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Research Article

Blow-Up Results for a Nonlinear Hyperbolic Equation with Lewis Function

Faramarz Tahamtani

Department of Mathematics, Shiraz University, Shiraz 71454, Iran

Correspondence should be addressed to Faramarz Tahamtani, tahamtani@susc.ac.ir

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The initial boundary value problem for a nonlinear hyperbolic equation with Lewis function in a bounded domain is considered. In this work, the main result is that the solution blows up in finite time if the initial data possesses suitable positive energy. Moreover, the estimates of the lifespan of solutions are also given.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the initial boundary value problem for a nonlinear hyperbolic equation with Lewis function $\alpha(x)$ which depends on spacial variable:

$$\alpha(x)u_{tt} - \rho\Delta u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = f(u), \quad x \in \Omega,\ t \geq 0, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

where $\alpha(x) \geq 0$, $\rho > 0$, $m \geq 2$, and f is a continuous function.

The large time behavior of solutions for nonlinear evolution equations has been considered by many authors (for the relevant references one may consult with [1–14].)

In the early 1970s, Levine [3] considered the nonlinear wave equation of the form

$$Pu_{tt} = Au + h(u) \tag{1.4}$$

in a Hilbert space where P are A are positive linear operators defined on some dense subspace of the Hilbert space and h is a gradient operator. He introduced the concavity method and showed that solutions with negative initial energy blow up in finite time. This method was later improved by Kalantarov and Ladyzheskaya [4] to accommodate more general cases.

Very recently, Zhou [10] considered the initial boundary value problem for a quasilinear parabolic equation with a generalized Lewis function which depends on both spacial variable and time. He obtained the blowup of solutions with positive initial energy. In the case with zero initial energy Zhou [11] obtained a blow-up result for a nonlinear wave equation in $\mathbb{R}^n$. A global nonexistence result for a semilinear Petrovsky equation was given in [14].

In this work, we consider blow-up results in finite time for solutions of problem (1.1)-(1.3) if the initial datas possesses suitable positive energy and obtain a precise estimate for the lifespan of solutions. The proof of our technique is similar to the one in [10]. Moreover, we also show the blowup of solution in finite time with nonpositive initial energy.

Throughout this paper $\|\cdot\|_X$ denotes the usual norm of $L_X(\Omega)$.

The source term $f(u)$ in (1.1) with the primitive

$$F(u)=\int_0^u f(\xi)d\xi \tag{1.5}$$

satisfies

$$\left|f(u)\right|\le c_0|u|^{p-1},\quad c_0>0,\ p>m\ge 2, \tag{1.6}$$

$$\beta_1 mF(u)+\beta_2 m|\nabla u|^{m-1}\nabla u_t\le pF(u)<uf(u),\quad \beta_1>1,\ \beta_2>0. \tag{1.7}$$

Let $\mathbb{B}$ be the best constant of Sobolev embedding inequality

$$\|u\|_p\le \mathbb{B}\|\nabla u\|_m \tag{1.8}$$

from $W_0^{1,m}(\Omega)$ to $L_P(\Omega)$.

We need the following lemma in [4, Lemma 2.1].

Lemma 1.1. *Suppose that a positive, twice differentiable function $\Psi(t)$ satisfies for $t\ge 0$ the inequality*

$$\Psi''\Psi-(1+\sigma)(\Psi')^2\ge 0,\quad \sigma>0. \tag{1.9}$$

If $\Psi(0)>0$, $\Psi'(0)>0$, then

$$\Psi\longrightarrow+\infty\quad as\ t\longrightarrow t_1<t_2=\frac{\Psi(0)}{\sigma\Psi'(0)}. \tag{1.10}$$

2. Blow-Up Results

We set

$$\lambda_0 = (c_0\mathbb{B}^m)^{-1/(p-m)}, \qquad E_0 = \frac{p-m}{pm}(c_0\mathbb{B}^p)^{-m/(p-m)}. \tag{2.1}$$

The corresponding energy to the problem (1.1)-(1.3) is given by

$$E(t) = \frac{1}{m}\int_\Omega |\nabla u|^m dx + \frac{1}{2}\int_\Omega \alpha(x)u_t^2 dx - \int_\Omega F(u)dx, \tag{2.2}$$

and one can find that $E(t) \leq E(0)$ easily from

$$E'(t) = -\rho\|\nabla u\|_2^2 \leq 0, \tag{2.3}$$

whence

$$E(t) = E(0) - \rho\int_0^t \|\nabla u_\tau\|_2^2 d\tau. \tag{2.4}$$

We note that from (1.6) and (1.7), we have

$$E(t) \geq \frac{1}{m}\|\nabla u\|_m^m - \frac{c_0}{p}\|u\|_p^p, \quad t \geq 0, \tag{2.5}$$

and by Sobolev inequality (1.8), $E(t) \leq G(\|u\|_p)$, $t \geq 0$, where

$$G(\lambda) = (m\mathbb{B}^m)^{-1}\lambda^m - c_0 p^{-1}\lambda^p. \tag{2.6}$$

Note that $G(\lambda)$ has the maximum value E_0 at λ_0 which are given in (2.1).

