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Research Article

Existence of Weak Solutions for a Nonlinear Elliptic System

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We investigate the existence of weak solutions to the following Dirichlet boundary value problem, which occurs when modeling an injection molding process with a partial slip condition on the boundary. We have $-\Delta\theta = k(\theta)|\nabla p|^r + q(x)$ in Ω; $-\text{div}\{(k(\theta)|\nabla p|^{r-2} + \beta(x)|\nabla p|^{r_0-2})\nabla p\} = 0$ in Ω; $\theta = \theta_0$, and $p = p_0$ on $\partial\Omega$.

1. Introduction

Injection molding is a manufacturing process for producing parts from both thermoplastic and thermosetting plastic materials. When the material is in contact with the mold wall surface, one has three choices: (i) no slip (which implies that the material sticks to the surface) (ii) partial slip, and (iii) complete slip [1–5]. Navier [6] in 1827 first proposed a partial slip condition for rough surfaces, relating the tangential velocity v_α to the local tangential shear stress $\tau_{\alpha 3}$

$$v_\alpha = -\beta\tau_{\alpha 3}, \tag{1.1}$$

where β indicates the amount of slip. When $\beta = 0$, (1.1) reduces to the no-slip boundary condition. A nonzero β implies partial slip. As $\beta \to \infty$, the solid surface tends to full slip.

There is a full description of the injection molding process in [3] and in our paper [7]. The formulation of this process as an elliptic system is given here in after.

Problem 1. Find functions θ and p defined in Ω such that

$$-\Delta\theta = k(\theta)\left|\nabla p\right|^{r} + q(x) \quad \text{in } \Omega, \tag{1.2}$$

$$-\operatorname{div}\left\{\left(k(\theta)\left|\nabla p\right|^{r-2} + \beta(x)\left|\nabla p\right|^{r_0-2}\right)\nabla p\right\} = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\theta = \theta_0, \quad p = p_0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Here we assume that Ω is a bounded domain in $\mathbb{R}^N$ with a C^1 boundary. We assume also that q, θ_0, p_0, β, and k are given functions, while r is a given positive constant related to the power law index; p is the pressure of the flow, and θ is the temperature. The leading order term $\beta(x)|\nabla p|^{r_0-2}$ of the PDE (1.3) is derived from a nonlinear slip condition of Navier type. Similar derivations based on the Navier slip condition occur elsewhere, for example, [8, 9], [10, equation (2.4)].

The mathematical model for this system was established in [7]. Some related papers, both rigorous and formal, are [3, 11–13]. In [11, 13], existence results in no-slip surface, $\beta = 0$, are obtained, while in [3, 7], Navier's slip conditions, $\beta \neq 0$ and $r_0 = 0$, are investigated, and numerical, existence, uniqueness, and regularity results are given. Although the physical models are two dimensional, we shall carry out our proofs in the case of N dimension.

In Section 2, we introduce some notations and lemmas needed in later sections. In Section 3, we investigate the existence, uniqueness, stability, and continuity of solution p to the nonlinear equation (1.3). In Section 4, we study the existence of weak solutions to Problem 1.

Using Rothe's method of time discretization and an existence result for Problem 1, one can establish existence of week solutions to the following *time-dependent* problem.

Problem 2. Find functions θ and p defined in Ω_T such that

$$\begin{gathered}
\theta_t - \Delta\theta = k(\theta)\left|\nabla p\right|^{r} + q(x) \quad \text{in } \Omega_T, \\
-\operatorname{div}\left\{\left(k(\theta)\left|\nabla p\right|^{r-2} + \beta(x)\left|\nabla p\right|^{r_0-2}\right)\nabla p\right\} = 0 \quad \text{in } \Omega_T, \\
\theta = \theta_0, \quad p = p_0 \quad \text{on } \partial\Omega \times (0,T), \\
\theta = \varphi \quad \text{on } \Omega \times \{0\}.
\end{gathered} \tag{1.5}$$

The proof is only a slight modification of the proofs given in [11, 13] and is omitted here.

