ *for any* $x \in \partial\Omega$, *then solutions to* (1.1)–(1.3) *with small initial data exist globally in time.*

Once we have characterized for which exponents and weights the solution to problem (1.1)–(1.3) can or cannot blow up, we want to study the way the blowing up solutions behave as approaching the blowup time. To this purpose, the first step usually consists in deriving a bound for the blowup rate. For this bound estimate, we will use the classical method initially proposed in Friedman and McLeod [24]. The use of the maximum principle in that process forces us to give the following hypothesis technically.

(H) *There exists a constant* $0 < \delta < 1$, *such that* $\Delta u_0 + (1-\delta)v_0^p \geq 0$, $\Delta v_0 + (1-\delta)u_0^q \geq 0$.

However, it seems that such an assumption is necessary to obtain the estimates of type (1.5) or (1.10) unless some additional restrictions on parameters p, q are imposed (for the related problem, we refer to the recent work of Matano and Merle [25]).

Here to obtain the precise blowup rates, we shall devote to establishing some relationship between the two components u and v as our problem involves a system, but we encounter the typical difficulties arising from the integral boundary condition. The following theorem shows that we have partially succeeded in this precise blowup characterization.

Theorem 1.4. *Suppose that* $pq > 1$, $p, q \geq 1$, $f(x, y) = g(x, y)$, $\int_\Omega f(x, y)dy \leq 1$, *and assumption (H) holds. If the solution* (u, v) *of* (1.1)–(1.3) *with positive initial data* (u_0, v_0) *blows up in finite time* T, *then*

$$C_1^{-1} \leq \max_{x\in\overline{\Omega}} u(x,t)(T-t)^{(p+1)/(pq-1)} \leq C_1, \qquad C_2^{-1} \leq \max_{x\in\overline{\Omega}} v(x,t)(T-t)^{(q+1)/(pq-1)} \leq C_2, \tag{1.10}$$

where C_1, C_2 *are both positive constants.*

Remark 1.5. If $q = p$ and $u_0 = v_0$, then Theorem 1.4 implies that for the blowup solution of problem (1.6), we have the following precise blowup rate estimate:

$$C_1^{-1}(T-t)^{-1/(p-1)} \leq \max_{x\in\overline{\Omega}} u(x,t) \leq C_1(T-t)^{-1/(p-1)}, \tag{1.11}$$

which improves the estimate (1.7). Moreover, we relax the restriction on f.

Remark 1.6. By comparing with Proposition 1.1(ii), Theorem 1.4 could be explained as the small perturbation of homogeneous Dirichlet boundary, which leads to the appearance of blowup, does not influence the precise asymptotic behavior of solutions near the blowup time and the blowup rate exponents $(p+1)/(pq-1)$ and $(q+1)/(pq-1)$ are just determined by the corresponding ODE system $u_t = v^p$, $v_t = u^q$. Similar phenomena are also noticed in our previous work [18], where the single porous medium equation is studied.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries, which include the comparison principle related to system (1.1)–(1.3). In Section 3, we will study the conditions for the solution to blow up and exist globally and hence prove Theorems 1.2 and 1.3. Proof of Theorem 1.4 is given in Section 4.

2. Preliminaries

In this section, we give some basic preliminaries. For convenience, we denote $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\overline{Q_T} = \overline{\Omega} \times [0,T)$. We begin with the definition of the super- and subsolution of system (1.1)–(1.3).

Definition 2.1. A pair of functions $u, v \in C^{2,1}(Q_T) \bigcap C(\overline{Q_T})$ is called a subsolution of (1.1)–(1.3) if

$$\begin{gathered} u_t - \Delta u \le v^p, \quad v_t - \Delta v \le u^q, \quad (x,t) \in Q_T, \\ u(x,t) \le \int_\Omega f(x,y) u(y,t)\,dy, \quad v(x,t) \le \int_\Omega g(x,y) v(y,t)\,dy, \quad (x,t) \in S_T, \\ u(x,0) \le u_0(x), \quad v(x,0) \le v_0(x), \quad x \in \Omega. \end{gathered} \tag{2.1}$$

A supersolution is defined with each inequality reversed.

