Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 540863, 20 pages
doi:10.1155/2009/540863

*Research Article*

# Recent Existence Results for Second-Order Singular Periodic Differential Equations

**Jifeng Chu[1, 2] and Juan J. Nieto[3]**

[1] *Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China*
[2] *Department of Mathematics, Pusan National University, Busan 609-735, South Korea*
[3] *Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Compostela, Spain*

Correspondence should be addressed to Jifeng Chu, jifengchu@126.com

Received 12 February 2009; Accepted 29 April 2009

Recommended by Donal O'Regan

We present some recent existence results for second-order singular periodic differential equations. A nonlinear alternative principle of Leray-Schauder type, a well-known fixed point theorem in cones, and Schauder's fixed point theorem are used in the proof. The results shed some light on the differences between a strong singularity and a weak singularity.



## 1. Introduction

The main aim of this paper is to present some recent existence results for the positive $T$-periodic solutions of second order differential equation

$$x'' + a(t)x = f(t,x) + e(t), \tag{1.1}$$

where $a(t), e(t)$ are continuous and $T$-periodic functions. The nonlinearity $f(t,x)$ is continuous in $(t,x)$ and $T$-periodic in $t$. We are mainly interested in the case that $f(t,x)$ has a repulsive singularity at $x = 0$:

$$\lim_{x\to 0^+} f(t,x) = +\infty, \quad \text{uniformly in } t. \tag{1.2}$$

It is well known that second order singular differential equations describe many problems in the applied sciences, such as the Brillouin focusing system [1] and nonlinear elasticity [2]. Therefore, during the last two decades, singular equations have attracted many researchers, and many important results have been proved in the literature; see, for example, [3–10]. Recently, it has been found that a particular case of (1.1), the Ermakov-Pinney equation

$$x'' + a(t)x = \frac{1}{x^3} \tag{1.3}$$

plays an important role in studying the Lyapunov stability of periodic solutions of Lagrangian equations [11–13].

In the literature, two different approaches have been used to establish the existence results for singular equations. The first one is the variational approach [14–16], and the second one is topological methods. Because we mainly focus on the applications of topological methods to singular equations in this paper, here we try to give a brief sketch of this problem. As far as the authors know, this method was started with the pioneering paper of Lazer and Solimini [17]. They proved that a necessary and sufficient condition for the existence of a positive periodic solution for equation

$$x'' = \frac{1}{x^\lambda} + e(t) \tag{1.4}$$

is that the mean value of $e$ is negative, $\overline{e} < 0$, here $\lambda \geq 1$, which is a strong force condition in a terminology first introduced by Gordon [18]. Moreover, if $0 < \lambda < 1$, which corresponds to a weak force condition, they found examples of functions $e$ with negative mean values and such that periodic solutions do not exist. Since then, the strong force condition became standard in the related works; see, for instance, [2, 8–10, 13, 19–21], and the recent review [22]. With a strong singularity, the energy near the origin becomes infinity and this fact is helpful for obtaining the a priori bounds needed for a classical application of the degree theory. Compared with the case of a strong singularity, the study of the existence of periodic solutions under the presence of a weak singularity by topological methods is more recent but has also attracted many researchers [4, 6, 23–28]. In [27], for the first time in this topic, Torres proved an existence result which is valid for a weak singularity whereas the validity of such results under a strong force assumption remains as an open problem. Among topological methods, the method of upper and lower solutions [6, 29, 30], degree theory [8, 20, 31], some fixed point theorems in cones for completely continuous operators [25, 32–34], and Schauder's fixed point theorem [27, 35, 36] are the most relevant tools.

In this paper, we select several recent existence results for singular equation (1.1) via different topological tools. The remaining part of the paper is organized as follows. In Section 2, some preliminary results are given. In Section 3, we present the first existence result for (1.1) via a nonlinear alternative principle of Leray-Schauder. In Section 4, the second existence result is established by using a well-known fixed point theorem in cones. The condition imposed on $a(t)$ in Sections 3 and 4 is that the Green function $G(t,s)$ associated with the linear periodic equations is positive, and therefore the results cannot cover the critical case, for example, when $a$ is a constant, $a(t) = k^2$, $0 < k < \sqrt{\lambda_1} = \pi/T$, and $\lambda_1$ is the first eigenvalue of the linear problem with Dirichlet conditions $x(0) = x(T) = 0$. Different from Sections 3 and 4, the results obtained in Section 5, which are established by Schauder's fixed point theorem, can cover the critical case because we only need that the Green function $G(t,s)$ is nonnegative. All results in Sections 3–5 shed some lights on the differences between a strong singularity and a weak singularity.

To illustrate our results, in Sections 3–5, we have selected the following singular equation:

$$x'' + a(t)x = x^{-\alpha} + \mu x^{\beta} + e(t), \tag{1.5}$$

here $a, e \in \mathbb{C}[0,T]$, $\alpha,\ \beta > 0$, and $\mu \in \mathbb{R}$ is a given parameter. The corresponding results are also valid for the general case

$$x'' + a(t)x = \frac{b(t)}{x^{\alpha}} + \mu c(t)x^{\beta} + e(t), \tag{1.6}$$

with $b, c \in \mathbb{C}[0,T]$. Some open problems for (1.5) or (1.6) are posed.

In this paper, we will use the following notation. Given $\psi \in L^1[0,T]$, we write $\psi \succ 0$ if $\psi \geq 0$ for a.e. $t \in [0,T]$, and it is positive in a set of positive measure. For a given function $p \in L^1[0,T]$ essentially bounded, we denote the essential supremum and infimum of $p$ by $p^*$ and $p_*$, respectively.

## 2. Preliminaries

Consider the linear equation

$$x'' + a(t)x = p(t) \tag{2.1}$$

with periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{2.2}$$

In Sections 3 and 4, we assume that

(A) the Green function $G(t,s)$, associated with (2.1)–(2.2), is positive for all $(t,s) \in [0,T] \times [0,T]$.

In Section 5, we assume that

(B) the Green function $G(t,s)$, associated with (2.1)–(2.2), is nonnegative for all $(t,s) \in [0,T] \times [0,T]$.

