Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 637243, 9 pages
doi:10.1155/2009/637243

Research Article

A Complement to the Fredholm Theory of Elliptic Systems on Bounded Domains

Patrick J. Rabier

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

Correspondence should be addressed to Patrick J. Rabier, rabier@imap.pitt.edu

Received 24 March 2009; Revised 4 June 2009; Accepted 11 June 2009

Recommended by Peter Bates

We fill a gap in the L^p theory of elliptic systems on bounded domains, by proving the p-independence of the index and null-space under "minimal" smoothness assumptions. This result has been known for long if additional regularity is assumed and in various other special cases, possibly for a limited range of values of p. Here, p-independence is proved in full generality.

1. Introduction

Although important issues are still being investigated today, the bulk of the Fredholm theory of linear elliptic boundary value problems on bounded domains was completed during the 1960s. (For pseudodifferential operators, the literature is more recent and begins with the work of Boutet de Monvel [1]; see also [2] for a more complete exposition.) While this was the result of the work and ideas of many, the most extensive treatment in the L^p framework is arguably contained in the 1965 work of Geymonat [3]. This note answers a question explicitly left open in Geymonat's paper which seems to have remained unresolved.

We begin with a brief partial summary of [3] in the case of a single scalar equation. Let Ω be a bounded connected open subset of $\mathbb{R}^N$, $N \geq 2$, and let $\mathcal{D}$ denote a differential operator on Ω of order $2m$, $m \geq 1$, with complex coefficients,

$$\mathcal{D} = \sum_{|\alpha| \leq 2m} a_\alpha(x)\partial^\alpha. \tag{1.1}$$

Next, let $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_m)^\top$ be a system of boundary differential operators on $\partial\Omega$ with $\mathcal{B}_\ell$ of order $\mu_\ell \geq 0$ also with complex coefficients,

$$\mathcal{B}_\ell = \sum_{|\beta| \leq \mu_\ell} b_{\ell\beta}(x)\partial^\beta. \tag{1.2}$$

With $M := \max\{2m, \mu_1 + 1, \ldots, \mu_m + 1\}$ and $\kappa \geq 0$ denoting a chosen integer, introduce the following regularity hypotheses:

(H1; κ) $\overline{\Omega}$ is a $C^{M+\kappa}$ ∂-submanifold of $\mathbb{R}^N$ (i.e., $\partial\Omega$ is a $C^{M+\kappa}$ submanifold of $\mathbb{R}^N$ and Ω lies on one side of $\partial\Omega$);

(H2; κ) the coefficients a_α are in $C^{M-2m+\kappa}(\overline{\Omega})$ if $|\alpha| = 2m$ and in $W^{M-2m+\kappa,\infty}(\Omega)$ otherwise;

(H3; κ) the coefficients $b_{\ell\beta}$ are of class $C^{M-\mu_\ell+\kappa}(\partial\Omega)$ if $|\beta| = \mu_\ell$ and in $W^{M-\mu_\ell+\kappa,\infty}(\partial\Omega)$ otherwise.

Then, for $k \in \{0, \ldots, \kappa\}$, the operator $\mathcal{P}$ maps continuously $W^{M+k,p}(\Omega)$ into $W^{M-2m+k,p}(\Omega)$ and $\mathcal{B}_\ell$ maps continuously $W^{M+k,p}(\Omega)$ into $W^{M-\mu_\ell+k-1/p,p}(\partial\Omega)$ for every $p \in (1,\infty)$

$$\mathcal{T}_{p,k} := (\mathcal{P}, \mathcal{B}) : W^{M+k,p}(\Omega) \longrightarrow W^{M-2m+k,p}(\Omega) \times \prod_{\ell=1}^{m} W^{M-\mu_\ell+k-1/p,p}(\partial\Omega) \tag{1.3}$$

is a well-defined bounded linear operator. Geymonat's main result [3, Teorema 3.4 and Teorema 3.5] reads as follows.

Theorem 1.1. *Suppose that (H1; κ), (H2; κ), and (H3; κ) hold for some $\kappa \geq 0$. Then,*

(i) *if $p \in (1,\infty)$ and $k \in \{0, \ldots, \kappa\}$, the operator $\mathcal{T}_{p,k}$ is Fredholm if and only if $\mathcal{P}$ is uniformly elliptic in $\overline{\Omega}$ and $(\mathcal{P}, \mathcal{B})$ satisfies the Lopatinskii-Schapiro condition (see below);*

(ii) *if also $\kappa \geq 1$ and $\mathcal{T}_{p,k}$ is Fredholm for some $p \in (1,\infty)$ and some $k \in \{0, \ldots, \kappa\}$ (and hence for every such p and k by (i)), both the index and null-space of $\mathcal{T}_{p,k}$ are independent of p and k.*

The assumptions made in Theorem 1.1 are nearly optimal. In fact, most subsequent expositions retain more smoothness of the boundary and leading coefficients to make the parametrix calculation a little less technical.

