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Research Article

Existence of Global Attractors in L^p for m-Laplacian Parabolic Equation in R^N

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We study the long-time behavior of solution for the m-Laplacian equation $u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) + \lambda|u|^{m-2}u + f(x,u) = g(x)$ in $R^N \times R^+$, in which the nonlinear term $f(x,u)$ is a function like $f(x,u) = -h(x)|u|^{q-2}u$ with $h(x) \geq 0$, $2 \leq q < m$, or $f(x,u) = a(x)|u|^{\alpha-2}u - h(x)|u|^{\beta-2}u$ with $a(x) \geq h(x) \geq 0$ and $\alpha > \beta \geq m$. We prove the existence of a global $(L^2(R^N), L^p(R^N))$-attractor for any $p > m$.

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1. Introduction

In this paper we are interested in the existence of a global $(L^2(R^N), L^p(R^N))$-attractor for the m-Laplacian equation

$$u_t - \Delta_m u + \lambda|u|^{m-2}u + f(x,u) = g(x), \quad x \in R^N,\ t \in R^+, \tag{1.1}$$

with initial data condition

$$u(x,0) = u_0(x), \quad x \in R^N, \tag{1.2}$$

where the m-Laplacian operator $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $2 \leq m < N$, $\lambda > 0$.

For the case $m = 2$, the existence of global $(L^2(R^N), L^2(R^N))$-attractor for (1.1)-(1.2) is proved by Wang in [1] under appropriate assumptions on f and g. Recently, Khanmamedov [2] studied the existence of global $(L^2(R^N), L^{m^*}(R^N))$-attractor for (1.1)-(1.2) with $m^* = mN/(N-m)$. Yang et al. in [3] investigated the global $(L^2(R^N), L^p(R^N) \cap W^{1,m}(R^N))$-attractor

$\mathcal{A}_p$ under the assumptions $f(x,u)u \ge a_1|u|^p - a_2|u|^m - a_3(x)$ and $f_u(x,u) \ge a_4(x)$ with the constants $a_1, a_2 > 0$ and the functions $a_3, a_4 \in L^1(R^N) \cap L^\infty(R^N)$. We note that the global attractor $\mathcal{A}_p$ in [3] is related to the p-order polynomial of u on $f(x,u)$. In [4], we consider the existence of global $(L^2(R^N), L^p(R^N))$-attractor for (1.1)-(1.2), which the term $\lambda|u|^{m-2}u$ is replaced by λu. We derive L^∞ estimate of solutions by Moser's technique as in [5–7], and due to this, we need not to make the assumption like $f_u(x,u) \ge a_4(x)$ to show the uniqueness. For a typical example is $f(x,u) = a(x)|u|^{\alpha-2}u - h(x)|u|^{\beta-2}u$ with $a(x) \ge h(x) \ge 0$, $\alpha > \beta \ge 2$, $h(x) \in L^2(R^N) \cap L^\infty(R^N)$. In [4], we assume that $f(x,u)$ satisfies

$$0 \le \int_0^u f(x,\eta)\,d\eta + L(x)|u| \le k_2\big(f(x,u)u + L(x)|u|\big) \tag{1.3}$$

with some $k_2 > 0$ and $L(x) \in L^2(R^N) \cap L^\infty(R^N)$.

Obviously, the nonlinear function $f(x,u) = -h(x)|u|^{q-2}u$ with $h(x) \ge 0$, $q \ge 1$ does not satisfy the assumption (1.3).

In this paper, motivated by [2–4], we are interested in the global $(L^2(R^N), L^p(R^N))$-attractor $\mathcal{A}_p$ for the problem (1.1)-(1.2) with any $p > m$, in which p is independent of the order of polynomial for u on $f(x,u)$.

Our assumptions on $f(x,u)$ is different from that in [2–4]. To obtain the continuity of solution of (1.1)-(1.2) in $L^p(R^N)$, $p \ge 2$, we derive L^∞ estimate of solutions by Moser's technique as in [4, 6, 7]. We will prove that the existence of the global attractor $\mathcal{A}_p$ in $L^p(R^N)$ under weaker conditions.

The paper is organized as follows. In Section 2, we derive some estimates and prove some lemmas for the solution of (1.1)-(1.2). By the a priori estimates in Section 2, the existence of global $(L^2(R^N), L^p(R^N))$-attractor for (1.1)-(1.2) is established in Section 3.

2. Preliminaries

We denote by L^p and $W^{1,m}$ the space $L^p(R^N)$ and $W^{1,m}(R^N)$, and the relevant norms by $\|\cdot\|_p$ and $\|\cdot\|_{1,m}$, respectively. It is well known that $W^{1,m}(R^N) = W_0^{1,m}(R^N)$. In general, $\|\cdot\|_E$ denotes the norm of the Banach space E.

For the proof of our results, we will use the following lemmas.

Lemma 2.1 ([8–10] (Gagliardo-Nirenberg)). *Let $\beta \ge 0$, $1 \le r \le q \le m(1+\beta)N/(N-m)$ when $N > m$ and $1 \le r \le q \le \infty$ when $N \le m$. Suppose $u \in L^r$ and $|u|^\beta u \in W^{1,m}$. Then there exists C_0 such that*

$$\|u\|_q \le C_0^{1/(\beta+1)} \|u\|_r^{1-\theta} \left\| \nabla(|u|^\beta u) \right\|_m^{\theta/(\beta+1)} \tag{2.1}$$

with $\theta = (1+\beta)(r^{-1} - q^{-1})/(N^{-1} - m^{-1} + (1+\beta)r^{-1})$, where C_0 is a constant independent of q, r, β, and θ if $N \ne m$ and a constant depending on $q/(1+\beta)$ if $N = m$.

Lemma 2.2 ([7]). *Let $y(t)$ be a nonnegative differentiable function on $(0,T]$ satisfying*

$$y'(t) + At^{\lambda\theta-1}y^{1+\theta}(t) \le Bt^{-k}y(t) + Ct^{-\delta}, \quad 0 < t \le T, \tag{2.2}$$

with $A, \theta > 0$, $\lambda\theta \geq 1$, $B, C \geq 0$, $k \leq 1$, and $0 \leq \delta < 1$. Then one has

$$y(t) \leq A^{-1/\theta}\left(2\lambda + 2BT^{1-k}\right)^{1/\theta} t^{-\lambda} + 2C\left(\lambda + BT^{1-k}\right)^{-1} t^{1-\delta}, \quad 0 < t \leq T. \tag{2.3}$$

Lemma 2.3 ([11]). *Let $y(t)$ be a nonnegative differential function on $(0, \infty)$ satisfying*

$$y'(t) + Ay^{1+\mu}(t) \leq B, \quad t > 0 \tag{2.4}$$

with A, $\mu > 0$, $B \geq 0$. Then one has

$$y(t) \leq \left(BA^{-1}\right)^{1/(1+\mu)} + (A\mu t)^{-1/\mu}, \quad t > 0. \tag{2.5}$$

First, the following assumptions are listed.