When $a(t) = k^2$, condition (A) is equivalent to $0 < k^2 < \lambda_1 = (\pi/T)^2$ and condition (B) is equivalent to $0 < k^2 \leq \lambda_1$. In this case, we have

$$G(t,s) = \begin{cases} \dfrac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leq s \leq t \leq T, \\[2ex] \dfrac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leq t \leq s \leq T. \end{cases} \tag{2.3}$$

For a nonconstant function $a(t)$, there is an $L^p$-criterion proved in [37], which is given in the following lemma for the sake of completeness. Let $K(q)$ denote the best Sobolev constant in the following inequality:

$$C\|u\|_q^2 \leq \|u'\|_2^2, \quad \forall u \in H_0^1(0,T). \tag{2.4}$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \dfrac{2\pi}{qT^{1+2/q}}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2 & \text{if } 1 \leq q < \infty, \\ \dfrac{4}{T}, & \text{if } q = \infty, \end{cases} \tag{2.5}$$

where $\Gamma$ is the Gamma function; see [21, 38]

**Lemma 2.1.** *Assume that $a(t) \succ 0$ and $a \in L^p[0,T]$ for some $1 \leq p \leq \infty$. If*

$$\|a\|_p < \mathbf{K}(2\widetilde{p}), \tag{2.6}$$

*then the condition (A) holds. Moreover, condition (B) holds if*

$$\|a\|_p \leq \mathbf{K}(2\widetilde{p}). \tag{2.7}$$

When the hypothesis (A) is satisfied, we denote

$$m = \min_{0 \leq s,t \leq T} G(t,s), \quad M = \max_{0 \leq s,t \leq T} G(t,s), \quad \sigma = \frac{m}{M}. \tag{2.8}$$

Obviously, $M > m > 0$ and $0 < \sigma < 1$.

Throughout this paper, we define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) = \int_0^T G(t,s)e(s)ds, \tag{2.9}$$

which corresponds to the unique $T$-periodic solution of

$$x'' + a(t)x = e(t). \tag{2.10}$$

## 3. Existence Result (I)

In this section, we state and prove the first existence result for (1.1). The proof is based on the following nonlinear alternative of Leray-Schauder, which can be found in [39]. This part can be regarded as the scalar version of the results in [4].

**Lemma 3.1.** *Assume $\Omega$ is a relatively compact subset of a convex set $K$ in a normed space $X$. Let $T : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:*

(a) *$T$ has at least one fixed point in $\overline{\Omega}$;*

(b) *thereexist $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda T x$.*

**Theorem 3.2.** *Suppose that $a(t)$ satisfies (A) and $f(t, x)$ satisfies the following.*

($H_1$) *There exist constants $\sigma > 0$ and $\nu \geq 1$ such that*

$$f(t,x) \geq \sigma x^{-\nu}, \quad \forall t \in [0,T], \quad \forall 0 < x \ll 1. \tag{3.1}$$

($H_2$) *There exist continuous, nonnegative functions $g(x)$ and $h(x)$ such that*

$$0 \leq f(t,x) \leq g(x) + h(x) \quad \forall (t,x) \in [0,T] \times (0,\infty), \tag{3.2}$$

*$g(x) > 0$ is nonincreasing and $h(x)/g(x)$ is nondecreasing in $x \in (0,\infty)$.*

($H_3$) *There exists a positive number $r$ such that $\sigma r + \gamma_* > 0$, and*

$$\frac{r}{g(\sigma r + \gamma_*)\{1 + h(r + \gamma^*)/g(r + \gamma^*)\}} > \omega^*, \quad \text{here } \omega(t) = \int_0^T G(t,s)ds. \tag{3.3}$$

*Then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, (1.1) has at least one positive periodic solution $x$ with $x(t) > \gamma(t)$ for all $t$ and $0 < \|x - \gamma\| < r$.*

*Proof.* The existence is proved using the Leray-Schauder alternative principle, together with a truncation technique. The idea is that we show that

$$x'' + a(t)x = f(t, x(t) + \gamma(t)) \tag{3.4}$$

has a positive periodic solution $x$ satisfying $x(t) + \gamma(t) > 0$ for $t$ and $0 < \|x\| < r$. If this is true, it is easy to see that $u(t) = x(t) + \gamma(t)$ will be a positive periodic solution of (1.1) with $0 < \|u - \gamma\| < r$ since

$$u'' + a(t)u = x'' + \gamma'' + a(t)x + a(t)\gamma = f(t, x + \gamma) + e(t) = f(t,u) + e(t). \tag{3.5}$$

Since ($H_3$) holds, we can choose $n_0 \in \{1, 2, \cdots\}$ such that $1/n_0 < \sigma r + \gamma_*$ and

$$\omega^* g(\sigma r + \gamma_*)\left\{1 + \frac{h(r + \gamma^*)}{g(r + \gamma^*)}\right\} + \frac{1}{n_0} < r. \tag{3.6}$$

Let $N_0 = \{n_0, n_0 + 1, \cdots\}$. Consider the family of equations

$$x'' + a(t)x = \lambda f_n(t, x(t) + \gamma(t)) + \frac{a(t)}{n}, \tag{3.7}$$

where $\lambda \in [0,1], n \in N_0$, and

$$f_n(t,x) = \begin{cases} f(t,x), & \text{if } x \geq \dfrac{1}{n}, \\ f\left(t, \dfrac{1}{n}\right), & \text{if } x \leq \dfrac{1}{n}. \end{cases} \tag{3.8}$$

Problem (3.7) is equivalent to the following fixed point problem:

$$x = \lambda T_n x + \frac{1}{n}, \tag{3.9}$$

where $T_n$ is defined by

$$(T_n x)(t) = \lambda \int_0^T G(t,s) f_n(s, x(s) + \gamma(s)) ds + \frac{1}{n}. \tag{3.10}$$

We claim that any fixed point $x$ of (3.9) for any $\lambda \in [0,1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that $x$ is a fixed point of (3.9) for some $\lambda \in [0,1]$ such that $\|x\| = r$. Note that

$$\begin{aligned} x(t) - \frac{1}{n} &= \lambda \int_0^T G(t,s) f_n(s, x(s) + \gamma(s)) ds \\ &\geq \lambda m \int_0^T f_n(s, x(s) + \gamma(s)) ds \\ &= \sigma M \lambda \int_0^T f_n(s, x(s) + \gamma(s)) ds \\ &\geq \sigma \max_{t \in [0,T]} \left\{ \lambda \int_0^T G(t,s) f_n(s, x(s) + \gamma(s)) ds \right\} \\ &= \sigma \left\| x - \frac{1}{n} \right\|. \end{aligned} \tag{3.11}$$

By the choice of $n_0$, $1/n \leq 1/n_0 < \sigma r + \gamma_*$. Hence, for all $t \in [0,T]$, we have

$$x(t) \geq \sigma \left\| x - \frac{1}{n} \right\| + \frac{1}{n} \geq \sigma \left( \|x\| - \frac{1}{n} \right) + \frac{1}{n} \geq \sigma r. \tag{3.12}$$

Therefore,

$$x(t) + \gamma(t) \geq \sigma r + \gamma_* > \frac{1}{n}. \tag{3.13}$$

Thus we have from condition $(\mathrm{H}_2)$, for all $t \in [0,T]$,

$$\begin{aligned} x(t) &= \lambda \int_0^T G(t,s) f_n(s, x(s)+\gamma(s))ds + \frac{1}{n} \\ &= \lambda \int_0^T G(t,s) f(s, x(s)+\gamma(s))ds + \frac{1}{n} \\ &\le \int_0^T G(t,s) f(s, x(s)+\gamma(s))ds + \frac{1}{n} \\ &\le \int_0^T G(t,s) g(x(s)+\gamma(s)) \left\{1 + \frac{h(x(s)+\gamma(s))}{g(x(s)+\gamma(s))}\right\} ds + \frac{1}{n} \\ &\le g(\sigma r + \gamma_*) \left\{1 + \frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\} \int_0^T G(t,s)ds + \frac{1}{n} \\ &\le g(\sigma r + \gamma_*) \left\{1 + \frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\} \omega^* + \frac{1}{n_0}. \end{aligned} \tag{3.14}$$

Therefore,

$$r = \|x\| \le g(\sigma r + \gamma_*) \left\{1 + \frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\} \omega^* + \frac{1}{n_0}. \tag{3.15}$$

This is a contradiction to the choice of $n_0$, and the claim is proved.