Lemma 2.2. *Suppose that c_1, c_2, f, and g are nonnegative functions. If $w_1, w_2 \in C^{2,1}(Q_T) \bigcap C(\overline{Q_T})$ satisfy*

$$\begin{gathered} w_{1t} - \Delta w_1 \ge c_1(x,t) w_2, \quad w_{2t} - \Delta w_2 \ge c_2(x,t) w_1, \quad (x,t) \in Q_T, \\ w_1(x,t) \ge \int_\Omega f(x,y) w_1(y,t)\,dy, \quad w_2(x,t) \ge \int_\Omega g(x,y) w_2(y,t)\,dy, \quad (x,t) \in S_T, \\ w_1(x,0) > 0, \quad w_2(x,0) > 0, \quad x \in \overline{\Omega}, \end{gathered} \tag{2.2}$$

then $w_1, w_2 > 0$ on $\overline{Q_T}$.

Proof. Set $t_1 := \sup\{t \in (0,T) : w_i(x,t) > 0, \ (i = 1,2)\}$. Since $w_1(x,0), w_2(x,0) > 0$, by continuity, there exists $\delta > 0$ such that $w_1(x,t), w_2(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times [0,\delta)$. Thus $t_1 \in (\delta, T]$.

We claim that $t_1 < T$ will lead to a contradiction. Indeed, $t_1 < T$ suggests that $w_1(x_1,t_1) = 0$ or $w_2(x_1,t_1) = 0$ for some $x_1 \in \overline{\Omega}$. Without loss of generality, we suppose that $w_1(x_1,t_1) = 0 = \inf_{\overline{Q_{t_1}}} w_1$.

If $x_1 \in \Omega$, we first notice that

$$w_{1t} - \Delta w_1 \ge c_1 w_2 \ge 0, \quad (x,t) \in \Omega \times (0,t_1]. \tag{2.3}$$

In addition, it is clear that $w_1 \ge 0$ on boundary $\partial\Omega$ and at the initial state $t = 0$. Then it follows from the strong maximum principle that $w_1 \equiv 0$ in Q_{t_1}, which contradicts to $w_1(x,0) > 0$.

If $x_1 \in \partial\Omega$, we shall have a contradiction:

$$0 = w_1(x_1,t_1) \ge \int_\Omega f(x_1,y) w_1(y,t_1)\,dy > 0. \tag{2.4}$$

In the last inequality, we have used the facts that $f(x,\cdot)\not\equiv 0$ for any $x \in \partial\Omega$ and $w_1(y,t_1) > 0$ for any $y \in \Omega$, which is a direct result of the previous case.

Therefore, the claim is true and thus $t_1 = T$, which implies that $w_1, w_2 > 0$ on $\overline{Q_T}$. □

Remark 2.3. If $\int_\Omega f(x,y)dy \le 1$ and $\int_\Omega g(x,y)dy \le 1$ for any $x \in \partial\Omega$ in Lemma 2.2, we can obtain $(w_1, w_2) \ge (0,0)$ in $\overline{Q_T}$ under the assumption that $(w_1(x,0), w_2(x,0)) \ge (0,0)$ for $x \in \overline{\Omega}$. Indeed, for any $\epsilon > 0$, we can conclude that $(w_1(x,t) + \epsilon e^t, w_2(x,t) + \epsilon e^t) > (0,0)$ in $\overline{Q_T}$ as the proof of Lemma 2.2. Then the desired result follows from the limit procedure $\epsilon \to 0$.

From the above lemma, we can obtain the following comparison principle by the standard argument.

Proposition 2.4. *Let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be a subsolution and supersolution of* (1.1)–(1.3) *in* Q_T, *respectively. If* $(\underline{u}(x,0), \underline{v}(x,0)) < (\overline{u}(x,0), \overline{v}(x,0))$ *for* $x \in \overline{\Omega}$, *then* $(\underline{u}, \underline{v}) < (\overline{u}, \overline{v})$ *in* $\overline{Q_T}$.