($\mathbf{A_1}$) *Let $f(x,u) \in C^1(R^{N+1})$, $f(x,0) = 0$ and there exist the nontrivial nonnegative functions $h(x) \in L^{q_1} \cap L^\infty$ and $h_1(x) \in L^1$, such that $F(x,u) \leq k_1 f(x,u)u$ and*

$$-h(x)|u|^q \leq f(x,u)u \leq h(x)|u|^q + h_1(x), \tag{2.6}$$

$$(f(x,u) - f(x,v))(u - v) \geq -k_2\left(1 + |u|^{q-2} + |v|^{q-2}\right)|u - v|^2, \tag{2.7}$$

where $F(x,u) = \int_0^u f(x,s)ds$, $2 \leq q < m$, $q_1 = m/(m-q)$ and some constants $k_1, k_2 \geq 0$.

($\mathbf{A_2}$) *Let $f(x,u) \in C^1(R^{N+1})$, $f(x,0) = 0$ and there exists the nontrivial nonnegative function $h_1(x) \in L^1$, such that $F(x,u) \leq k_1 f(x,u)u$ and*

$$\begin{gathered} a_1|u|^\alpha - a_2|u|^m \leq f(x,u)u \leq b_1|u|^\alpha + b_2|u|^m + h_1(x), \\ (f(x,u) - f(x,v))(u - v) \geq -k_4\left(1 + |u|^{\alpha-2} + |v|^{\alpha-2}\right)|u - v|^2, \end{gathered} \tag{2.8}$$

where $a_2 < \lambda$, $m < \alpha < m + 2m/N$, and $a_1, b_1, b_2 > 0$, $k_1, k_2 \geq 0$.

A typical example is $f(x,u) = a(x)|u|^{\alpha-2}u - h(x)|u|^{\beta-2}u$ with $a(x), h(x) \geq 0$, and $\alpha > \beta \geq m$. The assumption ($\mathbf{A_2}$) is similar to [3, (1.3)–(1.7)].

Remark 2.4. If $f(x,u) = -h(x)|u|^{q-2}u$, $q > m$, the problem (1.1)-(1.2) has no nontrivial solution for some $h(x) \geq 0$, see [12].

We first establish the following theorem.

Theorem 2.5. *Let $g \in L^{m'} \cap L^\infty$ and $u_0 \in L^2$. If ($\mathbf{A_1}$) holds, then the problem (1.1)-(1.2) admits a unique solution $u(t)$ satisfying*

$$\begin{gathered} u(t) \in \mathbf{X} \equiv \mathbf{C}\left([0,\infty), L^2\right) \cap L^m_{\mathrm{loc}}\left([0,\infty), W^{1,m}\right) \cap L^\infty_{\mathrm{loc}}\left([0,\infty), L^2\right), \\ u_t \in L^m_{\mathrm{loc}}\left([0,\infty), W^{-1,m'}\right), \end{gathered} \tag{2.9}$$

and the following estimates:

$$\|u(t)\|_2^2 \le C_0\left(\|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1}\right)t + \|u_0\|_2^2, \quad t \ge 0, \tag{2.10}$$

$$\|\nabla u(t)\|_m^m + \lambda\|u(t)\|_m^m \le C_0\left(\|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1} + \|h_1\|_1\right) + t^{-1}\|u_0\|_2^2, \quad t > 0, \tag{2.11}$$

$$\int_s^t \|u_t(\tau)\|_2^2 d\tau \le C_0\left(\|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1} + \|h_1\|_1\right) + s^{-1}\|u_0\|_2^2, \quad 0 < s \le t, \tag{2.12}$$

$$\|u(t)\|_\infty \le C_1 t^{-s_0}, \qquad s_0 = N(2m + (m-2)N)^{-1}, \quad 0 < t \le T \tag{2.13}$$

with $m' = m/(m-1)$. The constant C_0 depends only on m, N, q, λ, and C_1 depends on h, g, u_0, and T.

Proof. For any $T > 0$, the existence and uniqueness of solution $u(t)$ for (1.1)-(1.2) in the class

$$\mathbf{X}_T \equiv \mathbf{C}\left([0,T], L^2\right) \cap L^m\left([0,T], W^{1,m}\right) \cap L^\infty\left([0,T], L^2\right) \tag{2.14}$$

can be obtained by the standard Faedo-Galerkin method, see, for example, [10, Theorem 7.1, page 232], or by the pseudomonotone operator method in [2]. Further, we extend the solution $u(t)$ for all $t \ge 0$ by continuity and bounded over L^2 such that $u(t) \in \mathbf{X}$.

In the following, we will derive the estimates (2.10)–(2.13). The solution is in fact given as limits of smooth solutions of approximate equations (see [5, 6]), we may assume for our estimates that the solutions under consideration are appropriately smooth. We begin with the estimate of $\|u(t)\|_2$.

We multiply (1.1) by u and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_m^m + \lambda\|u(t)\|_m^m = \int_{R^N} (g(x) - f(x,u))u\,dx. \tag{2.15}$$

Since

$$\begin{gathered} -\int_{R^N} f(x,u(t))u(t)dx \le \int_{R^N} h(x)|u(t)|^q dx \le \lambda_0\|u(t)\|_m^m + C_0\|h\|_{q_1}^{q_1}, \\ \int_{R^N} g(x)u(t)dx \le \lambda_0\|u(t)\|_m^m + C_0\|g\|_{m'}^{m'} \end{gathered} \tag{2.16}$$

with $\lambda_0 = \lambda/4$. We have from (2.15) that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_m^m + 2\lambda_0\|u(t)\|_m^m \le C_0\left(\|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1}\right). \tag{2.17}$$

Integrating (2.17) with respect to t, we obtain

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \left(\|\nabla u(\tau)\|_m^m + 2\lambda_0\|u(\tau)\|_m^m\right)d\tau \le C_0\left(\|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1}\right)t + \frac{1}{2}\|u_0\|_2^2. \tag{2.18}$$

This implies (2.10) and the existence of $t^* \in (0,t)$ such that

$$\|\nabla u(t^*)\|_m^m + 2\lambda_0 \|u(t^*)\|_m^m \le C_0\Big(\|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1}\Big) + t^{-1}\|u_0\|_2^2, \quad t > 0. \tag{2.19}$$

On the other hand, multiplying (1.1) by u_t and integrating on $(s,t)\times R^N$, we get

$$\begin{aligned} \int_s^t \|u_t(\tau)\|_2^2 d\tau + \frac{1}{m}\|\nabla u(t)\|_m^m + \frac{\lambda}{m}\|u(t)\|_m^m + \int_{R^N}\big(F(x,u(t)) - g(x)u(t)\big)dx \\ = \frac{1}{m}\|\nabla u(s)\|_m^m + \frac{\lambda}{m}\|u(s)\|_m^m + \int_{R^N}\big(F(x,u(s)) - g(x)u(s)\big)dx. \end{aligned} \tag{2.20}$$