From this claim, the Leray-Schauder alternative principle guarantees that

$$x = T_n x + \frac{1}{n} \tag{3.16}$$

has a fixed point, denoted by $x_n$, in $B_r = \{x \in X : \|x\| < r\}$, that is, equation

$$x'' + a(t)x = f_n(t, x(t)+\gamma(t)) + \frac{a(t)}{n} \tag{3.17}$$

has a periodic solution $x_n$ with $\|x_n\| < r$. Since $x_n(t) \ge 1/n > 0$ for all $t \in [0,T]$ and $x_n$ is actually a positive periodic solution of (3.17).

In the next lemma, we will show that there exists a constant $\delta > 0$ such that

$$x_n(t) + \gamma(t) \ge \delta, \quad \forall t \in [0,T], \tag{3.18}$$

for $n$ large enough.

In order to pass the solutions $x_n$ of the truncation equations (3.17) to that of the original equation (3.4), we need the following fact:

$$\|x'_n\| \le H \tag{3.19}$$

for some constant $H > 0$ and for all $n \geq n_0$. To this end, by the periodic boundary conditions, $x'_n(t_0) = 0$ for some $t_0 \in [0, T]$. Integrating (3.17) from 0 to $T$, we obtain

$$\int_0^T a(t)x_n(t)dt = \int_0^T \left[ f_n\left(t, x_n(t) + \gamma(t)\right) + \frac{a(t)}{n} \right] dt. \tag{3.20}$$

Therefore

$$\begin{aligned}
\|x'_n\| &= \max_{0\le t\le T} |x'_n(t)| = \max_{0\le t\le T} \left| \int_{t_0}^t x''_n(s)ds \right| \\
&= \max_{0\le t\le T} \left| \int_{t_0}^t \left[ f_n\left(s, x_n(s) + \gamma(s)\right) + \frac{a(s)}{n} - a(s)x_n(s) \right] ds \right| \\
&\le \int_0^T \left[ f_n\left(s, x_n(s) + \gamma(s)\right) + \frac{a(s)}{n} \right] ds + \int_0^T a(s)x_n(s)ds \\
&= 2\int_0^T a(s)x_n(s)ds < 2r\|a\|_1 = H.
\end{aligned} \tag{3.21}$$

The fact $\|x_n\| < r$ and (3.19) show that $\{x_n\}_{n\in N_0}$ is a bounded and equicontinuous family on $[0, T]$. Now the Arzela-Ascoli Theorem guarantees that $\{x_n\}_{n\in N_0}$ has a subsequence, $\{x_{n_k}\}_{k\in\mathbb{N}}$, converging uniformly on $[0, T]$ to a function $x \in X$. Moreover, $x_{n_k}$ satisfies the integral equation

$$x_{n_k}(t) = \int_0^T G(t,s)f\left(s, x_{n_k}(s) + \gamma(s)\right)ds + \frac{1}{n_k}. \tag{3.22}$$

Letting $k \to \infty$, we arrive at

$$x(t) = \int_0^T G(t,s)f\left(s, x(s) + \gamma(s)\right)ds, \tag{3.23}$$

where the uniform continuity of $f(t,x)$ on $[0,T] \times [\delta, r + \gamma^*]$ is used. Therefore, $x$ is a positive periodic solution of (3.4). □

**Lemma 3.3.** *There exist a constant $\delta > 0$ and an integer $n_2 > n_0$ such that any solution $x_n$ of* (3.17) *satisfies* (3.18) *for all $n \geq n_2$.*

*Proof.* The lower bound in (3.18) is established using the strong force condition $(\mathrm{H}_1)$ of $f(t,x)$. By condition $(\mathrm{H}_1)$, there exists $c_0 \in (0,1)$ small enough such that

$$f(t,x) \geq \sigma c_0^{-\nu} > \max\left\{ r\|a\|_1, a^*\left(r + \gamma^*\right) + e^* \right\}, \quad \forall 0 \le t \le T, \quad 0 < x \le c_0. \tag{3.24}$$

Take $n_1 \in N_0$ such that $1/n_1 \leq c_0$ and let $N_1 = \{n_1, n_1+1, \cdots\}$. For $n \in N_1$, let

$$\alpha_n = \min_{0\leq t\leq T}\left[x_n(t) + \gamma(t)\right], \qquad \beta_n = \max_{0\leq t\leq T}\left[x_n(t) + \gamma(t)\right]. \tag{3.25}$$

We claim first that $\beta_n > c_0$ for all$n \in N_1$. Otherwise, suppose that $\beta_n \leq c_0$ for some $n \in N_1$. Then from (3.24), it is easy to verify

$$f_n\left(t, x_n(t) + \gamma(t)\right) > r\|a\|_1. \tag{3.26}$$

Integrating (3.17) from 0 to $T$, we deduce that

$$\begin{aligned}
0 &= \int_0^T \left[x_n''(t) + a(t)x_n(t) - f_n\left(t, x_n(t) + \gamma(t)\right) - \frac{a(t)}{n}\right]dt \\
&= \int_0^T a(t)x_n(t)dt - \left(\frac{1}{n}\right)\int_0^T a(t)dt - \int_0^T f_n\left(t, x_n(t) + \gamma(t)\right)dt \\
&< \int_0^T a(t)x_n(t)dt - r\|a\|_1 \leq 0.
\end{aligned} \tag{3.27}$$

This is a contradiction. Thus $\beta_n > c_0$ for $n \in N_1$.