The best known version of the Lopatinskii-Schapiro (LS) condition is probably the combination of proper ellipticity and of the so-called "complementing condition." Since we will not use it explicitly, we simply refer to the standard literature (e.g., [3–5]) for details.

We will fill the obvious "gap" between (i) and (ii) of Theorem 1.1 by proving what follows.

Theorem 1.2. *Theorem 1.1(ii) remains true if $\kappa = 0$.*

Note that $k = 0$ corresponds to the most general hypotheses about the boundary and the coefficients, which is often important in practice.

From now on, we set $\mathcal{T}_p := \mathcal{T}_{p,0}$ for simplicity of notation. The reason why $\kappa \geq 1$ is required in part (ii) of Theorem 1.1 is that the proof uses part (i) with κ replaced by $\kappa - 1$. By a different argument, a weaker form of Theorem 1.2 was proved in [3, Proposizione 4.2] (p-independence for p in some bounded open interval around the value $p = 2$, under additional technical conditions).

If $\mathcal{T}_p + (\lambda, 0)$ is invertible for some $\lambda \in \mathbb{C}$ and every $p \in (1,\infty)$, then Theorem 1.2 is a straightforward by-product of the Sobolev embedding theorems and, in fact, index $\mathcal{T}_p = 0$ in this case. However, this invertibility can only be obtained under more restrictive ellipticity hypotheses (such as strong ellipticity) and/or less general boundary conditions (Agmon [6], Browder [7], Denk et al. [8, Theorem 8.2, page 102]).

The idea of the proof of Theorem 1.2 is to derive the case $\kappa = 0$ from the case $\kappa \geq 1$ by regularization of the coefficients and stability of the Fredholm index. The major obstacle in doing so is the mere C^M regularity of $\partial\Omega$, since Theorem 1.1 with $\kappa \geq 1$ can only be used if $\partial\Omega$ is C^{M+1} or better. This will be overcome in a somewhat nonstandard way in these matters, by artificially increasing the smoothness of the boundary with the help of the following lemma.

Lemma 1.3. *Suppose that Ω is a bounded open subset of $\mathbb{R}^N$ and that $\overline{\Omega}$ is a ∂-submanifold of $\mathbb{R}^N$ of class C^M with $M \geq 2$. Then, there is a bounded open subset $\widetilde{\Omega}$ of $\mathbb{R}^N$ such that $\overline{\widetilde{\Omega}}$ is a ∂-submanifold of $\mathbb{R}^N$ of class C^∞ (even C^ω) and that $\overline{\Omega}$ and $\overline{\widetilde{\Omega}}$ are C^M diffeomorphic (as ∂-manifolds).*

The next section is devoted to the (simple) proof of Theorem 1.2 based on Lemma 1.3 and to a useful equivalent formulation (Corollary 2.1). Surprisingly, we have been unable to find any direct or indirect reference to Lemma 1.3 in the classical differential topology or PDE literature. It does not follow from the related and well-known fact that every ∂-manifold X of class C^M with $M \geq 1$ is C^M diffeomorphic to a ∂-manifold Y of class C^∞ since this does not ensure that both can always be embedded in the same euclidian space. It is also clearly different from the results just stating that Ω can be approximated by open subsets with a smooth boundary (as in [9]), which in fact need not even be homeomorphic to Ω. Accordingly, a proof of Lemma 1.3 is given in Section 3.

Based on the method of proof and the validity of Theorem 1.1 for systems after suitable modifications of the definition of $\mathcal{T}_{p,k}$ in (1.3) and of the hypotheses (H1; κ), (H2; κ), and (H3; κ), there is no difficulty in checking that Theorem 1.2 remains valid for most systems as well, but a brief discussion is given in Section 4 to make this task easier.

Remark 1.4. When the boundary $\partial\Omega$ is not connected, the system $\mathcal{B}$ of boundary conditions may be replaced by a collection of such systems, one for each connected component of $\partial\Omega$. Theorems 1.1 and 1.2 remain of course true in that setting, with the obvious modification of the target space in (1.3).

2. Proof of Theorem 1.2

As noted in [3, page 241], the p-independence of $\ker \mathcal{T}_p$ (recall $\mathcal{T}_p := \mathcal{T}_{p,0}$) follows from that of index $\mathcal{T}_p$, so that it will suffice to focus on the latter.

The problem can be reduced to the case when the lower-order coefficients in $\mathcal{P}$ and $\mathcal{B}_\ell$ vanish since the operator they account for is compact from the source space to the target space in (1.3), irrespective of $p \in (1,\infty)$. Thus, the lower-order terms have no impact on the existence of index $\mathcal{T}_p$ or on its value. It is actually more convenient to deal with the intermediate case when all the coefficients a_α are in $C^{M-2m}(\overline{\Omega})$ and all the coefficients $b_{\ell\beta}$ are in $C^{M-\mu_\ell}(\partial\Omega)$, which is henceforth assumed.