By (2.6), we have $F(x,u) \ge -h(x)|u|^q$ and

$$-\int_{R^N} F(x,u(t))dx \le \int_{R^N} h(x)|u(t)|^q dx \le \varepsilon\|u(t)\|_m^m + C_0\|h\|_{q_1}^{q_1} \tag{2.21}$$

with $0 < \varepsilon \le \lambda/2m$. Similarly, we have the following estimates by Young's inequality:

$$\begin{aligned} &\int_{R^N}\big|g(x)u(t)\big|dx \le \varepsilon\|u(t)\|_m^m + C_0\|g\|_{m'}^{m'}, \\ &\int_{R^N}\big|g(x)u(S)\big|dx \le \|u(s)\|_m^m + \|g\|_{m'}^{m'}, \\ &\int_{R^N} F(x,u(s))dx \le k_1\int_{R^N}\big(h(x)|u(s)|^q + h_1(x)\big)dx \\ &\qquad\qquad \le C_0\Big(\|u(s)\|_m^m + \|h\|_{q_1}^{q_1} + \|h_1\|_1\Big). \end{aligned} \tag{2.22}$$

Then, we have from (2.20) that

$$\int_s^t \|u_t(\tau)\|_2^2 d\tau + \frac{1}{m}\|\nabla u(t)\|_m^m + \frac{\lambda}{2m}\|u(t)\|_m^m \le C_0\big(\|\nabla u(s)\|_m^m + \|u(s)\|_m^m + M_1\big), \tag{2.23}$$

where

$$M_1 = \|g\|_{m'}^{m'} + \|h\|_{q_1}^{q_1} + \|h_1\|_1. \tag{2.24}$$

Further, we let $s = t^*$ in (2.23) and obtain from (2.19) that

$$\begin{aligned} &\|\nabla u(t)\|_m^m + \lambda\|u(t)\|_m^m \le C_0\Big(M_1 + t^{-1}\|u_0\|_2^2\Big), \quad t > 0, \\ &\int_s^t \|u_t(\tau)\|_2^2 d\tau \le C_0\Big(M_1 + s^{-1}\|u_0\|_2^2\Big), \quad 0 < s < t. \end{aligned} \tag{2.25}$$

Thus, the solution $u(t)$ satisfies (2.10)–(2.12). We now derive (2.13) by Moser's technique as in [5, 6]. In the sequel, we will write u^p instead of $|u|^{p-1}u$ when $p \geq 1$. Also, let C and C_j be the generic constants independent of p changeable from line to line.

Multiplying (1.1) by $|u|^{p-2}u$, $(p \geq 2)$, we get

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + C_1 p^{1-m}\left\|\nabla u^{(p+m-2)/m}\right\|_m^m + \lambda\|u(t)\|_{p+m-2}^{p+m-2} \leq \int_{R^N}\left(g(x) - f(x,u)\right)|u|^{p-2}u\,dx. \tag{2.26}$$

It follows from Young's inequality that

$$\begin{aligned}\int_{R^N}\left|g(x)\right||u|^{p-1}dx &\leq \lambda_0\|u\|_{p+m-2}^{p+m-2} + \lambda_0^{(1-p)/(m-1)}\left\|g\right\|_{\alpha_p}^{\alpha_p}\\ -\int_{R^N} f(x,u)|u|^{p-2}u\,dx &\leq \lambda_0\|u\|_{p+m-2}^{p+m-2} + \lambda_0^{(2-p-q)/(m-q)}\|h\|_{\beta_p}^{\beta_p}\end{aligned} \tag{2.27}$$

with $\lambda_0 = \lambda/4$, $\alpha_p = (p+m-2)/(m-1)$, $\beta_p = (p+m-2)/(m-q)$. Then, (2.26) becomes

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + C_1 p^{1-m}\left\|\nabla u^{(p+m-2)/m}\right\|_m^m + 2\lambda_0\|u(t)\|_{p+m-2}^{p+m-2}\\ &\quad\leq \lambda_0^{(1-p)/(m-1)}\left\|g\right\|_{\alpha_p}^{\alpha_p} + \lambda_0^{(2-p-q)/(m-q)}\|h\|_{\beta_p}^{\beta_p}.\end{aligned} \tag{2.28}$$

Let $R > m/2$, $p_1 = 2$, $p_n = Rp_{n-1} - (m-2)$, $n = 2, 3, \ldots$. Then, by Lemma 2.1, we see

$$\left\|\nabla u^{(p_n+m-2)/m}\right\|_m^m \geq C_0^{-m/\theta_n}\|u\|_{p_{n-1}}^{(p_n+m-2)(1-\theta_n^{-1})}\|u\|_{p_n}^{(p_n+m-2)\theta_n^{-1}}, \tag{2.29}$$

where

$$\theta_n = \frac{p_n+m-2}{m}\left(\frac{1}{p_{n-1}} - \frac{1}{p_n}\right)\left(\frac{1}{N} - \frac{1}{m} + \frac{p_n+m-2}{mp_{n-1}}\right)^{-1} = \frac{NR\left(1-p_{n-1}p_n^{-1}\right)}{m+N(R-1)}. \tag{2.30}$$

Inserting (2.29) into (2.28) $(p = p_n)$, we find

$$\frac{d}{dt}\|u(t)\|_{p_n}^{p_n} + C_1 C_0^{-m/\theta_n} p_n^{2-m}\|u\|_{p_n}^{p_n+r_n}\|u\|_{p_{n-1}}^{m-2-r_n} \leq p_n A_n, \tag{2.31}$$

where $r_n = (p_n+m-2)\theta_n^{-1} - p_n$ and

$$A_n = \lambda_0^{(2-p_n-q)/(m-q)}\|h\|_{\mu_n}^{\mu_n} + \lambda_0^{(1-p_n)/(m-1)}\left\|g\right\|_{\lambda_n}^{\lambda_n} \tag{2.32}$$

with $\lambda_n = (p_n+m-2)/(m-1)$, $\mu_n = (p_n+m-2)/(m-q)$, $n = 1, 2, \ldots$.

We claim that there exist the bounded sequences $\{\xi_n\}$ and $\{s_n\}$ such that

$$\|u(t)\|_{p_n} \le \xi_n t^{-s_n}, \quad 0 < t \le T. \tag{2.33}$$

Indeed, by (2.10), this holds for $n = 1$ if we take $s_1 = 0$, $\xi_1 = M_1 T^{1/2} + \|u_0\|_2$. If (2.33) is true for $n-1$, then we have from (2.31) that

$$y'(t) + At^{\tau_n \theta - 1} y^{1+\theta}(t) \le p_n A_n, \quad 0 < t \le T, \tag{2.34}$$

where $y(t) = \|u(t)\|_{p_n}^{p_n}$, $\tau_n = s_n p_n$ and

$$\theta = r_n p_n^{-1}, \qquad s_n = (1 + s_{n-1}(r_n - m + 2)) r_n^{-1}, \qquad A = C_1 C_0^{-m/\theta_n} p_n^{2-m} \xi_{n-1}^{m-2-r_n}. \tag{2.35}$$

Applying Lemma 2.2 to (2.34), we have (2.33) for n with

$$\xi_n = \xi_{n-1}\left(C_1^{-1} C_0^{m/\theta_n} p_n^{m-1} s_n^{-1}\right)^{1/r_n} + \left(2A_n s_n^{-1}\right)^{1/p_n} T^{1+s_n} \tag{2.36}$$

for $n = 2, 3, \ldots$.