3. Global Existence and Blowup in Finite Time

In this section, we will use the super and subsolution technique to get the global existence or finite time blowup of the solution to (1.1)–(1.3).

Proof of Theorem 1.2. As $0 < pq \le 1$, there exist $s, l \in (0,1)$ such that

$$\frac{1}{p} \ge \frac{l}{s}, \quad \frac{1}{q} \ge \frac{s}{l}. \tag{3.1}$$

Then we let $\phi(x,y)$ $(x \in \partial\Omega, y \in \overline{\Omega})$ be a continuous function satisfying $\phi(x,y) \ge \max\{f(x,y), g(x,y)\}$ and set

$$a(x) = \left(\int_\Omega \phi(x,y)dy\right)^{(1-s)/s}, \qquad b(x) = \left(\int_\Omega \phi(x,y)dy\right)^{(1-l)/l}, \quad x \in \partial\Omega. \tag{3.2}$$

We consider the following auxiliary problem:

$$\begin{gathered} w_t = \Delta w + kw, \quad x \in \Omega,\ t > 0, \\ w(x,t) = (a(x) + b(x) + 1)\left(\int_\Omega \left(\phi(x,y) + \frac{1}{|\Omega|}\right) w(y,t)dy\right), \quad x \in \partial\Omega, \\ w(x,0) = 1 + u_0^{1/s}(x) + v_0^{1/l}(x), \quad t > 0, \end{gathered} \tag{3.3}$$

where $|\Omega|$ is the measure of Ω and $k := 1/s + 1/l$. It follows from [13, Theorem 4.2] that $w(x,t)$ exists globally, and indeed $w(x,t) > 1$, $(x,t) \in \overline{\Omega} \times [0,\infty)$ (see [13, Theorem 2.1]).

Our intention is to show that $(\overline{u},\overline{v}) := (w^s, w^l)$ is a global supersolution of (1.1)–(1.3). Indeed, a direct computation yields

$$\begin{aligned}\overline{u}_t &= sw^{(s-1)}(\Delta w + kw) \ \geq \ sw^{(s-1)}\Delta w + w^s,\\ \Delta\overline{u} &= sw^{(s-1)}\Delta w + s(s-1)w^{(s-2)}|\nabla w|^2 \ \leq \ sw^{(s-1)}\Delta w,\end{aligned} \tag{3.4}$$

and thus

$$\overline{u}_t - \Delta\overline{u} \geq w^s = \left(w^l\right)^{s/l} \geq \overline{v}^p. \tag{3.5}$$

Here we have used the conclusion $w > 1$ and inequality (3.1). We still have to consider the boundary and initial conditions. When $x \in \partial\Omega$, in view of Hölder's inequality, we have

$$\begin{aligned}\overline{u}(x,t) &\geq (a(x))^s\left\{\int_\Omega \phi(x,y)w(y,t)dy\right\}^s\\ &= \left\{\int_\Omega \phi(x,y)dy\right\}^{1-s}\left\{\int_\Omega \phi(x,y)w(y,t)dy\right\}^s\\ &\geq \left\{\int_\Omega f(x,y)dy\right\}^{1-s}\left\{\int_\Omega f(x,y)w(y,t)dy\right\}^s\\ &= \left\{\int_\Omega \left(f^{1-s}(x,y)\right)^{1/1-s}dy\right\}^{1-s}\left\{\int_\Omega (f^s(x,y)w^s(y,t))^{1/s}dy\right\}^s\\ &\geq \int_\Omega f^{1-s}(x,y)(f(x,y)w(y,t))^s dy\\ &= \int_\Omega f(x,y)w^s(y,t)dy\\ &= \int_\Omega f(x,y)\overline{u}(y,t)dy.\end{aligned} \tag{3.6}$$

Similarly, we have also for $\overline{v}$ that

$$\begin{aligned}\overline{v}_t - \Delta\overline{v} &\geq \overline{u}^q, \quad x \in \Omega,\ t > 0,\\ \overline{v} &\geq \int_\Omega g(x,y)\overline{v}(y,t)dy, \quad x \in \partial\Omega,\ t > 0.\end{aligned} \tag{3.7}$$