First, $M \geq 2$ since $M \geq 2m$ and $m \geq 1$, so that by (H1; 0) and Lemma 1.3, there are a bounded open subset $\widetilde{\Omega}$ of $\mathbb{R}^N$ such that $\overline{\widetilde{\Omega}}$ is a ∂-submanifold of $\mathbb{R}^N$ of class C^∞ and a C^M diffeomorphism $\Phi : \overline{\widetilde{\Omega}} \to \overline{\Omega}$ mapping $\partial\widetilde{\Omega}$ onto $\partial\Omega$.

The pull-back $\Phi^* u := u \circ \Phi$ is a linear isomorphism of $W^{j,p}(\Omega)$ onto $W^{j,p}(\widetilde{\Omega})$ for every $j \in \{0,\ldots,M\}$ and of $W^{M-\mu_\ell-1/p,p}(\partial\Omega)$ onto $W^{M-\mu_\ell-1/p,p}(\partial\widetilde{\Omega})$ for every $1 \leq \ell \leq m$. Meanwhile,

$\mathcal{P}u = (\Phi^{-1})^* \widetilde{\mathcal{P}}(\Phi^* u)$ where $\widetilde{\mathcal{P}}$ is a differential operator of order $2m$ with coefficients $\widetilde{a}_\alpha$ of class C^{M-2m} on $\overline{\widetilde{\Omega}}$ and $\mathcal{B}_\ell u = (\Phi^{-1})^* \widetilde{\mathcal{B}}_\ell(\Phi^* u)$ where $\widetilde{\mathcal{B}}_\ell$ is a differential operator of order μ_ℓ with coefficients $\widetilde{b}_{\ell\beta}$ of class $C^{M-\mu_\ell}$ on $\partial\widetilde{\Omega}$.

From the above remarks, the operator (where $\widetilde{\mathcal{B}} := (\widetilde{\mathcal{B}}_1, \dots, \widetilde{\mathcal{B}}_m)$)

$$\widetilde{\mathcal{T}}_p := \left(\widetilde{\mathcal{P}}, \widetilde{\mathcal{B}}\right) : W^{M,p}\left(\widetilde{\Omega}\right) \longrightarrow W^{M-2m,p}\left(\widetilde{\Omega}\right) \times \prod_{\ell=1}^{m} W^{M-\mu_\ell-1/p,p}\left(\partial\widetilde{\Omega}\right) \tag{2.1}$$

has the form $\widetilde{\mathcal{T}}_p = \mathcal{U}_p \mathcal{T}_p \mathcal{V}_p$ where $\mathcal{U}_p$ and $\mathcal{V}_p$ are isomorphisms. As a result, $\widetilde{\mathcal{T}}_p$ is Fredholm with the same index as $\mathcal{T}_p$. Since the coefficients of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ and of $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ have the same smoothness, respectively, we may, upon replacing Ω by $\widetilde{\Omega}$ and $\mathcal{T}_p$ by $\widetilde{\mathcal{T}}_p$, continue the proof under the assumption that $\partial\Omega$ is a C^∞ submanifold of $\mathbb{R}^N$ (but the a_α are still $C^{M-2m}(\overline{\Omega})$ and the $b_{\ell\beta}$ still $C^{M-\mu_\ell}(\partial\Omega)$).

The coefficients a_α can be approximated in $C^{M-2m}(\overline{\Omega})$ by coefficients $a_\alpha^\infty \in C^\infty(\overline{\Omega})$ and the coefficients $b_{\ell\beta}$ can be approximated in $C^{M-\mu_\ell}(\partial\Omega)$ by C^∞ functions $b_{\ell\beta}^\infty$ on $\partial\Omega$ (since $\partial\Omega$ is C^∞; see, e.g., [10, Theorem 2.6, page 49]), which yields operators $\mathcal{P}^\infty$ and $\mathcal{B}_\ell^\infty$, $1 \le \ell \le m$, of order $2m$ and μ_ℓ, respectively, in the obvious way.

Let $p, q \in (1, \infty)$ be fixed. The corresponding operators $\mathcal{T}_p^\infty$ and $\mathcal{T}_q^\infty$ are arbitrarily norm-close to $\mathcal{T}_p$ and $\mathcal{T}_q$ if the approximation of the coefficients is close enough. If so, by the openness of the set of Fredholm operators and the local constancy of the index, it follows that $\mathcal{T}_p^\infty$ and $\mathcal{T}_q^\infty$ are Fredholm with index $\mathcal{T}_p^\infty$ = index $\mathcal{T}_p$ and index $\mathcal{T}_q^\infty$ = index $\mathcal{T}_q$. But since $\partial\Omega$ is now C^∞ and the coefficients a_α^∞ and $b_{\ell\beta}^\infty$ are C^∞, the hypotheses (H1; κ), (H2; κ), and (H3; κ) are satisfied by Ω, $\mathcal{P}^\infty$ and $\mathcal{B}^\infty$ and any $\kappa \ge 1$. Thus, index $\mathcal{T}_p^\infty$ = index $\mathcal{T}_q^\infty$ by part (ii) of Theorem 1.1, so that index $\mathcal{T}_p$ = index $\mathcal{T}_q$. This completes the proof of Theorem 1.2.

