

Research Article

Existence of Nontrivial Solution for a Nonlocal Elliptic Equation with Nonlinear Boundary Condition

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In this paper, we establish two different existence results of solutions for a nonlocal elliptic equations with nonlinear boundary condition. The first one is based on Galerkin method, and gives a priori estimate. The second one is based on Mountain Pass Lemma.

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1. Introduction

In this paper, we deal with the following elliptic equation with nonlinear boundary condition:

$$\begin{aligned} -\Delta u + u &= \frac{f(x, u)}{M(\int_{\Omega} |\nabla u|^2 + |u|^2 dx)}, & \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} &= g(x, u), & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $N > 2$, $\partial/\partial\gamma$ is the outer unite normal derivative, $M : R^+ \rightarrow R$ is continuous, $f : \Omega \times R \rightarrow R$, $g : \partial\Omega \times R \rightarrow R$ are Carathéodory functions.

For (1.1), if the nonlocal term $M(\int_{\Omega} |\nabla u|^2 + |u|^2 dx)$ is replaced by $M(\int_{\Omega} |\nabla u|^2 dx)$, then the equation

$$-M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), \quad \text{in } \Omega \tag{1.2}$$

is related to the stationary analog of the Kirchhoff equation:

$$u_{tt} - M\left(\int_{\Omega} |\nabla_x u|^2 dx\right) \Delta_x u = f(x, t), \quad (1.3)$$

where $M(s) = as + b$, $a, b > 0$. It was proposed by Kirchhoff [1] as an extension of the classical D'Alembert wave equations for free vibrations of elastic strings. The Kirchhoff model takes into account the length changes of the string produced by transverse vibrations. Equation (1.3) received much attention and an abstract framework to the problem was proposed after the work [2]. Some interesting and further results can be found in [3, 4] and the references therein. In addition, (1.2) has important physical and biological background. There are many authors who pay more attention to this equation. In particular, authors concerned with the existence of solutions for (1.2) with zero Dirichlet boundary condition via Galerkin method, and built the variational frame in [5, 6]. More recently, Perera and Zhang obtained solutions of a class of nonlocal quasilinear elliptic boundary value problems using the variational methods, invariant sets of descent flow, Yang index, and critical groups [7, 8].

If the nonlocal term $M(\int_{\Omega} |\nabla u|^2 + |u|^2 dx)$ is replaced by $M(\int_{\Omega} |u|^2 dx)$, then the equation

$$-M\left(\int_{\Omega} |u|^2 dx\right) \Delta u = f(x, u), \quad \text{in } \Omega \quad (1.4)$$

arises in numerous physical models such as systems of particles in thermodynamical equilibrium via gravitational (Coulomb) potential, 2-D fully turbulent behavior of real flow, thermal runaway in Ohmic Heating, shear bands in metal deformed under high strain rates, among others. Because of its importance, in [9, 10], the authors similarly studied the existence of solution for (1.4) with zero Dirichlet boundary condition.

On the other hand, elliptic equations with nonlinear boundary conditions have become rather an active area of research; see [11–15] and reference therein. Those references present necessary and sufficient conditions of solutions of elliptic equations with nonlinear boundary conditions. In [13], the authors study the elliptic equation

$$\Delta u = f(x, u), \quad \text{in } \Omega, \quad (1.5)$$

with the nonlinear boundary condition

$$\frac{\partial u}{\partial \gamma} = g(x, u), \quad \text{on } \partial\Omega. \quad (1.6)$$

They obtain various existence results applying coincidence degree theory and the method of upper and lower solutions.

Inspired by the above references, we deal with the existence of solutions for elliptic equation (1.1) with nonlinear boundary condition based on Galerkin method and the Mountain Pass Lemma.

The paper is organized as follows. In Section 2, we will give the existence of solution for (1.1) via Galerkin method. In Section 3, we will study the solution for (1.1) using the Mountain Pass Lemma.

2. Existence

In this section, we state and prove the main theorem via Galerkin method when Ω is bounded.

For convenience, we give the following hypotheses.

(H1) A typical assumption for M is that there exists an $m_0 > 0$ such that $M(s) \geq m_0$, for all $s \geq 0$.