It is clear that $u_0(x) < \overline{u}(x,0)$ and $v_0(x) < \overline{v}(x,0)$. Therefore, we get $(\overline{u},\overline{v})$ is a global supersolution of (1.1)–(1.3) and hence the solution to (1.1)–(1.3) exists globally by Proposition 2.4. □

Proof of Theorem 1.3. (i) Let $(\underline{u},\underline{v})$ be the solution to the homogeneous Dirichlet boundary problem (1.1), (1.4), and (1.3). Then it is well known that for sufficiently large initial data the solution $(\underline{u}, \underline{v})$ blows up in finite time when $pq > 1$ (see [4]). On the other hand, it is obvious that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1)–(1.3). Henceforth, the solution of (1.1)–(1.3) with large initial data blows up in finite time provided that $pq > 1$.

(ii) We consider the ODE system:

$$f'(t) = h^p(t), \quad h'(t) = f^q(t), \quad t > 0,$$
$$f(0) = a > 0, \qquad h(0) = b > 0, \tag{3.8}$$

where $a = (1/2)\min_{\overline{\Omega}} u_0(x),\ b = (1/2)\min_{\overline{\Omega}} v_0(x)$. Then $pq > 1$ implies that (f, h) blows up in finite time T (see [26]). Under the assumption that $\int_\Omega f(x,y)dy \geq 1$ and $\int_\Omega g(x,y)dy \geq 1$ for any $x \in \partial\Omega$, (f, h) is a subsolution of problem (1.1)–(1.3). Therefore, by Proposition 2.4, we see that the solution (u, v) of problem (1.1)–(1.3) satisfies $(u, v) \geq (f, h)$ and then (u, v) blows up in finite time.

(iii) Let $\varphi_1(x)$ be the positive solution of the linear elliptic problem:

$$-\Delta\varphi_1(x) = \epsilon_0, \quad x \in \Omega, \qquad \varphi_1(x) = \int_\Omega f(x,y)dy, \quad x \in \partial\Omega, \tag{3.9}$$

and let $\varphi_2(x)$ be the positive solution of the linear elliptic problem:

$$-\Delta\varphi_2(x) = \epsilon_0, \quad x \in \Omega, \qquad \varphi_2(x) = \int_\Omega g(x,y)dy, \quad x \in \partial\Omega, \tag{3.10}$$

where ϵ_o is a positive constant such that $0 \leq \varphi_i(x) \leq 1$ $(i = 1, 2)$. We remark that $\int_\Omega f(x,y)dy < 1$ and $\int_\Omega g(x,y)dy < 1$ ensure the existence of such ϵ_0.

Let

$$\overline{u}(x) = a\varphi_1(x), \qquad \overline{v}(x) = b\varphi_2(x), \tag{3.11}$$

where $a = \epsilon_0^{(p+1)/(pq-1)},\ b = \epsilon_0^{(q+1)/(pq-1)}$. We now show that $(\overline{u}, \overline{v})$ is a supsolution of problem (1.1)–(1.3) for small initial data (u_0, v_0). Indeed, it follows from $b\epsilon_0 = a^q,\ a\epsilon_0 = b^p$ that, for $x \in \Omega$,

$$\overline{u}_t - \Delta\overline{u} = a\epsilon_0 = b^p \geq \overline{v}^p, \qquad \overline{v}_t - \Delta\overline{v} = b\epsilon_0 = a^q \geq \overline{u}^q. \tag{3.12}$$

When $x \in \partial\Omega$,

$$\overline{u}(x) = a\int_\Omega f(x,y)dy \geq \int_\Omega f(x,y)a\varphi_1(y)dy = \int_\Omega f(x,y)\overline{u}(x)dy,$$
$$\overline{v}(x) = b\int_\Omega g(x,y)dy \geq \int_\Omega g(x,y)b\varphi_2(y)dy = \int_\Omega g(x,y)\overline{v}(x)dy. \tag{3.13}$$