Corollary 2.1. *Suppose that (H1; 0), (H2; 0), and (H3; 0) hold, that $\mathcal{P}$ is uniformly elliptic in $\overline{\Omega}$, and that $(\mathcal{P}, \mathcal{B})$ satisfies the LS condition. Let $p, q \in (1, \infty)$. If $u \in W^{M,p}(\Omega)$ and $(\mathcal{P}u, \mathcal{B}u) \in W^{M-2m,q}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/q,q}(\partial\Omega)$, then $u \in W^{M,q}(\Omega)$.*

Proof. Since the result is trivial if $p \ge q$, we assume $p < q$. Obviously, $(\mathcal{P}u, \mathcal{B}u) \in \operatorname{rge} \mathcal{T}_p$ and $\mathcal{T}_p$ is Fredholm by Theorem 1.1(i). Let Z denote a (finite-dimensional) complement of $\operatorname{rge} \mathcal{T}_p$ in $W^{M-2m,p}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/p,p}(\partial\Omega)$. Since $W^{M-2m,q}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/q,q}(\partial\Omega)$ is dense in $W^{M-2m,p}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/p,p}(\partial\Omega)$ and $\operatorname{rge} \mathcal{T}_p$ is closed, we may assume that $Z \subset W^{M-2m,q}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/q,q}(\partial\Omega)$. If not, approximate a basis of Z by elements of $W^{M-2m,q}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/q,q}(\partial\Omega)$. If the approximation is close enough, the approximate basis is linearly independent and its span Z' (of dimension $\dim Z$) intersects $\operatorname{rge} \mathcal{T}_p$ only at $\{0\}$ (by the closedness of $\operatorname{rge} \mathcal{T}_p$). Thus, Z may be replaced by Z' as a complement of $\operatorname{rge} \mathcal{T}_p$.

Since $\mathcal{T}_p$ and $\mathcal{T}_q$ have the same index and null-space by Theorem 1.2, their ranges have the same codimension. Now, $Z \cap \operatorname{rge} \mathcal{T}_q = \{0\}$ because Z is a complement of $\operatorname{rge} \mathcal{T}_p$ and $\operatorname{rge} \mathcal{T}_q \subset \operatorname{rge} \mathcal{T}_p$. This shows that Z is also a complement of $\operatorname{rge} \mathcal{T}_q$.

Therefore, since $(\mathcal{P}u, \mathcal{B}u) \in W^{M-2m,q}(\Omega) \times \prod_{\ell=1}^m W^{M-\mu_\ell-1/q,q}(\partial\Omega)$, there is $z \in Z$ such that $(\mathcal{P}u, \mathcal{B}u) - z := w \in \operatorname{rge} \mathcal{T}_q \subset \operatorname{rge} \mathcal{T}_p$. This yields $z = (\mathcal{P}u, \mathcal{B}u) - w \in \operatorname{rge} \mathcal{T}_p$, whence $z = 0$ and so $(\mathcal{P}u, \mathcal{B}u) = w \in \operatorname{rge} \mathcal{T}_q$. This means that $(\mathcal{P}u, \mathcal{B}u) = (\mathcal{P}v, \mathcal{B}v)$ for some $v \in W^{M,q}(\Omega) \subset W^{M,p}(\Omega)$. Thus, $\mathcal{T}_p(v - u) = 0$, that is, $v - u \in \ker \mathcal{T}_p$. Since $\ker \mathcal{T}_p = \ker \mathcal{T}_q \subset W^{M,q}(\Omega)$ by Theorem 1.2, it follows that $u \in W^{M,q}(\Omega)$. □

It is not hard to check that Corollary 2.1 is actually equivalent to Theorem 1.2. This was noted by Geymonat, along with the fact that Corollary 2.1 was only known to be true in special cases ([3, page 242]).

3. Proof of Lemma 1.3

Under the assumptions of Lemma 1.3, Ω has a finite number of connected components, each of which satisfies the same assumptions as Ω itself. Thus, with no loss of generality, we will assume that Ω is connected.

If X and Y are ∂-manifolds of class C^k with $k \geq 1$ and X and Y are C^1 diffeomorphic, they are also C^k diffeomorphic ([10, Theorem 3.5, page 57]). Thus, since $\overline{\Omega}$ is of class C^M with $M \geq 2$, it suffices to find a bounded open subset $\widetilde{\Omega}$ of $\mathbb{R}^N$ such that $\overline{\widetilde{\Omega}}$ is C^∞ and C^{M-1} diffeomorphic to $\overline{\Omega}$.