(H2) For all $s \in \mathbb{R}$, assume that the functions f, g satisfying

$$\begin{aligned} |f(x, s)| &\leq C_1(1 + |s|^{p_1-1}), & \text{a.e. in } \Omega, \\ |g(x, s)| &\leq C_2(1 + |s|^{p_2-1}), & \text{a.e. on } \partial\Omega, \end{aligned} \quad (2.1)$$

where $C_1, C_2 > 0$ are constants, $2 < p_1 < 2^* = 2N/(N-2)$, $2 < p_2 < \tilde{2}^* = 2(N-1)/(N-2)$.

(H3) The function $x \mapsto f(x, 0) + g(x, 0)$ is not identically zero.

Let $W^{1,2}(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$ be endowed with norm $\|u\|^2 = \int_{\Omega} (|\nabla u|^2 + |u|^2) dx$. Then $W^{1,2}(\Omega)$ is a Banach space.

A function $u \in W^{1,2}(\Omega)$ is a weak solution of (1.1) if

$$\int_{\Omega} \nabla u \nabla \varphi dx + \int_{\Omega} u \varphi dx - \int_{\partial\Omega} g(x, u) \varphi dx = \int_{\Omega} \frac{f}{M(\|u\|^2)} \varphi dx, \quad (2.2)$$

for all $\varphi \in W^{1,2}(\Omega)$.

Lemma 2.1. *Suppose that $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ on $|\xi| = r$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^m and $|\cdot|$ its related norm. Then, there exists $z_0 \in \overline{B_r}(0)$ such that $F(z_0) = 0$.*

Lemma 2.2 (see [16]). *Let Ω be a domain in \mathbb{R}^n satisfying the uniform C^m -regularity condition, and suppose that there exists a simple (m, p) -extension operator E for Ω . Also suppose that $mp < n$ and $p \leq q \leq p^* = (n-1)p/(n-mp)$. Then*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\partial\Omega). \quad (2.3)$$

If $mp = n$, then the embedding still holds for $p \leq q < \infty$. Moreover, if $1 < p \leq q < p^$, then the embedding is compact.*

Theorem 2.3. *Assume that (H1)–(H3) hold. In addition, we suppose that*

(H4) *there exist constants λ, η, μ, C_3 such that $f(x, u)u \leq \lambda|u|^2 + \eta|u|$, $\forall x \in \Omega, u \in \mathbb{R}$, $g(x, u)u \leq \mu(C_3 m_0)^{-1}|u|^2$, $\forall x$ on $\partial\Omega$ with*

$$\lambda + \mu < m_0. \quad (2.4)$$

Then problem (1.1) has at least one weak solution. Besides, any solution u satisfies the estimate

$$\|u\| \leq \frac{m_0 \eta |\Omega|^{1/2}}{(m_0 - \lambda - \mu)}. \quad (2.5)$$

Proof. Let $\{\psi_k\}$ be different complete orthonormal systems for $W^{1,2}(\Omega)$ and set

$$V_n = \text{Span}\{\psi_1, \dots, \psi_n\}. \quad (2.6)$$

Then V_n is isometric to R^n . Then, each $u \in V_n$ is uniquely associated to $\xi = (\xi_1, \dots, \xi_n)$ by the relation $u = \sum \xi_k \psi_k$. Since $\{\psi_k\}$ are, respectively, orthonormal in $W^{1,2}(\Omega)$, we get $\|u\|^2 = \|\xi\|_{R^n}^2$.