Here we used $\varphi_i(x) \leq 1$ $(i = 1, 2)$. The above inequalities show that $(\overline{u}, \overline{v})$ is a supsolution of problem (1.1)–(1.3) whenever $u_0(x) < a\varphi_1(x),\ v_0(x) < b\varphi_2(x)$. Therefore, system (1.1)–(1.3) has global solutions if $pq > 1$ and $\int_\Omega f(x,y)dy < 1$, $\int_\Omega g(x,y)dy < 1$ for any $x \in \partial\Omega$. □

4. Blowup Rate Estimate

In this section, we derive the precise blowup rate estimate. To this end, we first establish a partial relationship between the solution components $u(x,t)$ and $v(x,t)$, which will be very useful in the subsequent analysis. For definiteness, we may assume $p \geq q \geq 1$. If $q > p$, we can proceed in the same way by changing the role of u and v and then obtain the corresponding conclusion.

Lemma 4.1. *If $p \geq q$, $f(x,y) = g(x,y)$ and $\int_\Omega f(x,y)dy \leq 1$ for any $x \in \partial\Omega$, there exists a positive constant C_0 such that the solution (u,v) of problem (1.1)–(1.3) with positive initial data (u_0, v_0) satisfies*

$$u(x,t) \geq C_0 v^{(p+1)/(q+1)}(x,t), \quad (x,t) \in \overline{\Omega} \times [0,T). \tag{4.1}$$

Proof. Let $J(x,t) = u(x,t) - C_0 v^{(p+1)/(q+1)}(x,t)$, where C_0 is a positive constant to be chosen. For $(x,t) \in \Omega \times (0,T)$, a series of calculations show that

$$\begin{aligned} J_t - \Delta J &= u_t - C_0 \frac{p+1}{q+1} v^{(p-q)/(q+1)} v_t - \Delta u + C_0 \frac{(p+1)(p-q)}{(q+1)^2} |\nabla v|^2 + C_0 \frac{p+1}{q+1} v^{(p-q)/(q+1)} \Delta v \\ &\geq v^p - C_0 \frac{p+1}{q+1} v^{(p-q)/(q+1)} u^q \\ &= v^{(p-q)/(q+1)} \left(v^{q(p+1)/(q+1)} - C_0 \frac{p+1}{q+1} u^q \right) \\ &= v^{(p-q)/(q+1)} \left(\frac{1}{C_0{}^q} (u-J)^q - C_0 \frac{p+1}{q+1} u^q \right). \end{aligned} \tag{4.2}$$

If we choose C_0 such that $1/C_0{}^q \geq C_0(p+1)/(q+1)$, we have

$$J_t - \Delta J + v^{p-q/q+1} \theta(u,v) J \geq 0, \tag{4.3}$$

where $\theta(u,v)$ is a function of u and v and lies between $C_0(p+1/q+1)(u-J)$ and $C_0(p+1)/(q+1)u$.

When $(x,t) \in \partial\Omega \times (0,T)$, on the other hand, we have

$$J(x,t) = \int_\Omega f(x,y) u(y,t) dy - C_0 \left(\int_\Omega f(x,y) v(y,t) dy \right)^{(p+1)/(q+1)}. \tag{4.4}$$

Denote $H(x) := \int_\Omega f(x,y)dy \geq 0$, $(x \in \partial\Omega)$. Since $f(x,\cdot)\not\equiv 0$ for any $x \in \partial\Omega$, $H(x) > 0$. It follows from Jensen's inequality, $H(x) \leq 1$, and $(p+1)/(q+1) \geq 1$ that

$$\begin{aligned}
&\int_\Omega f(x,y)v^{(p+1)/(q+1)}(y,t)dy - \left(\int_\Omega f(x,y)v(y,t)dy\right)^{(p+1)/(q+1)} \\
&\quad\geq H(x)\left(\int_\Omega f(x,y)v(y,t)\frac{dy}{H(x)}\right)^{(p+1)/(q+1)} - \left(\int_\Omega f(x,y)v(y,t)dy\right)^{(p+1)/(q+1)} \\
&\quad\geq 0,
\end{aligned} \tag{4.5}$$

which implies that

$$\begin{aligned}
J(x,t) &\geq \int_\Omega f(x,y)u(y,t)dy - C_0\int_\Omega f(x,y)v^{(p+1)/(q+1)}(y,t)dy \\
&= \int_\Omega f(x,y)J(y,t)dy, \quad x \in \partial\Omega.
\end{aligned} \tag{4.6}$$

For the initial condition, we have

$$J(x,0) = u_0(x) - C_0 v_0^{(p+1)/(q+1)}(x) \geq 0, \quad x \in \overline{\Omega}, \tag{4.7}$$

provided that $C_0 \leq \inf_{x\in\overline{\Omega}}\{u_0(x)v_0^{-(p+1)/(q+1)}(x)\}$.