In a first step, we find a C^M function $g : \mathbb{R}^N \to \mathbb{R}$ such that $\partial\Omega = g^{-1}(0)$ and $\nabla g \neq 0$ on $\partial\Omega$ while $g < 0$ in Ω, $g > 0$ in $\mathbb{R}^N \setminus \partial\Omega$ and $\lim_{|x|\to\infty} g(x) = \infty$. This can be done in various ways and even when $M = 1$. However, since $M \geq 2$, the most convenient argument is to rely on the fact that the signed distance function

$$d(x) := \begin{cases} \operatorname{dist}(x, \partial\Omega), & \text{if } x \notin \Omega, \\ -\operatorname{dist}(x, \partial\Omega), & \text{if } x \in \Omega \end{cases} \tag{3.1}$$

is C^M in $\overline{U}_a$, where $a > 0$, and

$$U_a := \left\{ x \in \mathbb{R}^N : |d(x)| = \operatorname{dist}(x, \partial\Omega) < a \right\} \tag{3.2}$$

is an open neighborhood of $\partial\Omega$ in $\mathbb{R}^N$. This is shown in Gilbarg and Trudinger [11, page 355] and also in Krantz and Parks [12]. Both proofs reveal that $\nabla d(x) \neq 0$ when $x \in \partial\Omega$, that is, when $d(x) = 0$. (Without further assumptions, the C^M regularity of d breaks down when $M = 1$.)

Let $\chi \in C^\infty(\mathbb{R})$ be nondecreasing and such that $\chi(s) = s$ if $|s| \leq b/2$ and $\chi(s) = (\operatorname{sign} s)b$ if $|s| \geq b$, where $0 < b < a$ is given. Then, $g := \chi \circ d$ is C^M in U_a, vanishes only on $\partial\Omega$, and $\nabla g \neq 0$ on $\partial\Omega$. Furthermore, since $g = b$ on a neighborhood of $\partial(\Omega \cup U_a) = \{x \in \mathbb{R}^N : d(x) = a\}$ in $\overline{U}_a$ and $g = -b$ on a neighborhood of $\partial(\Omega \setminus U_a) = \{x \in \mathbb{R}^N : d(x) = -a\}$ in $\overline{U}_a$, g remains C^M after being extended to $\mathbb{R}^N$ by setting $g(x) = b$ if $x \in \mathbb{R}^N \setminus (\Omega \cup U_a)$, and $g(x) = -b$ if $x \in \Omega \setminus U_a$.

This g satisfies all the required conditions except $\lim_{|x|\to\infty} g(x) = \infty$. Since $g(x) = b > 0$ for $|x|$ large enough, this can be achieved by replacing $g(x)$ by $(1 + |x|^2)g(x)$. Since $g \neq 0$ off $\partial\Omega$, it follows from a classical theorem of Whitney [13, Theorem III] (with $\epsilon(x) := |g(x)|/2$ in that theorem) that there is a C^M function h on $\mathbb{R}^N$, of class C^ω in $\mathbb{R}^N \setminus \partial\Omega$ such that, if $|\gamma| \leq M$, then $\partial^\gamma h(x) = \partial^\gamma g(x)$ if $x \in \partial\Omega$ and $|\partial^\gamma h(x) - \partial^\gamma g(x)| < |g(x)|/2$ if $x \in \mathbb{R}^N \setminus \partial\Omega$.

Evidently, h does not vanish on $\mathbb{R}^N \setminus \partial\Omega$ and h has the same sign as g off $\partial\Omega$, that is, $h(x) < 0$ in Ω and $h(x) > 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$. Furthermore, $\nabla h(x) = \nabla g(x) \neq 0$ for every $x \in \partial\Omega$, so that $\nabla h(x) \neq 0$ for $x \in U_{2c}$ for some $c > 0$. Upon shrinking c, we may assume that

$\Omega \setminus U_{2c} \neq \emptyset$. Also, $\lim_{|x|\to\infty} h(x) = \lim_{|x|\to\infty} g(x) = \infty$. For convenience, we summarize the relevant properties of h below:

(i) h is C^M on $\mathbb{R}^N$ and C^ω off $\partial\Omega$,

(ii) $\nabla h(x) \neq 0$ for $x \in U_{2c}$,

(iii) $\Omega = \{x \in \mathbb{R}^N : h(x) < 0\}$,

(iv) $\partial\Omega = h^{-1}(0)$,

(v) $\lim_{|x|\to\infty} h(x) = \infty$.

Choose $\varepsilon > 0$. It follows from (v) that $K_\varepsilon := \{x \in \mathbb{R}^N : h(x) \leq \varepsilon\}$ is compact and, from (iii) and (iv), that $K_\varepsilon \subset \Omega \cup U_c$ if ε is small enough (argue by contradiction). Since $h^{-1}(\varepsilon) \cap \overline{\Omega} = \emptyset$ by (iii) and (iv) and since $h^{-1}(\varepsilon) \subset K_\varepsilon$, this implies $h^{-1}(\varepsilon) \subset U_c \setminus \partial\Omega$. Thus, by (i) and (ii), $h^{-1}(\varepsilon)$ is a C^ω submanifold of $\mathbb{R}^N$ and the boundary of the open set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : h(x) < \varepsilon\} \supset \overline{\Omega}$. In fact, $\overline{\Omega}_\varepsilon = K_\varepsilon$ is a ∂-manifold of class C^ω since, once again by (ii), Ω_ε lies on one side of its boundary.