We search for solutions $u_n \in V_n$ of the approximate problem

$$\begin{aligned} \int_{\Omega} \nabla u_n \nabla \psi_k dx + \int_{\Omega} u_n \psi_k dx - \int_{\Omega} \frac{f(x, u_n)}{M(\|u_n\|^2)} \psi_k dx - \int_{\partial\Omega} g(x, u_n) \psi_k dy = 0, \\ \forall \psi_k \in W^{1,2}(\Omega), \quad k = 1, 2, \dots, n. \end{aligned} \quad (2.7)$$

To solve this algebraic system we define the operator $P_n : R^n \rightarrow R^n$

$$\begin{aligned} (P_n u)_k = \int_{\Omega} \nabla u_n \nabla \psi_k dx + \int_{\Omega} u_n \psi_k dx - \int_{\Omega} \frac{f(x, u_n)}{M(\|u_n\|^2)} \psi_k dx \\ - \int_{\partial\Omega} g(x, u_n) \psi_k dy = 0, \quad \forall u \in V_n. \end{aligned} \quad (2.8)$$

By condition (H2), the growth of function f is subcritical, so $u \mapsto f(\cdot, u)$ defines a continuous Nemytskii mapping $N_f : L^{p_1}(\Omega) \rightarrow L^{p_1}(\Omega)$. Similarly, we also define a continuous mapping $N_g : L^{p_2}(\Omega) \rightarrow L^{p_2}(\Omega)$.

From the continuity of M and $f(x, u), g(x, u)$, with respect to u , we denote that P_n is continuous. Therefore, from (H1), (H2), (H4) and Hölder's inequality, we note that $u \in V_n$

$$\langle P_n u, u \rangle \geq \|u\|^2 - \frac{\lambda \|u\|^2 + \eta |\Omega|^{1/2} \|u\|}{m_0} - \int_{\partial\Omega} \left(\frac{\mu}{m_0 C_3} |u|^2 \right) dy. \quad (2.9)$$

On the other hand, by Lemma 2.2, we have

$$\int_{\partial\Omega} (\mu C_3^{-1} m_0^{-1} |u|^2) dy = \mu C_3^{-1} m_0^{-1} \|u\|_{L^2(\partial\Omega)}^2 \leq \mu (C_3 m_0)^{-1} C_3 \|u\|^2 = \frac{\mu}{m_0} \|u\|^2, \quad (2.10)$$

where $C_3 > 0$ is constant.

From (2.9) and (2.10), we can prove that

$$\langle P_n u, u \rangle \geq \left(1 - \frac{\lambda + \mu}{m_0} \right) \|u\|^2 - \frac{\eta}{m_0} |\Omega|^{1/2} \|u\|. \quad (2.11)$$

This shows that there exists $R > 0$, depending only on $m_0, \lambda, \eta, \mu, C_3, \Omega$, such that $\langle P_n u, u \rangle \geq 0$ if $\|u\| = R$. Then system (2.7) has a solution $u_n \in V_n$ satisfying

$$\|u_n\| \leq R. \quad (2.12)$$

From this bound estimate, going to a subsequence if necessary, there are v and u such that

$$\|u_n\|^2 \rightarrow v, \quad u_n \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega). \quad (2.13)$$

Besides, since $W^{1,2}(\Omega) \hookrightarrow L^{p_1}(\Omega)$, $W^{1,2}(\Omega) \hookrightarrow L^{p_2}(\partial\Omega)$ compactly and the mapping N_f, N_g is, respectively, continuous $L^{p_1}(\Omega) \rightarrow L^{p'_1}(\Omega)$ and $L^{p_2}(\partial\Omega) \rightarrow L^{p'_2}(\partial\Omega)$

$$\begin{aligned} u_n &\rightarrow u \quad \text{in } L^{p_1}(\Omega), & f(x, u_n) &\rightarrow f(x, u) \quad \text{in } L^{p'_1}(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^{p_2}(\partial\Omega), & g(x, u_n) &\rightarrow g(x, u) \quad \text{in } L^{p'_2}(\partial\Omega). \end{aligned} \quad (2.14)$$

Then fixing k in (2.7) and letting $n \rightarrow \infty$, we conclude that

$$\int_{\Omega} \nabla u \nabla \psi_k dx + \int_{\Omega} u \psi_k dx - \int_{\Omega} \frac{f(x, u)}{M(v)} \psi_k dx - \int_{\partial\Omega} g(x, u) \psi_k dy = 0. \quad (2.15)$$

From the completeness of ψ_k , identity holds with ψ_k replaced by any $\psi \in W^{1,2}(\Omega)$. In particular, when $\psi = u$, we get

$$\int_{\Omega} \nabla u \nabla \psi_k dx + \int_{\Omega} u \psi_k dx - \int_{\Omega} \frac{f(x, u)}{M(v)} \psi_k dx - \int_{\partial\Omega} g(x, u) \psi_k dy = 0. \quad (2.16)$$