Now we consider the minimum values $\alpha_n$. Let $n \geq n_1$. Without loss of generality, we assume that $\alpha_n < c_0$, otherwise we have (3.18). In this case,

$$\alpha_n = \min_{0\leq t\leq T}\left[x_n(t) + \gamma(t)\right] = x_n(t_n) + \gamma(t_n) < c_0 \tag{3.28}$$

for some $t_n \in [0, T]$. As $\beta_n > c_0$, there exists $c_n \in [0, 1]$ (without loss of generality, we assume $t_n < c_n$) such that $x_n(c_n) + \gamma(c_n) = c_0$ and $x_n(t) + \gamma(t) \leq c_0$ for $t_n \leq t \leq c_n$. By (3.24), it can be checked that

$$f_n\left(t, x_n(t) + \gamma(t)\right) > a(t)\left(x_n(t) + \gamma(t)\right) + e(t), \quad \forall t \in [t_n, c_n]. \tag{3.29}$$

Thus for $t \in (t_n, c_n]$, we have $x_n''(t) + \gamma''(t) > 0$. As $x_n'(t_n) + \gamma'(t_n) = 0$, $x_n'(t) + \gamma'(t) > 0$ for all $t \in (t_n, c_n]$ and the function $y_n := x_n + \gamma$ is strictly increasing on $[t_n, c_n]$. We use $\xi_n$ to denote the inverse function of $y_n$ restricted to $[t_n, c_n]$.

In order to prove (3.18) in this case, we first show that, for $n \in N_1$,

$$x_n(t) + \gamma(t) \geq \frac{1}{n}. \tag{3.30}$$

Otherwise, suppose that $\alpha_n < 1/n$ for some $n \in N_1$. Then there would exist $b_n \in (t_n, c_n)$ such that $x_n(b_n) + \gamma(b_n) = 1/n$ and

$$x_n(t) + \gamma(t) \le \frac{1}{n} \quad \text{for } t_n \le t \le b_n, \qquad \frac{1}{n} \le x_n(t) + \gamma(t) \le c_0 \quad \text{for } b_n \le t \le c_n. \tag{3.31}$$

Multiplying (3.17) by $x_n'(t) + \gamma'(t)$ and integrating from $b_n$ to $c_n$, we obtain

$$\begin{aligned} \int_{1/n}^{R_1} f(\xi_n(y), y)\,dy &= \int_{b_n}^{c_n} f(t, x_n(t) + \gamma(t))(x_n'(t) + \gamma'(t))\,dt \\ &= \int_{b_n}^{c_n} f_n(t, x_n(t) + \gamma(t))(x_n'(t) + \gamma'(t))\,dt \\ &= \int_{b_n}^{c_n} \left(x_n''(t) + a(t)x_n(t) - \frac{a(t)}{n}\right)(x_n'(t) + \gamma'(t))\,dt \\ &= \int_{b_n}^{c_n} x_n''(t)(x_n'(t) + \gamma'(t))\,dt \\ &\quad + \int_{b_n}^{c_n} \left(a(t)x_n(t) - \frac{a(t)}{n}\right)(x_n'(t) + \gamma'(t))\,dt. \end{aligned} \tag{3.32}$$

By the facts $\|x_n\| < r$ and $\|x_n'\| \le H$, one can easily obtain that the right side of the above equality is bounded. As a consequence, there exists $L > 0$ such that

$$\int_{1/n}^{R_1} f(\xi_n(y), y)\,dy \le L. \tag{3.33}$$

On the other hand, by the strong force condition $(\mathrm{H}_1)$, we can choose $n_2 \in N_1$ large enough such that

$$\int_{1/n}^{c_0} f(\xi_n(y), y)\,dy \ge \sigma \int_{1/n}^{c_0} y^{-\nu}\,dy > L \tag{3.34}$$

for all $n \in N_2 = \{n_2, n_2 + 1, \cdots\}$. So (3.30) holds for $n \in N_2$.

Finally, multiplying (3.17) by $x_n'(t) + \gamma'(t)$ and integrating from $t_n$ to $c_n$, we obtain

$$\begin{aligned} \int_{\alpha_n}^{c_0} f(\xi_n(y), y)\,dy &= \int_{t_n}^{c_n} f(t, x_n(t) + \gamma(t))(x_n'(t) + \gamma'(t))\,dt \\ &= \int_{t_n}^{c_n} f_n(t, x_n(t) + \gamma(t))(x_n'(t) + \gamma'(t))\,dt \\ &= \int_{t_n}^{c_n} \left(\frac{x_n''(t) + a(t)x_n(t) - a(t)}{n}\right)(x_n'(t) + \gamma'(t))\,dt. \end{aligned} \tag{3.35}$$

(We notice that the estimate (3.30) is used in the second equality above). In the same way, one may readily prove that the right-hand side of the above equality is bounded. On the other hand, if $n \in N_2$, by $(\mathrm{H}_1)$,

$$\int_{\alpha_n}^{c_0} f(\xi_n(y), y)dy \geq \sigma \int_{\alpha_n}^{c_0} y^{-\nu}dy \longrightarrow +\infty \tag{3.36}$$

if $\alpha_n \to 0^+$. Thus we know that $\alpha_n \geq \delta$ for some constant $\delta > 0$. □

From the proof of Theorem 3.2 and Lemma 3.3, we see that the strong force condition $(\mathrm{H}_1)$ is only used when we prove (3.18). From the next theorem, we will show that, for the case $\gamma_* \geq 0$, we can remove the strong force condition $(\mathrm{H}_1)$, and replace it by one weak force condition.

**Theorem 3.4.** *Assume that* (A) *and* $(\mathrm{H}_2)$–$(\mathrm{H}_3)$ *are satisfied. Suppose further that*

$(\mathrm{H}_4)$ *for each constant $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that $f(t,x) \geq \phi_L(t)$ for all $(t,x) \in [0,T] \times (0,L]$.*

*Then for each $e(t)$ with $\gamma_* \geq 0$,* (1.1) *has at least one positive periodic solution $x$ with $x(t) > \gamma(t)$ for all $t$ and $0 < \|x - \gamma\| < r$.*

*Proof.* We only need to show that (3.18) is also satisfied under condition $(\mathrm{H}_4)$ and $\gamma_* \geq 0$. The rest parts of the proof are in the same line of Theorem 3.2. Since $(\mathrm{H}_4)$ holds, there exists a continuous function $\phi_{r+\gamma^*} \succ 0$ such that $f(t,x) \geq \phi_{r+\gamma^*}(t)$ for all $(t,x) \in [0,T] \times (0, r+\gamma^*]$. Let $x_{r+\gamma^*}$ be the unique periodic solution to the problems (2.1)–(2.2) with $h = \phi_{r+\gamma^*}$. That is

$$x_{r+\gamma^*}(t) = \int_0^T G(t,s)\phi_{r+\gamma^*}(s)ds. \tag{3.37}$$

Then we have

$$x_{r+\gamma^*}(t) + \gamma(t) = \int_0^T G(t,s)\phi_{r+\gamma^*}(s)ds + \gamma(t) \geq \Phi_* + \gamma_* > 0, \tag{3.38}$$

here

$$\Phi(t) = \int_0^T G(t,s)\phi_{r+\gamma^*}(s)ds. \tag{3.39}$$

□

**Corollary 3.5.** *Assume that $a(t)$ satisfies* (A) *and $\alpha > 0, \beta \geq 0, \mu > 0$. Then*

(i) *if $\alpha \geq 1, \beta < 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$,* (1.5) *has at least one positive periodic solution for all $\mu > 0$;*

(ii) *if $\alpha \geq 1, \beta \geq 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$,* (1.5) *has at least one positive periodic solution for each $0 < \mu < \mu_1$, here $\mu_1$ is some positive constant.*

(iii) *if* $\alpha > 0, \beta < 1$, *then for each* $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ *with* $\gamma_* \geq 0$, (1.5) *has at least one positive periodic solution for all* $\mu > 0$;

(iv) *if* $\alpha > 0, \beta \geq 1$, *then for each* $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ *with* $\gamma_* \geq 0$, (1.5) *has at least one positive periodic solution for each* $0 < \mu < \mu_1$.