Adapting the idea of Zhou [10], we have the following lemma.

Lemma 2.1. *Suppose that* $\|u(x,0)\|_p > \lambda_0$ *and* $E(0) \leq E_0$*. Then*

$$\|u(x,t)\|_p > \lambda_0, \qquad \|\nabla u(x,t)\|_m > (c_0\lambda_0^p)^{1/m} \tag{2.7}$$

for all $t \geq 0$*.*

Theorem 2.2. *For* $\alpha(x) \in L_\infty(\Omega)$*, suppose that* $u_0 \in W_0^{1,m}(\Omega)$ *and* $u_1 \in L_2(\Omega)$ *satisfy*

$$\mu(x) =: \int_\Omega \alpha(x)u_0u_1 dx > 0. \tag{2.8}$$

If $0 < E(0) \leq E_0$, *then the global solution of the problem* (1.1)–(1.3) *blows up in finite time and the lifespan*

$$T < \frac{2\left(\|\nabla u_0\|_2^2 - (p-2)\mu(x)\right)}{(p-2)^2(E_0 - E(0))}. \tag{2.9}$$

Proof. To prove the theorem, it suffices to show that the function

$$A(t) = \left\|\sqrt{\alpha(x)}u\right\|_2^2 + \rho\int_0^t \|\nabla u\|_2^2 d\tau + \rho(T_0 - t)\|\nabla u_0\|_2^2 + \gamma(t + t_0)^2 \tag{2.10}$$

satisfies the hypotheses of the Lemma 1.1, where $T_0 > t$, $t_0 > 0$ and $\gamma > 0$ to be determined later. To achieve this goal let us observe

$$\begin{aligned} 2\int_0^t\int_\Omega \nabla u \nabla u_\tau dx d\tau &= \int_0^t \frac{d}{d\tau}\|\nabla u\|_2^2 d\tau \\ &= \|\nabla u\|_2^2 - \|\nabla u_0\|_2^2. \end{aligned} \tag{2.11}$$

Hence,

$$\|\nabla u\|_2^2 = 2\int_0^t\int_\Omega \nabla u \nabla u_\tau dx d\tau + \|\nabla u_0\|_2^2. \tag{2.12}$$

Let us compute the derivatives $A'(t)$ and $A''(t)$. Thus one has

$$\begin{aligned} A'(t) &= 2\int_\Omega \alpha(x) u u_t dx + \rho\|\nabla u\|_2^2 - \rho\|\nabla u_0\|_2^2 + 2\gamma(t + t_0) \\ &= 2\int_\Omega \alpha(x) u u_t dx + 2\rho\int_0^t\int_\Omega \nabla u \nabla u_\tau dx d\tau + 2\gamma(t + t_0), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} A''(t) &= 2\left\|\sqrt{\alpha(x)}u_t\right\|_2^2 - 2\|\nabla u\|_m^m + 2\int_\Omega u f(u) dx + 2\gamma \\ &\geq 2\left\|\sqrt{\alpha(x)}u_t\right\|_2^2 - 2\|\nabla u\|_m^m + 2p\int_\Omega F(u) dx + 2\gamma \\ &\geq (p+2)\left\|\sqrt{\alpha(x)}u_t\right\|_2^2 + 2\left(\frac{p}{m} - 1\right)\|\nabla u\|_m^m - 2pE(t) + 2\gamma \\ &\geq (p+2)\left(\left\|\sqrt{\alpha(x)}u_t\right\|_2^2 + \rho\int_0^t \|\nabla u_\tau\|_2^2 d\tau\right) + 2\left(\frac{p}{m} - 1\right)\|\nabla u\|_m^m - 2pE(0) + 2\gamma \end{aligned} \tag{2.14}$$

for all $t \geq 0$. In the above assumption (1.7), the definition of energy functionals (2.2) and (2.4) has been used. Then, due to (2.1) and (2.7) and taking $\gamma = 2(E_0 - E(0))$,

$$A''(t) \geq (p+2)\left(\left\|\sqrt{\alpha(x)}u_t\right\|_2^2 + \rho\int_0^t \|\nabla u_\tau\|_2^2 d\tau + \gamma\right). \tag{2.15}$$