We now proceed to show that $\overline{\Omega}_\varepsilon$ is C^{M-1} diffeomorphic to $\overline{\Omega}$. This will be done by a variant of the procedure used to prove that nearby noncritical level sets on compact manifolds are diffeomorphic. However, since we are dealing with sublevel sets and since critical points will abound, the details are significantly different.

Let $\theta \in C_0^\infty(U_{2c})$ be such that $\theta \geq 0$ and $\theta = 1$ on U_c. Since $\nabla h \neq 0$ on U_{2c} by (ii), the function $\theta(\nabla h/|\nabla h|^2)$ extended by 0 outside $\operatorname{Supp}\theta$ is a bounded C^{M-1} vector field on $\mathbb{R}^N$. Since $M - 1 \geq 1$, the function $\varphi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\begin{gathered} \frac{\partial\varphi}{\partial t}(t,x) = -\theta\big(\varphi(t,x)\big)\frac{\nabla h\big(\varphi(t,x)\big)}{\big|\nabla h(\varphi(t,x))\big|^2}, \\ \varphi(0,x) = x, \end{gathered} \tag{3.3}$$

is well defined and of class C^{M-1} and $\varphi(t,\cdot)$ is an orientation-preserving C^{M-1} diffeomorphism of $\mathbb{R}^N$ for every $t \in \mathbb{R}$. We claim that $\varphi(\varepsilon,\cdot)$ produces the desired diffeomorphism from $\overline{\Omega}_\varepsilon$ to $\overline{\Omega}$.

It follows at once from (3.3) that $(d/dt)(h \circ \varphi) = -\theta \circ \varphi \leq 0$, so that h is decreasing along the flow lines and hence that $\varphi(t,\cdot)$ maps $\overline{\Omega}_\varepsilon$ into itself for every $t \geq 0$. Also, if $x \in \Omega$, then $h(\varphi(t,x)) \leq h(x) < 0$ for every $t \geq 0$, so that $\varphi(t,x) \in \Omega$ by (iii). If now $x \in \partial\Omega \subset U_c$, then $h(x) = 0$ and $h(\varphi(t,x))$ is strictly decreasing for $t > 0$ small enough. It follows that $h(\varphi(t,x)) < 0$, that is, $\varphi(t,x) \in \Omega$ for $t > 0$. Altogether, this yields $\varphi(\varepsilon,\overline{\Omega}) \subset \Omega$.

Suppose now that $x \in \overline{\Omega}_\varepsilon \setminus \overline{\Omega} = K_\varepsilon \setminus \overline{\Omega}$. Then, $x \in U_c$ and hence $\theta(x) = 1$. For $t > 0$ small enough, $\varphi(t,x) \in U_c$ and so $\theta(\varphi(t,x)) = 1$ for $t > 0$ small enough. In fact, it is obvious that $\theta(\varphi(t,x)) = 1$ until t is large enough that $\varphi(t,x) \notin U_c$. But since $\varphi(t,x) \in \overline{\Omega}_\varepsilon$ and $h \circ \varphi(\cdot,x)$ is decreasing along the flow lines, $\varphi(t,x) \notin U_c$ implies $\varphi(t,x) \in \Omega$. Since $x \notin \overline{\Omega}$, this means that $\varphi(\tau(x),x) \in \partial\Omega$ for some $\tau(x) \in (0,t)$. Call $\tau_*(x) > 0$ the first (and, in fact, only, but this is unimportant) time when $\varphi(\tau_*(x),x) \in \partial\Omega$. From the above, $\varphi(t,x) \in U_c$ for $t \in [0,\tau_*(x))$ and hence for $t \in [0,\tau_*(x)]$ since $\partial\Omega \subset U_c$. Then, $\theta(\varphi(t,x)) = 1$ for $t \in [0,\tau_*(x)]$, so that $h(\varphi(t,x)) = h(x) - t$ for $t \in [0,\tau_*(x)]$. In particular, since $\varphi(\tau_*(x),x) \in \partial\Omega$ and hence $h(\varphi(\tau_*(x),x)) = 0$, it follows that $h(x) - \tau_*(x) = 0$. In other words, $\tau_*(x) = h(x) \leq \varepsilon$. Thus,

$h(\varphi(\varepsilon, x)) \leq h(\varphi(\tau_*(x), x)) = 0$, that is, $\varphi(\varepsilon, x) \in \overline{\Omega}$. If $x \in \partial\Omega_\varepsilon$ (so that $h(x) = \varepsilon$ and hence $\tau_*(x) = \varepsilon$), this yields $\varphi(\varepsilon, x) \in \partial\Omega$. On the other hand, if $x \in \Omega_\varepsilon \setminus \overline{\Omega}$, then $\tau_*(x) = h(x) < \varepsilon$. Since $\varphi(\tau_*(x), x) \in \partial\Omega \subset U_c$, $h(\varphi(t, x))$ is strictly decreasing for t near $\tau_*(x)$ and so $h(\varphi(\varepsilon, x)) < h(\varphi(\tau_*(x), x)) = 0$, whence $\varphi(\varepsilon, x) \in \Omega$.