*Proof.* We apply Theorems 3.2 and 3.4. Take

$$g(x) = x^{-\alpha}, \qquad h(x) = \mu x^{\beta}, \tag{3.40}$$

then $(H_2)$ is satisfied, and the existence condition $(H_3)$ becomes

$$\mu < \frac{r(\sigma r + \gamma_*)^{\alpha} - \omega^*}{\omega^* (r + \gamma^*)^{\alpha+\beta}} \tag{3.41}$$

for some $r > 0$. Note that condition $(H_1)$ is satisfied when $\alpha \geq 1$, while $(H_4)$ is satisfied when $\alpha > 0$. So (1.5) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{r(\sigma r + \gamma_*)^{\alpha} - \omega^*}{\omega^* (r + \gamma^*)^{\alpha+\beta}}. \tag{3.42}$$

Note that $\mu_1 = \infty$ if $\beta < 1$ and $\mu_1 < \infty$ if $\beta \geq 1$. Thus we have (i)–(iv). □

## 4. Existence Result (II)

In this section, we establish the second existence result for (1.1) using a well-known fixed point theorem in cones. We are mainly interested in the superlinear case. This part is essentially extracted from [24].

First we recall this fixed point theorem in cones, which can be found in [40]. Let $K$ be a cone in $X$ and $D$ is a subset of $X$, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

**Theorem 4.1** (see [40]). *Let* $X$ *be a Banach space and* $K$ *a cone in* $X$. *Assume* $\Omega^1, \Omega^2$ *are open bounded subsets of* $X$ *with* $\Omega^1_K \neq \emptyset, \overline{\Omega}^1_K \subset \Omega^2_K$. *Let*

$$T : \overline{\Omega}^2_K \longrightarrow K \tag{4.1}$$

*be a completely continuous operator such that*

(a) $\|Tx\| \leq \|x\|$ *for* $x \in \partial_K \Omega^1$,

(b) *There exists* $v \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda v$ for all $x \in \partial_K \Omega^2$ *and all* $\lambda > 0$.

*Then* $T$ *has a fixed point in* $\overline{\Omega}^2_K \setminus \Omega^1_K$.

In applications below, we take $X = \mathbb{C}[0,T]$ with the supremum norm $\|\cdot\|$ and define

$$K = \left\{ x \in X : x(t) \geq 0 \quad \forall t \in [0,T], \quad \min_{0\leq t\leq T} x(t) \geq \sigma \|x\| \right\}. \tag{4.2}$$

**Theorem 4.2.** *Suppose that* $a(t)$ *satisfies* (A) *and* $f(t,x)$ *satisfies* ($H_2$)–($H_3$). *Furthermore, assume that*

($H_5$) *there exist continuous nonnegative functions* $g_1(x), h_1(x)$ *such that*

$$f(t,x) \geq g_1(x) + h_1(x), \quad \forall (t,x) \in [0,T] \times (0,\infty), \tag{4.3}$$

$g_1(x) > 0$ *is nonincreasing and* $h_1(x)/g_1(x)$ *is nondecreasing in* $x$;

($H_6$) *there exists* $R > 0$ *with* $\sigma R > r$ *such that*

$$\frac{\sigma R}{g_1(R + \gamma^*)\{1 + h_1(\sigma R + \gamma_*)/g_1(\sigma R + \gamma_*)\}} \leq \omega_*. \tag{4.4}$$

*Then* (1.1) *has one positive periodic solution* $\tilde{x}$ *with* $r < \|\tilde{x} - \gamma\| \leq R$.

*Proof.* As in the proof of Theorem 3.2, we only need to show that (3.4) has a positive periodic solution $u \in X$ with $u(t) + \gamma(t) > 0$ and $r < \|u\| \leq R$.

Let $K$ be a cone in $X$ defined by (4.2). Define the open sets

$$\Omega^1 = \{x \in X : \|x\| < r\}, \qquad \Omega^2 = \{x \in X : \|x\| < R\}, \tag{4.5}$$

and the operator $T : \overline{\Omega}^2_K \to K$ by

$$(Tx)(t) = \int_0^T G(t,s) f(s, x(s) + \gamma(s)) ds, \quad 0 \leq t \leq T. \tag{4.6}$$

For each $x \in \overline{\Omega}^2_K \setminus \Omega^1_K$, we have $r \leq \|x\| \leq R$. Thus $0 < \sigma r + \gamma_* \leq x(t) + \gamma(t) \leq R + \gamma^*$ for all $t \in [0,T]$. Since $f : [0,T] \times [\sigma r + \gamma_*, R + \gamma^*] \to [0,\infty)$ is continuous, then the operator $T : \overline{\Omega}^2_K \setminus \Omega^1_K \to K$ is well defined and is continuous and completely continuous. Next we claim that:

(i) $\|Tx\| \leq \|x\|$ for $x \in \partial_K \Omega^1$, and

(ii) there exists $v \in K \setminus \{0\}$ such that $x \neq Tx + \lambda v$ for all $x \in \partial_K \Omega^2$ and all $\lambda > 0$.