On the other hand, let $\psi_k = u_n$ in (2.7) and passing to the limit, we get

$$v - \int_{\Omega} \frac{f(x, u)u}{M(v)} dx - \int_{\partial\Omega} g(x, u)u dy = 0. \quad (2.17)$$

Then we conclude that $v = \|u\|^2$, which shows that u is a solution of (1.1). Finally, if u is any solution of (1.1) and u is nontrivial, then

$$\begin{aligned} \|u\|^2 - \int_{\Omega} \frac{f(x, u)u}{M(\|u\|^2)} dx - \int_{\partial\Omega} g(x, u)u dy &= 0, \\ \|u\| &\leq \frac{\eta|\Omega|^{1/2}}{(m_0 - \lambda - \mu)}. \end{aligned} \quad (2.18)$$

The proof is complete. □

3. Variational Method

In this section, we consider the following problem:

$$\begin{aligned} M\left(\int_{\Omega} |\nabla u|^2 + |u|^2 dx\right) (-\Delta u + u) &= au^{p_1-1} + b, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} &= cu^{p_2-1} + d, \quad \text{on } \partial\Omega, \end{aligned} \quad (3.1)$$

where a, b, c, d are constants, and p_1, p_2 are defined in (H2).

The nontrivial solution of (3.1) comes from the Mountain Pass Lemma in [17].

Lemma 3.1 (Mountain Pass Lemma). *Let E be a Banach space and let $I \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Suppose also that*

- (i) $I(0) = 0$,
- (ii) there exist constants $r, a > 0$ such that $I(u) \geq a$, if $\|u\| = r$,
- (iii) there exists an element $v \in H$ with $\|v\| > r$, $I(v) \leq 0$.

Define $\Gamma := \{g \in C[0, 1]; H : g(0) = 0, g(1) = v\}$. Then

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I[g(t)] \quad (3.2)$$

is a critical value of I .

Theorem 3.2. *Assume the conditions (H1)–(H3) hold. In addition, the function M satisfies*

- (H5) there exist $m_1 \geq m_0$ with $(m_0/2) - (m_1/p) > 0$ and $t_0 > 0$, such that $M(t) = m_1, \forall t \geq t_0$, where $p = \min\{p_1, p_2\}$.

Then (3.1) has a nontrivial solution.

Proof. The weak solutions of (3.1) are critical points of the functional $J : W^{1,2}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \widehat{M}(\|u\|^2) - \frac{1}{p_1} \int_{\Omega} au^{p_1} dx - \frac{1}{p_2} \int_{\partial\Omega} cu^{p_2} dy - \int_{\Omega} bu dx - \int_{\partial\Omega} du dy, \quad (3.3)$$

where $\widehat{M}(t) = \int_0^t M(s) ds$.

Let us check the (PS) condition. Let $\varphi \in W^{1,2}$, we have

$$\begin{aligned} J'(u)\varphi &= M(\|u\|^2) \left(\int_{\Omega} (\nabla u \nabla \varphi + u\varphi) dx \right) - \int_{\Omega} au^{p_1-1} \varphi dx \\ &\quad - \int_{\partial\Omega} cu^{p_2-1} \varphi dy - \int_{\Omega} b\varphi dx - \int_{\partial\Omega} d\varphi dy. \end{aligned} \quad (3.4)$$