The above shows that $\varphi(\varepsilon, \cdot)$ maps $\overline{\Omega}_\varepsilon$ into $\overline{\Omega}$, $\partial\Omega_\varepsilon$ into $\partial\Omega$, and Ω_ε into Ω. That it actually maps Ω_ε *onto* Ω follows from a Brouwer's degree argument: Ω is connected and no point of Ω is in $\varphi(\varepsilon, \partial\Omega_\varepsilon)$ since, as just noted, $\varphi(\varepsilon, \partial\Omega_\varepsilon) \subset \partial\Omega$. Thus, for $y \in \Omega$, $\deg(\varphi(\varepsilon, \cdot), \Omega_\varepsilon, y)$ is defined and independent of y. Now, choose $y_0 \in \Omega \setminus U_{2c} \neq \emptyset$, so that $\theta(y_0) = 0$. Then, $\varphi(t, y_0) = y_0$ for every $t \geq 0$ and so $\varphi(\varepsilon, y_0) = y_0$. Since $\varphi(\varepsilon, \cdot)$ is one to one and orientation-preserving, it follows that $\deg(\varphi(\varepsilon, \cdot), \Omega_\varepsilon, y_0) = 1$ and so $\deg(\varphi(\varepsilon, \cdot), \Omega_\varepsilon, y) = 1$ for every $y \in \Omega$. Thus, there is $x \in \Omega_\varepsilon$ such that $\varphi(\varepsilon, x) = y$, which proves the claimed surjectivity.

At this stage, we have shown that $\varphi(\varepsilon, \cdot)$ is a C^{M-1} diffeomorphism of $\mathbb{R}^N$ mapping $\overline{\Omega}_\varepsilon$ into $\overline{\Omega}$, $\partial\Omega_\varepsilon$ into $\partial\Omega$, and Ω_ε into and onto Ω. It is straightforward to check that such a diffeomorphism also maps $\partial\Omega_\varepsilon$ onto $\partial\Omega$ (approximate $x \in \partial\Omega$ by a sequence from Ω) and hence it is a boundary-preserving diffeomorphism of $\overline{\Omega}_\varepsilon$ onto $\overline{\Omega}$. This completes the proof of Lemma 1.3.

Remark 3.1. The C^{M-1} diffeomorphism $\varphi(\varepsilon, \cdot)$ above is induced by a diffeomorphism of $\mathbb{R}^N$, but this does not mean that the same thing is true of the C^M diffeomorphism of Lemma 1.3.

4. Systems

Suppose now that $\mathcal{D} := (\mathcal{D}_{ij})$, $1 \leq i, j \leq n$, is a system of n^2 differential operators on Ω, which is properly elliptic in the sense of Douglis and Nirenberg [14]. We henceforth assume some familiarity with the nomenclature and basic assumptions of [4, 14]. Recall that Douglis-Nirenberg ellipticity is equivalent to a more readily usable condition due to Volevič [15]. See [5] for a statement and simple proof.

Let $\{s_1, \ldots, s_n\} \subset \mathbb{Z}$ and $\{t_1, \ldots, t_n\} \subset \mathbb{Z}$ be two sets of Douglis-Nirenberg numbers, so that $\operatorname{order} \mathcal{D}_{ij} \leq s_i + t_j$, that have been normalized so that $\max\{s_1, \ldots, s_n\} = 0$ and $\min\{t_1, \ldots, t_n\} \geq 0$.

It is well known that since $N \geq 2$, proper ellipticity implies $\Sigma_{i=1}^n (s_i + t_i) = 2m$ with $m \geq 0$. We assume that a system $\mathcal{B} := (\mathcal{B}_{\ell j})$, $1 \leq \ell \leq m$, $1 \leq j \leq n$ of boundary differential operators is given, with $\operatorname{order} \mathcal{B}_{\ell j} \leq r_\ell + t_j$ for some $\{r_1, \ldots, r_m\} \subset \mathbb{Z}$.

Let

$$R := \max\{0, r_1 + 1, \ldots, r_m + 1\}, \qquad M := R + \max\{t_1, \ldots, t_n\}, \tag{4.1}$$

and call $a_{ij\alpha}$ and $b_{\ell j\beta}$ the (complex valued) coefficients of $\mathcal{D}_{ij}$ and $\mathcal{B}_{\ell j}$, respectively. Given an integer $\kappa \geq 0$, introduce the following hypotheses (generalizing those for a single equation in the Introduction).