We start with (i). In fact, if $x \in \partial_K \Omega^1$, then $\|x\| = r$ and $\sigma r + \gamma_* \le x(t) + \gamma(t) \le r + \gamma^*$ for all $t \in [0, T]$. Thus we have

$$
\begin{aligned}
(Tx)(t) &= \int_0^T G(t,s) f(s, x(s) + \gamma(s)) ds \\
&\le \int_0^T G(t,s) g(x(s) + \gamma(s)) \left\{ 1 + \frac{h(x(s) + \gamma(s))}{g(x(s) + \gamma(s))} \right\} ds \\
&\le g(\sigma r + \gamma_*) \left\{ 1 + \frac{h(r + \gamma^*)}{g(r + \gamma^*)} \right\} \int_0^T G(t,s) ds \\
&\le g(\sigma r + \gamma_*) \left\{ 1 + \frac{h(r + \gamma^*)}{g(r + \gamma^*)} \right\} \omega^* < r = \|x\|.
\end{aligned}
\tag{4.7}
$$

Next we consider (ii). Let $v(t) \equiv 1$, then $v \in K \setminus \{0\}$. Next, suppose that there exists $x \in \partial_K \Omega^2$ and $\lambda > 0$ such that $x = Tx + \lambda v$. Since $x \in \partial_K \Omega^2$, then $\sigma R + \gamma_* \le x(t) + \gamma(t) \le R + \gamma^*$ for all $t \in [0, T]$. As a result, it follows from ($H_5$) and ($H_6$) that, for all $t \in [0, T]$,

$$
\begin{aligned}
x(t) = (Tx)(t) + \lambda &= \int_0^T G(t,s) f(s, x(s) + \gamma(s)) ds + \lambda \\
&\ge \int_0^T G(t,s) g_1(x(s) + \gamma(s)) \left\{ 1 + \frac{h_1(x(s) + \gamma(s))}{g_1(x(s) + \gamma(s))} \right\} ds + \lambda \\
&\ge g_1(R + \gamma^*) \left\{ 1 + \frac{h_1(\sigma R + \gamma_*)}{g_1(\sigma R + \gamma_*)} \right\} \int_0^T G(t,s) ds + \lambda \\
&\ge g_1(R + \gamma^*) \left\{ 1 + \frac{h_1(\sigma R + \gamma_*)}{g_1(\sigma R + \gamma_*)} \right\} \omega_* + \lambda \\
&> g_1(R + \gamma^*) \left\{ 1 + \frac{h_1(\sigma R + \gamma_*)}{g_1(\sigma R + \gamma_*)} \right\} \omega_* \ge \sigma R.
\end{aligned}
\tag{4.8}
$$

Hence $\min_{0 \le t \le T} x(t) > \sigma R$, this is a contradiction and we prove the claim. □

Now Theorem 4.1 guarantees that $T$ has at least one fixed point $x \in \overline{\Omega}_K^2 \setminus \Omega_K^1$ with $r \le \|x\| \le R$. Note $\|x\| \ne r$ by (4.7).

Combined Theorem 4.2 with Theorems 3.2 or 3.4, we have the following two multiplicity results.

**Theorem 4.3.** *Suppose that $a(t)$ satisfies* (A) *and $f(t, x)$ satisfies* ($H_1$)–( $H_3$) *and* ($H_5$)–( $H_6$)*. Then* (1.1) *has two different positive periodic solutions $x$ and $\tilde{x}$ with* $0 < \|x - \gamma\| < r < \|\tilde{x} - \gamma\| \le R$.

**Theorem 4.4.** *Suppose that $a(t)$ satisfies* (A) *and $f(t, x)$ satisfies* ($H_2$)–( $H_6$)*. Then* (1.1) *has two different positive periodic solutions $x$ and $\tilde{x}$ with* $0 < \|x - \gamma\| < r < \|\tilde{x} - \gamma\| \le R$.

**Corollary 4.5.** *Assume that $a(t)$ satisfies* (A) *and $\alpha > 0, \beta > 1, \mu > 0$. Then*

(i) *if $\alpha \geq 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$,* (1.5) *has at least two positive periodic solutions for each $0 < \mu < \mu_1$;*

(ii) *if $\alpha > 0$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* \geq 0$,* (1.5) *has at least two positive periodic solutions for each $0 < \mu < \mu_1$.*

*Proof.* Take $g_1(x) = x^{-\alpha}, h_1(x) = \mu x^{\beta}$. Then $(H_5)$ is satisfied and the existence condition $(H_6)$ becomes

$$\mu \geq \frac{\sigma R\left(R + \gamma^*\right)^{\alpha} - \omega_*}{\omega_*\left(\sigma R + \gamma_*\right)^{\alpha+\beta}}. \tag{4.9}$$

Since $\beta > 1$, it is easy to see that the right-hand side goes to 0 as $R \to +\infty$. Thus, for any given $0 < \mu < \mu_1$, it is always possible to find such $R \gg r$ that (4.9) is satisfied. Thus, (1.5) has an additional positive periodic solution $\widetilde{x}$. □

## 5. Existence Result (III)

In this section, we prove the third existence result for (1.1) by Schauder's fixed point theorem. We can cover the critical case because we assume that the condition (B) is satisfied. This part comes essentially from [35], and the results for the vector version can be found in [4].

**Theorem 5.1.** *Assume that conditions* (B) *and* $(H_2)$, $(H_4)$ *are satisfied. Furthermore, suppose that*

$(H_7)$ *there exists a positive constant $R > 0$ such that $R > \Phi_*, \Phi_* + \gamma_* > 0$ and $R \geq g(\Phi_* + \gamma_*)\{1 + h(R+\gamma^*)/g(R+\gamma^*)\}\omega^*$, here $\Phi_* = \min_t \Phi(t)$, $\quad \Phi(t) = \int_0^T G(t,s)\phi_{R+\gamma^*}(s)ds$.*

*Then* (1.1) *has at least one positive $T$-periodic solution.*

*Proof.* A $T$-periodic solution of (1.1) is just a fixed point of the map $T : X \to X$ defined by (4.6). Note that $T$ is a completely continuous map.

Let $R$ be the positive constant satisfying $(H_7)$ and $r = \Phi_* > 0$. Then we have $R > r > 0$. Now we define the set

$$\Omega = \{x \in X : r \leq x(t) \leq R \quad \forall t\}. \tag{5.1}$$

Obviously, $\Omega$ is a closed convex set. Next we prove $T(\Omega) \subset \Omega$.

In fact, for each $x \in \Omega$, using that $G(t,s) \geq 0$ and condition $(H_4)$,

$$(Tx)(t) \geq \int_0^T G(t,s)\phi_{R+\gamma^*}(s)ds \geq \Phi_* = r > 0. \tag{5.2}$$

On the other hand, by conditions ($\mathrm{H}_2$) and ($\mathrm{H}_7$), we have

$$\begin{aligned}(Tx)(t) &\leq \int_0^T G(t,s)g(x(s)+\gamma(s))\left\{1+\frac{h(x(s)+\gamma(s))}{g(x(s)+\gamma(s))}\right\}ds \\ &\leq g(\Phi_* + \gamma_*)\left\{1+\frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^* \leq R.\end{aligned} \tag{5.3}$$

In conclusion, $T(\Omega) \subset \Omega$. By a direct application of Schauder's fixed point theorem, the proof is finished. □

As an application of Theorem 5.1, we consider the case $\gamma_* = 0$. The following corollary is a direct result of Theorem 5.1.