It is not difficult to show that $s_n \to s_0 = N(2m + (m-2)N)^{-1}$, as $n \to \infty$ and $\{\xi_n\}$ is bounded, see [6]. Then, (2.13) follows from (2.33) as $n \to \infty$.

We now consider the uniqueness and continuity of the solution for (1.1)-(1.2) in L^2. Let u_1, u_2 be two solutions of (1.1)-(1.2), which satisfy (2.10)–(2.13). Denote $u(t) = u_1(t) - u_2(t)$. Then $u(t)$ solves

$$u_t - (\Delta_m u_1 - \Delta_m u_2) + \lambda\left(|u_1|^{m-2} u_1 - |u_2|^{m-2} u_2\right) = f(x, u_2) - f(x, u_1). \tag{2.37}$$

Multiplying (2.37) by u, we get from (2.7) and (2.13) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \gamma_0 \|\nabla u(t)\|_m^m + \gamma_1 \|u(t)\|_m^m &\le k_2 \int_{R^N} \left(1 + |u_1|^{q-2} + |u_2|^{q-2}\right) u^2 dx \\ &\le k_2 \int_{R^N} \left(1 + \|u_1(t)\|_\infty^{q-2} + \|u_2(t)\|_\infty^{q-2}\right) u^2 dx \le C_0\left(1 + t^{-s_0(q-2)}\right)\|u(t)\|_2^2 \end{aligned} \tag{2.38}$$

with some $\gamma_0, \gamma_1 > 0$. Since $s_0(q-2) < 1$ and $u(0) = 0$, (2.38) implies that $\|u(t)\|_2 \equiv 0$ in $[0, T]$ and $u_1(t) = u_2(t)$ in $[0, T]$.

Further, let $t > s \ge 0$. Note that

$$\|u(t) - u(s)\|_2^2 = \int_{R^N} \left(\int_s^t u_t(\tau) d\tau\right)^2 dx \le \int_s^t \|u_t(\tau)\|_2^2 (t - s). \tag{2.39}$$

This shows that $\|u(t) - u(s)\|_2^2 \to 0$ as $t \to s$ and $u(t) \in C([0, T], L^2)$. Then the proof of Theorem 2.5 is completed. □

Remark 2.6. By (2.23), we know that if $u_0 \in W^{1,m}$, then

$$\int_0^t \|u_t(\tau)\|_2^2 d\tau + \frac{1}{m}\|\nabla u(t)\|_m^m + \frac{\lambda}{2m}\|u(t)\|_m^m \le C_0\|u_0\|_{1,m}^m + M_1, \quad t \ge 0, \tag{2.40}$$

where M_1 is given in (2.24). Hence, we have

Theorem 2.7. *Assume* $(\mathbf{A_1})$ *and* $g \in L^{m'} \cap L^\infty$. *Suppose also* $u_0(x) \in W^{1,m}$. *Then, the unique solution* $u(t)$ *in Theorem 2.5 also satisfies*

$$u(t) \in Y \equiv L^\infty\Big([0,+\infty), W^{1,m}\Big), \qquad u_t \in L^2\Big([0,+\infty), L^2\Big), \tag{2.41}$$

and the estimate (2.40).

Now consider the assumption $(\mathbf{A_2})$. *Since* $m < \alpha < m + 2m/N$, *one has* $s_0(\alpha - 2) = N(\alpha - 2)/(2m + (m-2)N) < 1$. *By a similar argument in the proof of Theorem 2.5, one can establish the following theorem.*

Theorem 2.8. *Assume* $(\mathbf{A_2})$ *and* $g \in L^{m'} \cap L^\infty$, $u_0 \in L^2$. *Then the problem* (1.1)-(1.2) *admits a unique solution* $u(t)$ *which satisfies*

$$\begin{gathered} u(t) \in \mathbf{X} \equiv \mathbf{C}\Big([0,\infty), L^2\Big) \cap L^m_{\mathrm{loc}}\Big([0,\infty), W^{1,m}\Big) \cap L^\infty_{\mathrm{loc}}\Big([0,\infty), L^2\Big), \\ u_t \in L^m_{\mathrm{loc}}\Big([0,\infty), W^{-1,m'}\Big), \end{gathered} \tag{2.42}$$

and the following estimates:

$$\begin{gathered} \|u(t)\|_2^2 \le C_0 t \|g\|_{m'}^{m'} + \|u_0\|_2^2, \quad t \ge 0, \\ \|\nabla u(t)\|_m^m + \lambda\|u(t)\|_m^m + \|u(t)\|_\alpha^\alpha \le C_0\Big(\|g\|_{m'}^{m'} + \|h_1\|_1\Big) + t^{-1}\|u_0\|_2^2, \quad t > 0, \\ \int_s^t \|u_t(\tau)\|_2^2 d\tau \le C_0\Big(\|g\|_{m'}^{m'} + \|h_1\|_1\Big) + s^{-1}\|u_0\|_2^2, \quad 0 < s \le t, \\ \|u(t)\|_\infty \le C_1 t^{-s_0}, \quad s_0 = N(2m + (m-2)N)^{-1}, \quad 0 < t \le T. \end{gathered} \tag{2.43}$$

Further, if $u_0 \in W^{1,m}$, *the unique solution* $u(t)(\in Y)$ *satisfies*

$$\int_0^t \|u_t(\tau)\|_2^2 d\tau + \|\nabla u(t)\|_m^m + \|u(t)\|_m^m + \|u(t)\|_\alpha^\alpha \le C_0\Big(\|u_0\|_{1,m}^m + \|h_1\|_1 + \|g\|_{m'}^{m'}\Big), \tag{2.44}$$

where C_0 *depends only on* m, N, λ, α, *and* C_1 *on the given data* g, h_1, u_0, *and* $T > 0$.

So, by Theorems 2.5–2.8, one obtains that the solution operator $S(t)u_0 = u(t)$, $t \ge 0$ of the problem (1.1)-(1.2) generates a semigroup on L^2 or on $W^{1,m}$, which has the following properties:

(1) $S(t) : L^2 \to L^2$ for $t \geq 0$, and $S(0)u_0 = u_0$ for $u_0 \in L^2$ or $S(t) : W^{1,m} \to W^{1,m}$ for $t \geq 0$, and $S(0)u_0 = u_0$ for $u_0 \in W^{1,m}$;

(2) $S(t+s) = S(t)S(s)$ for $t, s \geq 0$;

(3) $S(t)\theta \to S(s)\theta$ in L^2 as $t \to s$ for every $\theta \in L^2$.

From Theorems 2.5–2.8, one has the following lemma.