Let $\{u_n\}$ be a Palais-Smale sequence in $W^{1,2}(\Omega)$, that is, $J(u_n) \rightarrow \bar{c}$ and $J'(u_n) \rightarrow 0$ and assume the contradiction that $\|u_n\| \rightarrow +\infty$, then, from (H1), (H5), we have

$$\begin{aligned} J(u_n) - \frac{1}{p} J'(u_n)u_n &\geq \left(\frac{m_0}{2} - \frac{m_1}{p}\right) \|u_n\|^2 - \left(\frac{1}{p_1} - \frac{1}{p}\right) \int_{\Omega} a u_n^{p_1} dx \\ &\quad - \left(\frac{1}{p_2} - \frac{1}{p}\right) \int_{\partial\Omega} c u_n^{p_2} dy - \left(1 - \frac{1}{p}\right) \int_{\Omega} b u_n dx - \left(1 - \frac{1}{p}\right) \int_{\partial\Omega} d u_n dy, \end{aligned} \quad (3.5)$$

where $a, b, c, d > 0$. Then by the Sobolev embedding theorem and Lemma 2.2, we can select $C > 0$ such that

$$C + C\|u_n\| \geq m_0 \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2, \quad (3.6)$$

which is a contradiction with $\|u_n\| \rightarrow +\infty$. Hence $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$. So $\{u_n\}$ admits a weakly convergence subsequence. From (H2), all the growth of f, g is subcritical, so the standard argument shows that $\{u_n\}$ admits a strongly convergence subsequence.

Next we will verify the hypotheses of Lemma 3.1. By Hölder's inequality, Sobolev embedding theorem, and Lemma 2.2, we have

$$\begin{aligned} \int_{\Omega} a|u|^{p_1} dx &= a\|u\|_{L^{p_1}(\Omega)}^{p_1} \leq N_1\|u\|^{p_1}, & \int_{\partial\Omega} c|u|^{p_2} dy &= c\|u\|_{L^{p_2}(\partial\Omega)}^{p_2} \leq N_2\|u\|^{p_2}, \\ \int_{\Omega} b|u| dx &\leq bN_3\|u\|, & \int_{\partial\Omega} d|u| dy &\leq dN_4\|u\|. \end{aligned} \quad (3.7)$$

So we obtain

$$J(u) \geq \frac{1}{2}m_0\|u\|^2 - N_1\|u\|^{p_1} - N_2\|u\|^{p_2} - bN_3\|u\| - dN_4\|u\|. \quad (3.8)$$

Let $\|u\| < 1$, we get

$$J(u) \geq \frac{1}{2}m_0\|u\|^2 - N_5\|u\|^p - bN_3\|u\| - dN_4\|u\|. \quad (3.9)$$

Let $h(r) = (1/2)m_0r^2 - N_5r^p - N_6\varepsilon \cdot r$, then we take $r = r_0 = 3N_6\varepsilon/2m_0$ such that $h(r_0) = \bar{a} = (3N_6^2/m_0)\varepsilon^2 - N_53^pN_6^p\varepsilon^p > 0$, when ε is sufficient small.

So for b and d small enough, then we have $J(u) \geq \bar{a} > 0$ for all $\|u\| = r_0$.

On the other hand, take $\omega_0 \in W^{1,2}(\Omega)$ with $\int_{\Omega} a\omega_0^{p_1} dx = 1$ for $k > 0$, we have

$$\begin{aligned} J(k\omega_0) &= \frac{1}{2} \widehat{M}(\|k\omega_0\|^2) - \frac{k^{p_1}}{p_1} \int_{\Omega} a\omega_0^{p_1} dx - \frac{1}{p_2} \int_{\partial\Omega} c(k\omega_0)^{p_2} dy - \int_{\Omega} bk\omega_0 dx - \int_{\partial\Omega} kd\omega_0 dy \\ &\leq \frac{1}{2} m_1 \|k\omega_0\|^2 - \frac{k^{p_1}}{p_1} \int_{\Omega} a\omega_0^{p_1} dx - \frac{1}{p_2} \int_{\partial\Omega} c(k\omega_0)^{p_2} dy - \int_{\Omega} bk\omega_0 dx - \int_{\partial\Omega} kd\omega_0 dy. \end{aligned} \quad (3.10)$$

Since $p_1, p_2 > 2$, we obtain $J(k\omega_0) \rightarrow -\infty$ when $k \rightarrow +\infty$.

Let $\omega = k\omega_0$, with k large enough, we have $\|\omega\| > \max\{t_0, r_0\}$ and $J(\omega) < \bar{a}$. So by the Mountain Pass Lemma and (H3), we have a nontrivial solution $u(x)$ for (3.1). The proof is complete. \square

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