Summarily, if we take $C_0 = \min\{\inf_{x\in\overline{\Omega}} u_0(x)v_0^{-(p+1)/(q+1)}(x), (q+1)/(p+1)^{1/q+1}\}$, then it follows from Theorem 2.1 in [13] that $J(x,t) \geq 0$, that is,

$$u(x,t) \geq C_0 v^{(p+1)/(q+1)}(x,t), \quad (x,t) \in \overline{\Omega} \times [0,T), \tag{4.8}$$

which is desired. □

Using this lemma, we could establish our blowup rate estimate. To derive our conclusion, we shall use some ideas of [3].

Proof of Theorem 1.4. For simplicity, we introduce $\alpha = (p+1)/(pq-1)$, $\beta = (q+1)/(pq-1)$. Let $F(x,t) = u_t - \delta v^p$ and $G(x,t) = v_t - \delta u^q$. A direct computation yields

$$F_t - \Delta F \geq p v^{p-1} G, \quad G_t - \Delta G \geq q u^{q-1} F, \quad x \in \Omega,\ 0 < t < T. \tag{4.9}$$

For $(x,t) \in \partial\Omega \times (0,T)$, we have from the boundary conditions that

$$\begin{aligned}
F(x,t) &= u_t - \delta v^p \\
&= \int_\Omega f(x,y)u_t(y,t)dy - \delta\left[\int_\Omega f(x,y)v(y,t)dy\right]^p \\
&= \int_\Omega f(x,y)(F+\delta v^p)(y,t)dy - \delta\left[\int_\Omega f(x,y)v(y,t)dy\right]^p \\
&= \int_\Omega f(x,y)F(y,t)dy + \delta\left\{\int_\Omega f(x,y)v^p(y,t)dy - \left[\int_\Omega f(x,y)v(y,t)dy\right]^p\right\}.
\end{aligned} \tag{4.10}$$

It follows from $\int_\Omega f(x,y)dy \le 1$ and Jensen's inequality that the difference in the last brace is nonnegative and thus

$$F(x,t) \ge \int_\Omega f(x,y)F(y,t)dy, \quad x \in \partial\Omega. \tag{4.11}$$

By similar arguments, we have

$$G(x,t) \ge \int_\Omega f(x,y)G(y,t)dy, \quad (x,t) \in \partial\Omega \times (0,T). \tag{4.12}$$

On the other hand, the hypothesis (H) implies that

$$F(x,0) \ge 0, \quad G(x,0) \ge 0 \quad x \in \Omega. \tag{4.13}$$

Hence, from (4.9)–(4.13) and the comparison principle (see Remark 2.3), we get

$$F(x,t) \ge 0, \quad G(x,t) \ge 0, \quad (x,t) \in \Omega \times (0,T). \tag{4.14}$$

That is,

$$u_t \ge \delta v^p, \quad v_t \ge \delta u^q, \quad (x,t) \in \Omega \times (0,T). \tag{4.15}$$

Let $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$, $V(t) = \max_{x\in\overline{\Omega}} v(x,t)$. Then $U(t)$ and $V(t)$ are Lipschitz continuous and thus are differential almost everywhere (see e.g., [24]). Moreover, we have from equations (1.1) that

$$U'(t) \le V^p(t), \quad V'(t) \le U^q(t), \quad \text{a.e. } t \in [0,T). \tag{4.16}$$