Hence $A''(t) \geq 0$ for all $t \geq 0$ and by assumption (2.8) we have

$$A'(0) = 2\left(\mu(x) + \gamma t_0\right) > 0. \tag{2.16}$$

Therefore $A'(t) \geq 0$ for all $t \geq 0$ and by the construction of $A(t)$, it is clearly that

$$A(t) \geq \left\|\sqrt{\alpha(x)}u\right\|_2^2 + \rho\int_0^t \|\nabla u\|_2^2 d\tau + \gamma(t+t_0)^2, \tag{2.17}$$

whence, $A(0) > 0$. Thus for all $(a,b) \in \mathbb{R}^2$, from (2.13), (2.15), and (2.17) we obtain

$$\begin{aligned}
a^2A(t) + abA'(t) + (p+2)^{-1}b^2A''(t) \geq{}& a^2\left(\left\|\sqrt{\alpha(x)}u\right\|_2^2 + \rho\int_0^t \|\nabla u\|_2^2 d\tau + \gamma(t+t_0)^2\right) \\
&+ 2ab\left(\int_\Omega \alpha(x)uu_t dx + \rho\int_0^t\int_\Omega \nabla u\nabla u_\tau dx d\tau + \gamma(t+t_0)\right) \\
&+ b^2\left(\left\|\sqrt{\alpha(x)}u_t\right\|_2^2 + \rho\int_0^t \|\nabla u_\tau\|_2^2 d\tau + \gamma\right) \\
={}& \left\|\sqrt{\alpha(x)}(au+bu_t)\right\|_2^2 \\
&+ \rho\int_0^t \|a\nabla u + b\nabla u_\tau\|_2^2 d\tau + \gamma\left(a(t+t_0)+b\right)^2 \\
\geq{}& 0,
\end{aligned} \tag{2.18}$$

which implies

$$\left(A'(t)\right)^2 - \frac{4}{p+2}A(t)A''(t) \leq 0. \tag{2.19}$$

Then using Lemma 1.1, one obtain that $A(t) \longrightarrow +\infty$ as

$$t \longrightarrow \frac{4A(0)}{(p-2)A'(0)} = \frac{2\left(\left\|\sqrt{\alpha(x)}u_0\right\|_2^2 + T_0\|\nabla u_0\|_2^2 + \gamma t_0^2\right)}{(p-2)\left(\mu(x) + \gamma t_0\right)}. \tag{2.20}$$

Now, we are in a position to choose suitable t_0 and T_0. Let t_0 be a number that depends on p, $(E_0 - E(0))$, $\|\nabla u_0\|_{L_2(\Omega)}$, and $\mu(x)$ as

$$t_0 > \frac{2\|\nabla u_0\|_2^2 - (p-2)\mu(x)}{(p-2)\gamma}. \tag{2.21}$$

To choose T_0, we may fix t_0 as

$$\begin{aligned} T_0 &= \frac{2\|\sqrt{\alpha(x)}u_0\|_2^2 + 2T_0\|\nabla u_0\|_2^2 + 2\gamma t_0^2}{(p-2)(\mu(x)+\gamma t_0)} \\ &= \frac{2\|\sqrt{\alpha(x)}u_0\|_2^2 + \gamma t_0^2}{(p-2)(\mu(x)+\gamma t_0) - 2\|\nabla u_0\|_2^2}. \end{aligned} \tag{2.22}$$

Thus, for $t \geq t_0$ the lifespan T is estimated by

$$\begin{aligned} T &< \frac{2\|\sqrt{\alpha(x)}u_0\|_2^2 + 2\gamma t^2}{(p-2)(\mu(x)+\gamma t) - 2\|\nabla u_0\|_2^2} \\ &< \frac{2\|\nabla u_0\|_2^2 - (p-2)\mu(x)}{(p-2)^2(E_0 - E(0))}, \end{aligned} \tag{2.23}$$

which completes the proof. □

Theorem 2.3. *Assume that $\alpha(x) \in L_\infty(\Omega)$ and the following conditions are valid:*

$$u_0 \in W_0^{1,m}, \quad u_1 \in L_2(\Omega), \; E(0) \leq 0. \tag{2.24}$$

Then the corresponding solution to (1.1)–(1.3) *blows up in finite time.*

Proof. Let

$$B(t) = \left\|\sqrt{\alpha(x)}u\right\|_2^2 + \rho\int_0^t \|\nabla u\|_2^2 d\tau, \tag{2.25}$$

then

$$B'(t) = 2\int_{\Omega} \alpha(x) u u_t dx + \rho \|\nabla u\|_2^2, \tag{2.26}$$