Lemma 2.9. *Suppose* $(\mathbf{A_1})$ *(or* $(\mathbf{A_2})$*) and* $g \in L^{m'} \cap L^\infty$. *Let* $\mathcal{B}_0$ *be a bounded subset of* L^2. *Then, there exists* $T_0 = T_0(\mathcal{B}_0)$ *such that* $S(t)\mathcal{B}_0 \subset \mathfrak{D}$ *for every* $t \geq T_0$, *where*

$$\mathfrak{D} = \left\{ u \in W^{1,m} \mid \|\nabla u\|_m^m + \lambda \|u\|_m^m \leq M_1 \right\} \tag{2.45}$$

with $M_1 = \|h\|_{q_1}^{q_1} + \|h_1\|_1 + \|g\|_{m'}^{m'}$ *if* $(\mathbf{A_1})$ *holds, and* $M_1 = \|h_1\|_1 + \|g\|_{m'}^{m'}$ *if* $(\mathbf{A_2})$ *holds.*

Now it is a position of Theorem 2.5 to establish some continuity of $S(t)$ *with respect to the initial data* u_0, *which will be needed in the proof for the existence of attractor.*

Lemma 2.10. *Assume that all the assumptions in Theorem 2.5 are satisfied. Let* $S(t)\phi_n$ *and* $S(t)\phi$ *be the solutions of problem (1.1)-(1.2) with the initial data* ϕ_n *and* ϕ, *respectively. If* $\phi_n \to \phi$ *in* $L^p (p \geq 2)$ *as* $n \to \infty$, *then* $S(t)\phi_n$ *uniformly converges to* $S(t)\phi$ *in* L^p *for any compact interval* $[0, T]$ *as* $n \to \infty$.

Proof. Let $u_n(t) = S(t)\phi_n$, $u(t) = S(t)\phi$, $n = 1, 2, \ldots$. Then, $w_n(t) = u_n(t) - u(t)$ solves

$$w_{nt} - (\Delta_m u_n - \Delta_m u) + \lambda \left(|u_n|^{m-2} u_n - |u|^{m-2} u \right) = f(x, u) - f(x, u_n) \tag{2.46}$$

and $w_n(x, 0) = \phi_n(x) - \phi(x)$.

Multiplying (2.46) by $|w_n|^{p-2} w_n$, we get from [8, Chapter 1, Lemma 4.4] and (2.13) that

$$\begin{aligned}
&\frac{1}{p} \frac{d}{dt} \|w_n(t)\|_p^p + \gamma_0 \int_{R^N} |\nabla w_n|^m |w_n|^{p-2} dx + \lambda \|w_n(t)\|_{p+m-2}^{p+m-2} \\
&\quad \leq k_2 \int_{R^N} \left(1 + |u|^{q-2}(t) + |u_n|^{q-2}(t) \right) |w_n(t)|_p^p dx \\
&\quad \leq C_0 \left(1 + \|u_n(t)\|_\infty^{q-2} + \|u(t)\|_\infty^{q-2} \right) \|w_n(t)\|_p^p \\
&\quad \leq C_0 \left(1 + t^{-s_0(q-2)} \right) \|w_n(t)\|_p^p, \quad 0 \leq t \leq T,
\end{aligned} \tag{2.47}$$

for some $\gamma_0 > 0$, depending on m, N. This implies that

$$\begin{aligned}
\|w_n(t)\|_p &\leq \|w_n(0)\|_p \exp\left(C_0 \left(T + (1 - s_0(q-2))^{-1} T^{1-s_0(q-2)} \right) \right) \\
&= \|\phi_n - \phi\|_p \exp\left(C_0 \left(T + (1 - s_0(q-2))^{-1} T^{1-s_0(q-2)} \right) \right), \quad 0 \leq t \leq T,
\end{aligned} \tag{2.48}$$

with $s_0(q-2) = N(q-2)((m-2)N+2m)^{-1} < 1$. Letting $n \to \infty$, we obtain the desired result. □

Lemma 2.11. *Suppose that all the assumptions in Theorem 2.5 are satisfied. Let $u(t)$ be the solution of (1.1)-(1.2) with $u_0 \in L^2$, $\|u_0\|_2 \le M_0$. Then, $\exists T_0 > 0$, such that for any $p > m$, one has*

$$\|u(t)\|_p \le A_p + B_p(t-T_0)^{-1/p\alpha_0}, \quad t > T_0, \tag{2.49}$$

where $\alpha_0 = (m-2+m^2/N)/(p-m)$ and $A_p, B_p > 0$, which depend only on p, N, m and the given data $\|g\|_{\alpha_p}$, $\|h\|_{\beta_p}$, M_0 with $\alpha_p = (p+m-2)/(m-1)$, $\beta_p = (p+m-2)/(m-q)$.

Proof. Multiplying (1.1) by $|u|^{p-2}u$, we have

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \gamma_p \left\| \nabla\left(|u|^{(p-2)/m}u\right)\right\|_m^m + \lambda\|u\|_{p+m-2}^{p+m-2} \le \int_{R^N} (g(x) - f(x,u))u|u|^{p-2}dx \tag{2.50}$$

with $\gamma_p = m^m(p-1)(m+p-2)^{-m}$. Note that

$$\begin{aligned} \int_{R^N} g(x)|u|^{p-2}u\,dx &\le \varepsilon\|u\|_{p+m-2}^{p+m-2} + C_p\|g\|_{\alpha_p}^{\alpha_p}, \\ -\int_{R^N} f(x,u)u|u|^{p-2}dx \le \int_{R^N} h(x)|u|^{p+q-2}dx &\le \varepsilon\|u\|_{p+m-2}^{p+m-2} + C_p\|h\|_{\beta_p}^{\beta_p} \end{aligned} \tag{2.51}$$

with $0 < \varepsilon < \lambda/4$. Then (2.50) becomes

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \gamma_p \left\| \nabla\left(|u|^{(p-2)/m}u\right)\right\|_m^m + \frac{\lambda}{2}\|u\|_{p+m-2}^{p+m-2} \le C_p\left(\|h\|_{\beta_p}^{\beta_p} + \|g\|_{\alpha_p}^{\alpha_p}\right). \tag{2.52}$$

By Lemma 2.1, we get

$$\|\nabla(|u(t)|^\tau u(t))\|_m^m \ge C_0\|u(t)\|_p^{m(1+\tau)/\theta_1}\,\|u(t)\|_m^{\tau_1}, \tag{2.53}$$

with

$$\tau = \frac{p-2}{m}, \qquad \theta_1 = (1+\tau)\left(\frac{1}{m} - \frac{1}{p}\right)\left(\frac{1}{N} + \frac{\tau}{m}\right)^{-1}, \qquad \tau_1 = m\left(1-\theta_1^{-1}\right)(1+\tau) < 0. \tag{2.54}$$

By Lemma 2.9, $\exists T_0 > 0$, such that $t \ge T_0$, $\|u(t)\|_m \le M_1$. Therefore, we have from (2.52) and (2.53) that

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + C_0M_1^{\tau_1}\|u(t)\|_p^{p(1+\alpha_0)} \le A \equiv C_p\left(\|h\|_{\beta_p}^{\beta_p} + \|g\|_{\alpha_p}^{\alpha_p}\right), \quad t > T_0 \tag{2.55}$$

with

$$p(1+\alpha_0)=\frac{m(1+\tau)}{\theta_1},\qquad \tau_1=m-2-p\alpha_0<0,\qquad \alpha_0=\frac{m-2+m^2/N}{p-m}>0. \tag{2.56}$$