(H1; κ) $\overline{\Omega}$ is a $C^{M+\kappa}$ ∂-submanifold of $\mathbb{R}^N$.

(H2; κ) The coefficients $a_{ij\alpha}$ are in $C^{R-s_i+\kappa}(\overline{\Omega})$ if $|\alpha| = s_i + t_j$ and in $W^{R-s_i+\kappa,\infty}(\Omega)$ otherwise.

(H3; κ) The coefficients $b_{\ell j\beta}$ are in $C^{R-r_\ell+\kappa}(\partial\Omega)$ if $|\beta| = r_\ell + t_j$ and in $W^{R-r_\ell+\kappa,\infty}(\partial\Omega)$ otherwise.

For $p \in (1, \infty)$ and $k \in \{0, \dots, \kappa\}$, define

$$\mathcal{T}_{p,k} := (\mathcal{D}, \mathcal{B}) : \prod_{j=1}^{n} W^{R+t_j+k,p}(\Omega) \longrightarrow \prod_{i=1}^{n} W^{R-s_i+k,p}(\Omega) \times \prod_{\ell=1}^{m} W^{R-r_\ell+k-1/p,p}(\partial\Omega). \tag{4.2}$$

Then (as proved in [3]), Theorem 1.1 holds (once again, the LS condition amounts to proper ellipticity plus complementing condition and proper ellipticity is equivalent to ellipticity if $m > 0$ and $N \geq 3$) and it is straightforward to check that the proof of Theorem 1.2 carries over to this case if $M \geq 2$. If so, Corollary 2.1 is also valid, with a similar proof and an obvious modification of the function spaces.

Remark 4.1. If $m = 0$, there is no boundary condition (in particular, $R = 0$, and (H3; κ) is vacuous) and the system $\mathcal{D}u = f$ can be solved explicitly for u in terms of f and its derivatives. This is explained in [14, page 506]. If so, the smoothness of $\partial\Omega$ (i.e., (H1; κ)) is irrelevant, and Theorem 1.2 is trivially true regardless of M ($\mathcal{T}_p$ is an isomorphism). A special case when $m = 0$ arises if $t_1 = \cdots = t_n = 0$ (in particular, if $M = 0$), for then $s_1 = \cdots = s_n = 0$ from the conditions $2m = \Sigma_{i=1}^{n}(s_i + t_i) \geq 0$ and $s_i \leq 0$.

From the above, Theorem 1.2 may only fail if $m \geq 1$, $R = 0$, and $M = 1$. (The author was recently informed by H. Koch [16] that he could prove Lemma 1.3 when $M = 1$, so that Theorem 1.2 remains true in this case as well.)

References

[1] L. Boutet de Monvel, “Boundary problems for pseudo-differential operators,” *Acta Mathematica*, vol. 126, no. 1-2, pp. 11–51, 1971.

[2] S. Rempel and B.-W. Schulze, *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin, Germany, 1982.

[3] G. Geymonat, “Sui problemi ai limiti per i sistemi lineari ellittici,” *Annali di Matematica Pura ed Applicata*, vol. 69, pp. 207–284, 1965.

[4] S. Agmon, A. Douglis, and L. Nirenberg, “Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II,” *Communications on Pure and Applied Mathematics*, vol. 17, pp. 35–92, 1964.

[5] J. T. Wloka, B. Rowley, and B. Lawruk, *Boundary Value Problems for Elliptic Systems*, Cambridge University Press, Cambridge, UK, 1995.

[6] S. Agmon, “The L_p approach to the Dirichlet problem. I. Regularity theorems,” *Annali della Scuola Normale Superiore di Pisa*, vol. 13, pp. 405–448, 1959.

[7] F. E. Browder, “On the spectral theory of elliptic differential operators I,” *Mathematische Annalen*, vol. 142, pp. 22–130, 1961.

[8] R. Denk, M. Hieber, and J. Prüss, *$\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, vol. 788, Memoirs of the American Mathematical Society, Providence, RI, USA, 2003.

[9] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford Mathematical Monographs, Oxford University Press, Oxford, UK, 1987.

[10] M. W. Hirsch, *Differential Topology*, vol. 33 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1976.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, vol. 224 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 2nd edition, 1983.

[12] S. G. Krantz and H. R. Parks, “Distance to C^k hypersurfaces,” *Journal of Differential Equations*, vol. 40, no. 1, pp. 116–120, 1981.

[13] H. Whitney, “Analytic extensions of differentiable functions defined in closed sets,” *Transactions of the American Mathematical Society*, vol. 36, no. 1, pp. 63–89, 1934.

[14] A. Douglis and L. Nirenberg, "Interior estimates for elliptic systems of partial differential equations," *Communications on Pure and Applied Mathematics*, vol. 8, pp. 503–538, 1955.
[15] L. R. Volevič, "Solvability of boundary problems for general elliptic systems," *American Mathematical Society Translations*, vol. 67, pp. 182–225, 1968.
[16] H. Koch, Private communication.