$$\begin{aligned}
B''(t) &= 2\left\|\sqrt{\alpha(x)} u_t\right\|_2^2 + 2\int_{\Omega} \alpha(x) u u_{tt} dx + 2\rho \int_{\Omega} \nabla u \nabla u_t dx \\
&= 2\left\|\sqrt{\alpha(x)} u_t\right\|_2^2 - 2\|\nabla u\|_m^m + 2\int_{\Omega} u f(u) dx \\
&> 2\left\|\sqrt{\alpha(x)} u_t\right\|_2^2 - 2\|\nabla u\|_m^m + 2\beta_1 m \int_{\Omega} F(u) dx + 2\beta_2 m \int_{\Omega} |\nabla u|^{m-1} \nabla u_t dx \\
&> 2(\beta_1 + 1)\left\|\sqrt{\alpha(x)} u_t\right\|_2^2 + 2(\beta_1 - 1)\|\nabla u\|_m^m + 2\beta_2 \frac{d}{dt}\|\nabla u\|_m^m - 2\beta_1 m E(0) \\
&> 2(\beta_1 - 1)\|\nabla u\|_m^m + 2\beta_2 \frac{d}{dt}\|\nabla u\|_m^m - 2\beta_1 m E(0), \quad t > 0,
\end{aligned} \tag{2.27}$$

where the left-hand side of assumption (1.7) and the energy functional (2.2) have been used. Taking the inequality (2.27) and integrating this, we obtain

$$B'(t) > 2(\beta_1 - 1)\int_0^t \|\nabla u\|_m^m d\tau + 2\beta_2 \|\nabla u\|_m^m - 2\beta_1 m E(0) t + B'(0), \quad t > 0. \tag{2.28}$$

By using Poincare-Friedrich's inequality

$$\|u\|_2^2 \leq \lambda_1 \|\nabla u\|_2^2, \tag{2.29}$$

and Holder's inequality

$$\|\nabla u\|_m^m \geq (\lambda_1 M)^{-m/2} |\Omega|^{1-m/2} \left(\int_{\Omega} \alpha(x) u^2 dx\right)^{m/2}, \tag{2.30}$$

$$\int_0^t \|\nabla u\|_m^m d\tau \geq t^{1-m/2} \left(\int_0^t \|\nabla u\|_2^2 d\tau\right)^{m/2}, \tag{2.31}$$

where $M = \max_{\Omega}|\alpha(x)|$. Using (2.30) and (2.31), we find from (2.28) that

$$\begin{aligned} B'(t) &\geq 2\beta_2(\lambda_1 M)^{-m/2}|\Omega|^{1-m/2}\left(\int_{\Omega}\alpha(x)u^2dx\right)^{m/2} \\ &\quad + 2(\beta_1-1)t^{1-m/2}\left(\int_0^t\|\nabla u\|_2^2d\tau\right)^{m/2} - 2\beta_1 mE(0)t + B'(0) \\ &\geq 2\beta_2(\lambda_1 M)^{-m/2}|\Omega|^{1-m/2}t^{1-m/2}\left(\int_{\Omega}\alpha(x)u^2dx\right)^{m/2} \\ &\quad + 2(\beta_1-1)t^{1-m/2}\left(\int_0^t\|\nabla u\|_2^2d\tau\right)^{m/2} - 2\beta_1 mE(0)t + B'(0), \quad t > 1. \end{aligned} \tag{2.32}$$

Since $-2\beta_1 mE(0)t + B'(0) \to \infty$ as $t \to \infty$ so, there must be a $t_1 > 1$ such that

$$-2\beta_1 mE(0)t + B'(0) \geq 0 \quad \text{as } t > t_1. \tag{2.33}$$

By inequality

$$(a_1 + a_2)^r < 2^{r-1}(a_1^r + a_2^r), \quad r > 1 \tag{2.34}$$

and by virtue of (2.33) and using (2.32), we get

$$B'(t) \geq Ct^{1-m/2}(B(t))^{m/2}, \tag{2.35}$$

where

$$C = \min\left(2^{2-m/2}(\beta_1 - 1), 2^{2-m/2}\beta_2(\lambda_1 M)^{-m/2}|\Omega|^{1-m/2}\right). \tag{2.36}$$

Therefore, there exits a positive constant

$$T = \begin{cases} C\exp(t_1), & m = 2, \\ Ct_1^{(4-m)/(2-m)}, & m > 2, \end{cases} \tag{2.37}$$

such that

$$B(t) \longrightarrow \infty \quad \text{as } t \longrightarrow T^-. \tag{2.38}$$

This completes the proof. □

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