It follows from (2.55) and Lemma 2.3 that

$$\|u(t)\|_p^p\le\left(AM_1^{-\tau_1}C_0^{-1}\right)^{1/(1+\alpha_0)}+\left(C_0M_1^{\tau_1}\alpha_0(t-T_0)\right)^{-1/\alpha_0},\quad t>T_0. \tag{2.57}$$

This gives (2.49) and completes the proof of Lemma 2.11. □

By Lemma 2.11, we now establish

Lemma 2.12. *Assume that all the assumptions in Theorem 2.5 are satisfied. Let $\mathcal{B}_0$ be a bounded set in L^2 and $u(t)$ be a solution of* (1.1)-(1.2) *with $u_0\in\mathcal{B}_0$. Then, for any $\eta>0$ and $p>m$, $\exists r_0=r_0(\eta,\mathcal{B}_0)$, $T_1=T_1(\eta,\mathcal{B}_0)$, such that $r\ge r_0$, $t\ge T_1$,*

$$\int_{B_r^c}|u(t)|^p\,dx\le\eta,\quad \forall u_0\in\mathcal{B}_0, \tag{2.58}$$

where $B_r^c=\{x\in R^N\mid |x|\ge r\}$.

Proof. We choose a suitable cut-off function for the proof. Let

$$\phi_0(s)=\begin{cases}0, & 0\le s\le 1;\\ (n-k)^{-1}\left(n(s-1)^k-k(s-1)^n\right), & 1<s<2;\\ 1, & s\ge 2;\end{cases} \tag{2.59}$$

in which $n(>k>m)$ will be determined later. It is easy to see that $\phi_0(s)\in C^1[0,\infty)$, $0\le\phi_0(s)\le 1$, $0\le\phi_0'(s)\le\beta_0\phi_0^{1-1/k}(s)$ for $s\ge 0$, where $\beta_0=k(n/(n-k))^{1/k}$. For every $r>0$, denote $\phi=\phi(r,x)=\phi_0(|x|/r)$, $x\in R^N$. Then

$$\left|\nabla_x\phi(r,x)\right|\le\frac{\beta_1}{r}\phi^{1-\frac{1}{k}}(r,x),\quad x\in R^N, \tag{2.60}$$

with $\beta_1=N\beta_0$.

Multiplying (1.1) by $|u|^{p-2}u\phi$, $(p>m)$, we obtain

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{R^N}|u|^p\phi\,dx+\int_{R^N}|\nabla u|^{m-2}\nabla u\nabla\left(|u|^{p-2}u\phi\right)dx+\frac{\lambda}{2}\int_{R^N}|u|^{p+m-2}\phi\,dx\\&\quad\le C_p\left(\|h\|_{\beta_p}^{\beta_p}(B_r^c)+\|g\|_{\alpha_p}^{\alpha_p}(B_r^c)\right),\end{aligned} \tag{2.61}$$

where and in the sequel, we let $\|f\|_p^p(\Omega) = \int_\Omega |f(x)|^p dx$. Note that

$$D_1 = \int_{R^N} |\nabla u|^{m-2}\nabla u \nabla\left(|u|^{p-2}u\phi\right)dx = (p-1)\int_{R^N}|u|^{p-2}|\nabla u|^m\phi\, dx + D_2 \tag{2.62}$$

with

$$\begin{aligned} D_2 &= \int_{R^N} |\nabla u|^{m-2}\nabla u\nabla\phi|u|^{p-2}u\, dx \\ &\leq \int_{R^N}|\nabla u|^{m-1}\left|\nabla\phi\right||u|^{p-1}dx \\ &\leq \frac{\beta_1}{r}\int_{R^N}|\nabla u|^{m-1}|u|^{p-1}\phi^{1-1/k}dx \\ &\leq \frac{\beta_1}{r}\int_{R^N}\left(|\nabla u|^m|u|^{p-2}\phi + |u|^{p+m-2}\phi^{1-m/k}\right)dx. \end{aligned}\tag{2.63}$$

Therefore, if $r \geq 2\beta_1/(p-1)$,

$$D_1 \geq \frac{p-1}{2}\int_{R^N}|\nabla u|^m|u|^{p-2}\phi\, dx - \frac{\beta_1}{r}\int_{R^N}|u|^{p+m-2}\phi^{1-m/k}dx. \tag{2.64}$$

Further, we estimate the first term of the right-hand side in (2.64). Since

$$\begin{gathered} \frac{\partial}{\partial x_i}\left(\left|u\phi^{1/p}\right|^\tau u\phi^{1/p}\right) = (\tau+1)|u|^\tau\phi^{\tau/p}\left(\phi^{1/p}\frac{\partial u}{\partial x_i} + \frac{u}{p}\frac{\partial\phi}{\partial x_i}\phi^{1/p-1}\right), \quad i = 1,2,\ldots,N, \\ \left|\nabla\left(\left|u\phi^{1/p}\right|^\tau u\phi^{1/p}\right)\right|^2 = (\tau+1)^2|u|^{2\tau}\phi^{2\tau/p}\left(|\nabla u|^2\phi^{2/p} + \frac{u^2}{p^2}\left|\nabla\phi\right|^2\phi^{2/p-2} + \frac{2u}{p}\phi^{2/p-1}\nabla u\nabla\phi\right), \end{gathered}\tag{2.65}$$

we have

$$\begin{aligned} D_3 &= \left|\nabla\left(\left|u\phi^{1/p}\right|^\tau u\phi^{1/p}\right)\right|^m = \left[\left|\nabla\left(\left|u\phi^{1/p}\right|^\tau u\phi^{1/p}\right)\right|^2\right]^{m/2} \\ &\leq \lambda_0\left(|u|^{\tau m}|\nabla u|^m\phi^{m\tau_2} + |u|^{m\tau_0}|\nabla\phi|^m\phi^{m(\tau_2-1)} + |u|^{m\tau+m/2}\left(|\nabla u|\left|\nabla\phi\right|\right)^{m/2}\phi^{m\tau_2-m/2}\right), \end{aligned}\tag{2.66}$$

where $\tau_2 = \tau_0/p$, $\tau_0 = 1+\tau = (p-2+m)/m$ and with some constant $\lambda_0 > 0$. The second term of (2.66) is

$$(2.66)_2 \leq \frac{\beta_1^m}{r^m}|u|^{p-2+m}\phi^{1+(m-2)/p-m/k} \leq \frac{C_1}{r}|u|^{p-2+m}\phi^{1+(m-2)/p-m/k}, \quad r \geq 1, \tag{2.67}$$

and the third term of (2.66) is

$$\begin{aligned}(2.66)_3 &\le \frac{C_1}{r}|u|^{p-2+m/2}|\nabla u|^{m/2}\phi^{1+(m-2)/p-m/2k}\\ &\le \frac{C_1}{r}\Big(|u|^{p-2}|\nabla u|^m\phi + |u|^{p+m-2}\phi^{1+(2m-4)/p-m/k}\Big), \quad r\ge 1\end{aligned} \tag{2.68}$$

with some $C_1 > 0$. Thus, we let $k > pm/(2m-4)$ and have

$$D_3 \le C_1\Big(|u|^{p-2}|\nabla u|^m\phi + r^{-1}|u|^{p+m-2}\phi^{1+(m-2)/p-m/2k}\Big) \tag{2.69}$$

or

$$|u|^{p-2}|\nabla u|^m\phi \ge C_1^{-1}\Big|\nabla\Big(|u\phi^{1/p}|^{\tau}u\phi^{1/p}\Big)\Big|^m - r^{-1}|u|^{p+m-2}\phi^{1+(m-2)/p-m/2k}. \tag{2.70}$$