**Corollary 5.2.** *Assume that conditions* (B) *and* ($\mathrm{H}_2$), ($\mathrm{H}_4$) *are satisfied. Furthermore, assume that*

($\mathrm{H}_8$) *there exists a positive constant $R > 0$ such that $R > \Phi_*$ and*

$$R \geq g(\Phi_*)\left\{1+\frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^*. \tag{5.4}$$

*If $\gamma_* = 0$, then* (1.1) *has at least one positive T-periodic solution.*

**Corollary 5.3.** *Suppose that $a$ satisfies* (B) *and $0 < \alpha < 1$, $\beta \geq 0$, then for each $e(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* = 0$,one has the following:*

(i) *if $\alpha + \beta < 1 - \alpha^2$, then* (1.5) *has at least one positive periodic solution for each $\mu \geq 0$.*

(ii) *if $\alpha + \beta \geq 1 - \alpha^2$, then* (1.5) *has at least one positive T-periodic solution for each $0 \leq \mu < \mu_2$, where $\mu_2$ is some positive constant.*

*Proof.* We apply Corollary 3.5 and follow the same notation as in the proof of Corollary 3.5. Then ($\mathrm{H}_2$) and ($\mathrm{H}_4$) are satisfied, and the existence condition ($\mathrm{H}_8$) becomes

$$\mu < \frac{R\Phi_*^{\alpha} - \omega^*}{\omega^*(R+\gamma^*)^{\alpha+\beta}}, \tag{5.5}$$

for some $R > 0$ with $R > \Phi_*$. Note that

$$\Phi_* = (R+\gamma^*)^{-\alpha}{}_{\omega_*}. \tag{5.6}$$

Therefore, (5.5) becomes

$$\mu < \frac{R(R+\gamma^*)^{-\alpha^2}\omega_*^{\alpha} - \omega^*}{\omega^*(R+\gamma^*)^{\alpha+\beta}}, \tag{5.7}$$

for some $R > 0$.

So (1.5) has at least one positive $T$-periodic solution for

$$0 < \mu < \mu_2 = \sup_{R>0} \frac{R(R+\gamma^*)^{-\alpha^2}\omega_*^{\alpha} - \omega^*}{\omega^*(R+\gamma^*)^{\alpha+\beta}}. \tag{5.8}$$

Note that $\mu_2 = \infty$ if $\alpha + \beta < 1 - \alpha^2$ and $\mu_2 < \infty$ if $\alpha + \beta \geq 1 - \alpha^2$. We have the desired results (i) and (ii). □

*Remark 5.4.* The validity of (ii) in Corollary 5.3 under strong force conditions remains still open to us. Such an open problem has been partially solved by Corollary 3.5. However, we do not solve it completely because we need the positivity of $G(t,s)$ in Corollary 3.5, and therefore it is not applicable to the critical case. The validity for the critical case remains open to the authors.

The next results explore the case when $\gamma_* > 0$.

**Theorem 5.5.** *Suppose that $a(t)$ satisfies* (B) *and $f(t,x)$ satisfies condition* ($H_2$)*. Furthermore, assume that*

($H_9$) *there exists $R > \gamma^*$ such that*

$$g(\gamma_*)\left\{1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^* \leq R. \tag{5.9}$$

*If $\gamma_* > 0$, then* (1.1) *has at least one positive $T$-periodic solution.*

*Proof.* We follow the same strategy and notation as in the proof of Theorem 5.1. Let $R$ be the positive constant satisfying ($H_9$) and $r = \gamma_*$, then $R > r > 0$ since $R > \gamma^*$. Next we prove $T(\Omega) \subset \Omega$.

For each $x \in \Omega$, by the nonnegative sign of $G(t,s)$ and $f(t,x)$, we have

$$(Tx)(t) = \int_0^T G(t,s)f(s,x(s))ds + \gamma(t) \geq \gamma_* = r > 0. \tag{5.10}$$

On the other hand, by ($H_2$) and ($H_9$), we have

$$\begin{aligned}(Tx)(t) &\leq \int_0^T G(t,s)g(x(s)+\gamma(s))\left\{1 + \frac{h(x(s)+\gamma(s))}{g(x(s)+\gamma(s))}\right\}ds \\ &\leq g(\gamma_*)\left\{1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^* \leq R.\end{aligned} \tag{5.11}$$

In conclusion, $T(\Omega) \subset \Omega$, and the proof is finished by Schauder's fixed point theorem. □

**Corollary 5.6.** *Suppose that $a(t)$ satisfies* (B) *and $\alpha, \beta \geq 0$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\gamma_* > 0$, one has the following:*

(i) *if $\alpha + \beta < 1$, then* (1.5) *has at least one positive $T$-periodic solution for each $\mu \geq 0$;*

(ii) *if $\alpha + \beta \geq 1$, then* (1.5) *has at least one positive $T$-periodic solution for each $0 \leq \mu < \mu_3$, where $\mu_3$ is some positive constant.*

*Proof.* We apply Theorem 5.5 and follow the same notation as in the proof of Corollary 3.5. Then $(H_2)$ is satisfied, and the existence condition $(H_9)$ becomes

$$\mu < \frac{R\gamma^{*\alpha} - \omega^*}{\omega^* \left(R + \gamma^*\right)^{\alpha+\beta}} \tag{5.12}$$

for some $R > 0$. So (1.5) has at least one positive $T$-periodic solution for

$$0 < \mu < \mu_3 = \sup_{R>0} \frac{R\gamma^{*\alpha} - \omega^*}{\omega^* \left(R + \gamma^*\right)^{\alpha+\beta}}. \tag{5.13}$$

Note that $\mu_3 = \infty$ if $\alpha + \beta < 1$ and $\mu_3 < \infty$ if $\alpha + \beta \geq 1$. We have the desired results (i) and (ii). □

## Acknowledgments

The authors express their thanks to the referees for their valuable comments and suggestions. The research of J. Chu is supported by the National Natural Science Foundation of China (Grant no. 10801044) and Jiangsu Natural Science Foundation (Grant no. BK2008356). The research of J. J. Nieto is partially supported by Ministerio de Education y Ciencia and FEDER, Project MTM2007-61724, and by Xunta de Galicia and FEDER, project PGIDIT06PXIB207023PR.

## References

[1] T. Ding, "A boundary value problem for the periodic Brillouin focusing system," *Acta Scientiarum Naturalium Universitatis Pekinensis*, vol. 11, pp. 31–38, 1965 (Chinese).

[2] M. A. del Pino and R. F. Manásevich, "Infinitely many $T$-periodic solutions for a problem arising in nonlinear elasticity," *Journal of Differential Equations*, vol. 103, no. 2, pp. 260–277, 1993.

[3] J. Chu, X. Lin, D. Jiang, D. O'Regan, and R. P. Agarwal, "Multiplicity of positive periodic solutions to second order differential equations," *Bulletin of the Australian Mathematical Society*, vol. 73, no. 2, pp. 175–182, 2006.

[4] J. Chu, P. J. Torres, and M. Zhang, "Periodic solutions of second order non-autonomous singular dynamical systems," *Journal of Differential Equations*, vol. 239, no. 1, pp. 196–212, 2007.