We claim that

$$V'(t) \ge kV^{q(p+1)/(q+1)}(t), \quad \text{a.e. } t \in [0,T) \tag{4.17}$$

for some positive constant k. Indeed, if we let $(x(t), t)$ be the points at which v attains its maximum, then relation (4.1) means that

$$u(x(t), t) \geq C_0 V^{(p+1)/(q+1)}(t), \quad t \in [0, T). \tag{4.18}$$

At any point t_1 of differentiability of $V(t)$, if $t_2 > t_1$,

$$\frac{V(t_2) - V(t_1)}{t_2 - t_1} \geq \frac{v(x(t_1), t_2) - v(x(t_1), t_1)}{t_2 - t_1} = v_t(x(t_1), t_1) + o(1), \quad \text{as } t_2 \longrightarrow t_1. \tag{4.19}$$

From (4.15), (4.18), and (4.19), we can confirm our claim (4.17).

Integrating (4.17) on $[t, T)$ yields

$$V(t)(T - t)^{\beta} \leq \overline{k}, \quad t \in [0, T), \tag{4.20}$$

which gives the upper estimate for $V(t)$. Namely, there exists a constant $c_4 > 0$ such that

$$V(t) \leq c_4 (T - t)^{-\beta}, \quad t \in [0, T). \tag{4.21}$$

Then by (4.16) and (4.21), we get

$$U'(t) \leq V^p(t) \leq c_4^p (T - t)^{-p\beta}, \quad t \in [0, T). \tag{4.22}$$

Integrating this equality from 0 to t, we obtain

$$U(t) \leq c_2 (T - t)^{-\alpha}, \quad t \in [0, T) \tag{4.23}$$

for some positive constant c_2. Thus we have established the upper estimates for $U(t)$.

To obtain the lower estimate for $U(t)$, we notice that (4.16) and (4.18) lead to

$$U'(t) \leq k_2 U^{p(q+1)/(p+1)}(t) \tag{4.24}$$

for a constant k_2. Integrating above equality on $[t, T)$, we see there exists a positive constant c_1 such that

$$U(t) \geq c_1 (T - t)^{-\alpha}, \quad t \in [0, T). \tag{4.25}$$

Finally, we give the lower estimate for $V(t)$. Indeed, using the relationship (4.16), (4.23) and (4.25), we could prove that $V(t)(T - t)^{\beta}$ is bounded from below; that is, there exists a positive constant c_3 such that

$$V(t) \geq c_3 (T - t)^{-\beta}. \tag{4.26}$$

To see this, our approach is based on the contradiction arguments. Assume that there would exist two sequences $\{t_n\} \subset (0,T)$ with $t_n \to T^-$ and $\{d_n\}$ with $d_n \to 0$ as $n \to \infty$ such that

$$V(t_n) \le d_n(T-t_n)^{-\beta}, \quad n = 1,2,3,\ldots. \tag{4.27}$$

Then we could choose a corresponding sequence $\{s_n\}$ such that $t_n - s_n = k(T-t_n)$, where k is a positive constant to be determined later. As $U'(t) \le V^p(t)$, we have

$$U(t_n) \le U(s_n) + \int_{s_n}^{t_n} V^p(\tau)d\tau. \tag{4.28}$$

From (4.23) and (4.27), we obtain

$$\begin{aligned} U(t_n) &\le c_2(T-s_n)^{-\alpha} + V^p(t_n)(t_n - s_n) \\ &\le c_2(T-s_n)^{-\alpha} + d_n^p(T-t_n)^{-\beta p}(t_n - s_n) \\ &\le c_2(k+1)^{-\alpha}(T-t_n)^{-\alpha} + kd_n^p(T-t_n)^{-\alpha}. \end{aligned} \tag{4.29}$$

Choosing k such that $c_2(k+1)^{-\alpha} \le c_1/2$, one can get

$$U(t_n) \le \frac{c_1}{2}(T-t_n)^{-\alpha} + kd_n^p(T-t_n)^{-\alpha} = \left(\frac{c_1}{2} + kd_n^p\right)(T-t_n)^{-\alpha}, \tag{4.30}$$

which would contradict to (4.25) as n is large enough since $d_n \to 0$ as $n \to \infty$. □

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