This implies that

$$\int_{R^N}|u|^{p-2}|\nabla u|^m\phi\,dx \ge C_1^{-1}\Big\|\nabla\Big(|u\phi^{1/p}|^{\tau}u\phi^{1/p}\Big)\Big\|_m^m - r^{-1}\int_{R^N}|u|^{p+m-2}\phi^{1+(m-2)/p-m/2k}dx \tag{2.71}$$

and for $r \ge 1$,

$$D_1 \ge C_1^{-1}\Big\|\nabla\Big(|u\phi^{1/p}|^{\tau}u\phi^{1/p}\Big)\Big\|_m^m - C_p r^{-1}\int_{R^N}|u|^{p+m-2}\Big(\phi^{1+(m-2)/p-m/2k} + \phi^{1-m/k}\Big)dx. \tag{2.72}$$

On the other hand, we obtain by Lemma 2.9 that

$$\Big\|u(t)\phi^{1/p}\Big\|_m \le \|u(t)\|_m \le M_1, \quad t \ge T_0, \tag{2.73}$$

and then for $t \ge T_0$,

$$\Big\|\nabla\Big(|u\phi^{1/p}|^{\tau}u\phi^{1/p}\Big)\Big\|_m^m \ge C_0\Big\|u\phi^{1/p}\Big\|_p^{(m+m\tau)/\theta_1}\Big\|u\phi^{1/p}\Big\|_m^{\tau_1} \ge C_0M_1^{\tau_1}\Big\|u\phi^{1/p}\Big\|_p^{(m+m\tau)/\theta_1}, \tag{2.74}$$

where τ_1 and θ_1 are determined by (2.54). Hence we get from (2.61)–(2.74) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\Big\|u(t)\phi^{1/p}\Big\|_p^p + C_0M_1^{\tau_1}\Big\|u(t)\phi^{1/p}\Big\|_p^{p(1+\alpha_0)}\\ &\quad\le C_p\Big(\|h\|_{\beta_p}^{\beta_p}(B_r^c) + \|g\|_{\alpha_p}^{\alpha_p}(B_r^c) + r^{-1}\|u(t)\|_{p+m-2}^{p+m-2}(B_r^c)\Big), \quad t > T_0,\ r \ge 1.\end{aligned} \tag{2.75}$$

By Lemma 2.11, we know that there exist $\exists T_1 > T_0$ and $M_{p+m-2} > 0$, such that

$$\|u(t)\|_{p+m-2} \le M_{p+m-2}, \quad \text{for } t \ge T_1. \tag{2.76}$$

Then we obtain

$$\int_{R^N} |u|^p \phi \, dx \leq \left(H(r,t) \left(M_1^{T_1} C_0 \right)^{-1} \right)^{1/(1+\alpha_0)} + \left(C_0 M_1^{T_1} \alpha_0 (t - T_1) \right)^{-1/\alpha_0}, \quad t > T_1, \tag{2.77}$$

where

$$H(r,t) = C_p \left(\|h\|_{\beta_p}^{\beta_p} (B_r^c) + \|g\|_{\alpha_p}^{\alpha_p} (B_r^c) + r^{-1} M_{p+m-2}^{p+m-2} \right), \quad t > T_0, \; r \geq 1, \tag{2.78}$$

and $H(r,t) \to 0$ as $r \to \infty$. Then (2.77) implies (2.58) and the proof of Lemma 2.12 is completed. □

Remark 2.13. In fact, we see from the proof of Lemma 2.12 that if (2.73) and (2.76) are satisfied, then (2.77) and (2.58) hold.

Remark 2.14. In a similar argument, we can prove Lemmas 2.10–2.12 under the assumptions in Theorem 2.8.

3. Global Attractor in R^N

In this section, we will prove the existence of the global (L^2, L^p)-attractor for problem (1.1)-(1.2). To this end, we first give the definition about the bi-spaces global attractor, then, prove the asymptotic compactness of $\{S(t)\}_{t\geq 0}$ in L^p and the existence of the global (L^2, L^p)-attractor by a priori estimates established in Section 2.

Definition 3.1 ([2, 3, 13, 14]). A set $\mathcal{A}_p \subset L^p$ is called a global (L^2, L^p)-attractor of the semigroup $\{S(t)\}_{t\geq 0}$ generated by the solution of problem (1.1)-(1.2) with initial data $u_0 \in L^2$ if it has the following properties:

(1) $\mathcal{A}_p$ is invariant in L^p, that is, $S(t)\mathcal{A}_p = \mathcal{A}_p$ for every $t \geq 0$;

(2) $\mathcal{A}_p$ is compact in L^p;

(3) $\mathcal{A}_p$ attracts every bounded subset $\mathcal{B}$ of L^2 in the topology of L^p, that is,

$$\operatorname{dist}\left(S(t)\mathcal{B}, \mathcal{A}_p \right) = \sup_{v \in \mathcal{B}} \inf_{u \in \mathcal{A}_p} \|S(t)v - u\|_p \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \tag{3.1}$$

Now we can prove the main result.

Theorem 3.2. *Assume that all assumptions in Theorem 2.5 (Theorem 2.7) are satisfied. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by the solutions of the problem* (1.1)-(1.2) *with $u_0 \in L^2$ has a global (L^2, L^p)-attractor $\mathcal{A}_p$ for any $p > m$.*

Proof. We only consider the case in Theorem 2.5 and the other is similar and omitted. Define

$$\mathcal{A}_p = \bigcap_{\tau \geq 0} \mathcal{A}(\tau), \qquad \mathcal{A}(\tau) = \left[\bigcup_{t \geq \tau} S(t)\mathfrak{D} \right]_{L^p}, \tag{3.2}$$

where $\mathfrak{D}$ is defined in (2.45) and $[\mathbf{E}]_{\mathbf{L^p}}$ is the closure of $\mathbf{E}$ in L^p.

Obviously, $\mathcal{A}(\tau)$ is closed and nonempty and $\mathcal{A}(\tau_1) \subset \mathcal{A}(\tau_2)$ if $\tau_1 \geq \tau_2$. Thus, $\mathcal{A}_p$ is nonempty. We now prove that $\mathcal{A}_p$ is a global (L^2, L^p)-attractor for (1.1)-(1.2).