[5] J. Chu and J. J. Nieto, "Impulsive periodic solutions of first-order singular differential equations," *Bulletin of the London Mathematical Society*, vol. 40, no. 1, pp. 143–150, 2008.

[6] I. Rachůnková, M. Tvrdý, and I. Vrkoč, "Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems," *Journal of Differential Equations*, vol. 176, no. 2, pp. 445–469, 2001.

[7] S. Solimini, "On forced dynamical systems with a singularity of repulsive type," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 14, no. 6, pp. 489–500, 1990.

[8] P. Yan and M. Zhang, "Higher order non-resonance for differential equations with singularities," *Mathematical Methods in the Applied Sciences*, vol. 26, no. 12, pp. 1067–1074, 2003.
[9] M. Zhang, "A relationship between the periodic and the Dirichlet BVPs of singular differential equations," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 128, no. 5, pp. 1099–1114, 1998.
[10] M. Zhang, "Periodic solutions of damped differential systems with repulsive singular forces," *Proceedings of the American Mathematical Society*, vol. 127, no. 2, pp. 401–407, 1999.
[11] J. Chu and M. Zhang, "Rotation numbers and Lyapunov stability of elliptic periodic solutions," *Discrete and Continuous Dynamical Systems*, vol. 21, no. 4, pp. 1071–1094, 2008.
[12] J. Lei, X. Li, P. Yan, and M. Zhang, "Twist character of the least amplitude periodic solution of the forced pendulum," *SIAM Journal on Mathematical Analysis*, vol. 35, no. 4, pp. 844–867, 2003.
[13] M. Zhang, "Periodic solutions of equations of Emarkov-Pinney type," *Advanced Nonlinear Studies*, vol. 6, no. 1, pp. 57–67, 2006.
[14] S. Adachi, "Non-collision periodic solutions of prescribed energy problem for a class of singular Hamiltonian systems," *Topological Methods in Nonlinear Analysis*, vol. 25, no. 2, pp. 275–296, 2005.
[15] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular problems by variational methods," *Proceedings of the American Mathematical Society*, vol. 134, no. 3, pp. 817–824, 2006.
[16] A. Ambrosetti and V. Coti Zelati, *Periodic Solutions of Singular Lagrangian Systems*, Progress in Nonlinear Differential Equations and Their Applications, 10, Birkhäuser, Boston, Mass, USA, 1993.
[17] A. C. Lazer and S. Solimini, "On periodic solutions of nonlinear differential equations with singularities," *Proceedings of the American Mathematical Society*, vol. 99, no. 1, pp. 109–114, 1987.
[18] W. B. Gordon, "Conservative dynamical systems involving strong forces," *Transactions of the American Mathematical Society*, vol. 204, pp. 113–135, 1975.
[19] A. Fonda, R. Manásevich, and F. Zanolin, "Subharmonic solutions for some second-order differential equations with singularities," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 5, pp. 1294–1311, 1993.
[20] P. Habets and L. Sanchez, "Periodic solutions of some Liénard equations with singularities," *Proceedings of the American Mathematical Society*, vol. 109, no. 4, pp. 1035–1044, 1990.
[21] P. J. Torres and M. Zhang, "A monotone iterative scheme for a nonlinear second order equation based on a generalized anti-maximum principle," *Mathematische Nachrichten*, vol. 251, pp. 101–107, 2003.
[22] I. Rachůnková, S. Stanek, and M. Tvrdý, *Solvability of Nonlinear Singular Problems for Ordinary Differential Equations*, Hindawi, New York, NY, USA, 4th edition.
[23] A. Cabada, A. Lomtatidze, and M. Tvrdý, "Periodic problem involving quasilinear differential operator and weak singularity," *Advanced Nonlinear Studies*, vol. 7, no. 4, pp. 629–649, 2007.
[24] J. Chu and M. Li, "Positive periodic solutions of Hill's equations with singular nonlinear perturbations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 1, pp. 276–286, 2008.
[25] D. Franco and J. R. L. Webb, "Collisionless orbits of singular and non singular dynamical systems," *Discrete and Continuous Dynamical Systems*, vol. 15, no. 3, pp. 747–757, 2006.
[26] D. Jiang, J. Chu, and M. Zhang, "Multiplicity of positive periodic solutions to superlinear repulsive singular equations," *Journal of Differential Equations*, vol. 211, no. 2, pp. 282–302, 2005.
[27] P. J. Torres, "Weak singularities may help periodic solutions to exist," *Journal of Differential Equations*, vol. 232, no. 1, pp. 277–284, 2007.
[28] P. J. Torres, "Existence and stability of periodic solutions for second-order semilinear differential equations with a singular nonlinearity," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 137, no. 1, pp. 195–201, 2007.
[29] D. Bonheure and C. De Coster, "Forced singular oscillators and the method of lower and upper solutions," *Topological Methods in Nonlinear Analysis*, vol. 22, no. 2, pp. 297–317, 2003.
[30] P. Habets and F. Zanolin, "Upper and lower solutions for a generalized Emden-Fowler equation," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 3, pp. 684–700, 1994.
[31] J. Mawhin, "Topological degree and boundary value problems for nonlinear differential equations," in *Topological Methods for Ordinary Differential Equations (Montecatini Terme, 1991)*, M. Furi and P. Zecca, Eds., vol. 1537 of *Lecture Notes in Mathematics*, pp. 74–142, Springer, Berlin, Germany, 1993.
[32] J. Chu and D. O'Regan, "Multiplicity results for second order non-autonomous singular Dirichlet systems," *Acta Applicandae Mathematicae*, vol. 105, no. 3, pp. 323–338, 2009.
[33] D. Jiang, J. Chu, D. O'Regan, and R. P. Agarwal, "Multiple positive solutions to superlinear periodic boundary value problems with repulsive singular forces," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 2, pp. 563–576, 2003.

[34] P. J. Torres, "Non-collision periodic solutions of forced dynamical systems with weak singularities," *Discrete and Continuous Dynamical Systems*, vol. 11, no. 2-3, pp. 693–698, 2004.

[35] J. Chu and P. J. Torres, "Applications of Schauder's fixed point theorem to singular differential equations," *Bulletin of the London Mathematical Society*, vol. 39, no. 4, pp. 653–660, 2007.

[36] D. Franco and P. J. Torres, "Periodic solutions of singular systems without the strong force condition," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1229–1236, 2008.

[37] P. J. Torres, "Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem," *Journal of Differential Equations*, vol. 190, no. 2, pp. 643–662, 2003.

[38] M. Zhang and W. Li, "A Lyapunov-type stability criterion using $L^\alpha$ norms," *Proceedings of the American Mathematical Society*, vol. 130, no. 11, pp. 3325–3333, 2002.

[39] D. O'Regan, *Existence Theory for Nonlinear Ordinary Differential Equations*, vol. 398 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[40] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, The Netherlands, 1964.