We first prove $\mathcal{A}_p$ is invariant in L^p. Let $\phi \in \mathcal{A}_p$. Then, $\exists t_n \to +\infty$ and $\theta_n \in \mathfrak{D}$ such that $S(t_n)\theta_n \to \phi$ in L^p. Since $S(t)$ is continuous from $L^p \to L^p$ by Lemma 2.10, we obtain $S(t+t_n)\theta_n = S(t)(S(t_n)\theta_n) \to S(t)\phi$ in L^p. Note that

$$S(t+t_n)\theta_n \in \bigcup_{t \geq \tau} S(t)\mathfrak{D} \Longrightarrow S(t)\phi \in \mathcal{A}(\tau) \Longrightarrow S(t)\phi \in \bigcap_{\tau \geq 0} \mathcal{A}(\tau). \tag{3.3}$$

That is, $S(t)\phi \in \mathcal{A}_p$ and $S(t)\mathcal{A}_p \subset \mathcal{A}_p$.

On the other hand, let $\phi \in \mathcal{A}_p$. Suppose $t_n \to +\infty$ and $\theta_n \in \mathfrak{D}$ such that $S(t_n)\theta_n \to \phi$ in L^p. We claim that there exists $\psi \in \mathcal{A}_p$ such that $S(t)\psi = \phi$. This implies $\mathcal{A}_p \subset S(t)\mathcal{A}_p$.

First, since $\{\theta_n\}$ is bounded in $W^{1,m}$ by Lemma 2.9, so is $\{S(t_n - t)\theta_n\}$ by Theorem 2.7. That is, $\exists n_0 > 1, T_0 > 0, M_3 > 0$, such that

$$\|u_n\|_m \leq M_3, \quad \|\nabla u_n\|_m \leq M_3 \quad \text{for } n \geq n_0,\ t_n - t \geq T_0, \tag{3.4}$$

with $u_n(x) = S(t_n - t)\theta_n(x)$. Then,

$$\|u_n\|_{W^{1,m}(B_{r_0})} = \|\nabla u_n\|_m(B_{r_0}) + \|u_n\|_m(B_{r_0}) \leq h(r_0, M_3), \quad n \geq n_0, \tag{3.5}$$

where the constant $h(r_0, M_3)$ depends on r_0, M_3, and r_0 is from Lemma 2.12. By the compact embedding theorem, $\exists\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \to \psi$ in $L^p(B_{r_0})$ if $2 \leq p < m^*$. We extend $\psi(x)$ as zero when $|x| > r_0$. Then $u_{n_k} \to \psi$ in L^p, and $\psi \in \mathcal{A}(\tau)$, $\psi \in \mathcal{A}_p$. By the continuity of $S(t)$ in L^p, we have

$$S(t_{n_k})\theta_{n_k} = S(t)(S(t_{n_k} - t)\theta_{n_k}) \longrightarrow S(t)\psi \Longrightarrow \phi = S(t)\psi \quad \text{in } L^p. \tag{3.6}$$

So, $\mathcal{A}_p \subset S(t)\mathcal{A}_p$ and $\mathcal{A}_p$ is invariant in L^p for every $t \geq 0$.

For the case $p \geq m^*$, we take $\mu \in (m, m^*]$ and $u_{n_k} \to \psi$ in L^μ as the above proof. Thus $\{u_{n_k}\}$ is a Cauchy sequence in L^μ. We claim that $\{u_{n_k}\}$ is also a Cauchy sequence in L^p.

In fact, it follows from Lemma 2.11 that $\exists M_\rho$ and n_0 such that if $n \geq n_0$, then $t_n - t \geq T_0$ and

$$\|u_n\|_\rho \leq M_\rho, \quad \rho = \frac{(p-1)\mu}{\mu - 1}. \tag{3.7}$$

Notice that

$$\int_{R^N} \left|u_{n_i} - u_{n_j}\right|^p dx \leq \left\|u_{n_i} - u_{n_j}\right\|_\mu \left\|u_{n_i} - u_{n_j}\right\|_\rho^{p-1} \leq (2M_\rho)^{p-1} \left\|u_{n_i} - u_{n_j}\right\|_\mu \tag{3.8}$$

for $i, j \geq n_0$. This gives our claim. Therefore, $\exists \psi \in L^p$ such that $u_{n_k} = S(t_{n_k} - t)\theta_{n_k} \to \psi$ in L^p and $\phi = S(t)\psi$. Hence $\mathcal{A}_p \subset S(t)\mathcal{A}_p$ and $S(t)\mathcal{A}_p = \mathcal{A}_p$.

We now consider the compactness of $\mathcal{A}_p$ in L^p. In fact, from the proof of $\mathcal{A}_p \subset S(t)\mathcal{A}_p$, we know that $[\cup_{t \geq \tau} S(t)\mathfrak{D}]_{L^p}$ is compact in L^p, so is $\mathcal{A}_p$.

For claim (3), we argue by contradiction and assume that for some bounded set $\mathcal{B}_0$ of L^2, $\text{dist}_{L^p}(S(t)\mathcal{B}_0, \mathcal{A}_p)$ does not tend to 0 as $t \to +\infty$. Thus there exists $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$\text{dist}_{L^p}\left(S(t_n)\mathcal{B}_0, \mathcal{A}_p\right) \geq \frac{\delta}{2} > 0, \quad \text{for } n = 1, 2, \ldots. \tag{3.9}$$

For every $n = 1, 2, \ldots, \exists \theta_n \in \mathcal{B}_0$ such that

$$\text{dist}_{L^p}\left(S(t_n)\theta_n, \mathcal{A}_p\right) \geq \frac{\delta}{2} > 0. \tag{3.10}$$

By Lemma 2.9, $\mathfrak{D}$ is an absorbing set, and $S(t_n)\theta_n \subset \mathfrak{D}$ if $t_n \geq T_0$. By the aforementioned proof, we know that $\exists \phi \in L^p$ and a subsequence $\{S(t_{n_k})\theta_{n_k}\}$ of $\{s(t_n)\theta_n\}$ such that

$$\phi = \lim_{k \to \infty} S(t_{n_k})\theta_{n_k} = \lim_{k \to \infty} S(t_{n_k} - T_0)(S(T_0)\theta_{n_k}), \quad \text{in } L^p. \tag{3.11}$$

When $\theta_{n_k} \in \mathcal{B}_0$ and T_0 is large, we have from Lemma 2.9 that $S(T_0)\theta_{n_k} \in \mathfrak{D}$ and

$$S(t_{n_k} - T_0)(S(T_0)\theta_{n_k}) \in \bigcup_{t \geq \tau} S(t)\mathfrak{D}. \tag{3.12}$$

Thus, $\phi \in \mathcal{A}_p$ which contradicts (3.10). Then the proof of Theorem 3.2 is completed. □

Remark 3.3. Let $p = m^* = mN/(N - m)$. Theorem 3.2 gives the results in [2, Theorem 2] for the case $N > m > 2$ and improve the corresponding results in [3]. The attractor $\mathcal{A}_p$ in Theorem 3.2 is independent of the order of u on $f(x, u)$.

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