2. Notations and Preliminaries

2.1. Notations

In this paper, for $s > 1$, let $H^{1,s}(\Omega)$ and $H_0^{1,s}(\Omega)$ denote the usual Sobolev space equipped with the standard norm. Let

$$\sigma = \begin{cases} \dfrac{N}{N-1}, & \text{if } 1 < r < N, \\[2ex] \dfrac{r}{r-1}, & \text{if } r > N, \\[2ex] \dfrac{qN}{qN-q+N}, & \text{if } r = N, \end{cases} \tag{2.1}$$

where $N < q < \infty$. The conjugate exponent of σ is

$$\sigma^* = \begin{cases} N, & \text{if } 1 < r < N, \\ r, & \text{if } r > N, \\ \dfrac{qN}{q-N}, & \text{if } r = N. \end{cases} \tag{2.2}$$

We assume that the boundary values θ_0 and p_0 for Problem 1 can be extended to functions defined on Ω such that

$$\theta_0 \in H^{1,\sigma}(\Omega), \qquad p_0 \in H^{1,\tau}(\Omega). \tag{2.3}$$

We further assume that there exist positive numbers $k_2 > k_1 > 0$ and β_0 such that

$$\begin{gathered} k_1 < k(\theta) < k_2, \quad \forall \theta \in R^1, \\ 0 \le \beta(x) \le \beta_0. \end{gathered} \tag{2.4}$$

Finally, we assume that for $\theta_m, \theta \in H_0^{1,\sigma}(\Omega) + \theta_0$, $\lim_{m\to\infty}\theta_m = \theta$ a.e. in Ω indicates

$$\lim_{m\to\infty} k(\theta_m) = k(\theta) \quad \text{a.e. in } \Omega. \tag{2.5}$$

For the convenience of exposition, we assume that

$$1 < r_0 < r < \tau < \infty. \tag{2.6}$$

Next, we recall some previous results which will be needed in the rest of the paper.

2.2. Preliminaries

An important inequality (e.g., see [11, page 550]) in the study of p-Laplacian is as follows:

$$\left(|x|^{r-2}x-|y|^{r-2}y\right)(x-y)\geq\begin{cases}a|x-y|^{r}, & \text{if } r\geq 2,\\ \dfrac{a|x-y|^{2}}{(b+|x|+|y|)^{2-r}}, & \text{if } 1<r<2,\end{cases}\tag{2.7}$$

where $a>0$ and $b>0$ are certain constants.

To establish coercivity condition, we will use the following inequality:

$$(a+b)^{r}\leq 2^{r}(a^{r}+b^{r}),\tag{2.8}$$

where $r>0$, $a>0$, and $b>0$.

Using the Sobolev Embedding Theorem and Hölder's Inequality, we can derive the following results (for more details, see [11, Lemma 3.4] and [13, Lemma 4.2]).

Lemma 2.1. *The following statements hold*

(i) *For any positive numbers α and ς, if $u\in L^{\alpha}(\Omega)$ and $v\in L^{\varsigma}(\Omega)$, then*

$$uv\in L^{\gamma},\quad \text{where } \gamma=\left(\frac{1}{\alpha}+\frac{1}{\varsigma}\right)^{-1};\tag{2.9}$$

moreover, $\|uv\|_{L^{\gamma}(\Omega)}\leq\|u\|_{L^{\alpha}(\Omega)}\|v\|_{L^{\varsigma}(\Omega)}$.

(ii) *If $p\in H^{1,r}(\Omega)$ and $1<r<N$, then $p|\nabla p|^{r-2}\nabla p\in[L^{N/(N-1)}(\Omega)]^{N}$; moreover,*

$$\left\|p|\nabla p|^{r-2}\nabla p\right\|_{L^{N/(N-1)}(\Omega)}\leq\|p\|_{L^{Nr/(N-r)}(\Omega)}\|\nabla p\|^{r-1}_{L^{r}(\Omega)}.\tag{2.10}$$

(iii) *If $p\in H^{1,r}(\Omega)$ and $1<r<\infty$, then $|\nabla p|^{r-2}\nabla p\nabla p_0\in L^{\zeta}(\Omega)$, where*

$$\zeta=\left(\frac{1}{r^{*}}+\frac{1}{\tau}\right)^{-1},\tag{2.11}$$

and r^{*} denotes the conjugate of r, namely, $r^{*}=r/(r-1)$ for $1<r<\infty$; moreover,

$$\left\||\nabla p|^{r-2}\nabla p\nabla p_0\right\|_{L^{\zeta}(\Omega)}\leq\|\nabla p\|^{r-1}_{L^{r}(\Omega)}\|\nabla p_0\|_{L^{\tau}(\Omega)}.\tag{2.12}$$

(iv) *If $p\in H^{1,r}(\Omega)$ and $n\leq r<\infty$, then*

$$\begin{aligned}&p|\nabla p|^{r-2}\nabla p\in\left[L^{r^{*}}(\Omega)\right]^{n},\quad r>n,\\ &p|\nabla p|^{r-2}\nabla p\in[L^{s}(\Omega)]^{n},\quad r=n,\end{aligned}\tag{2.13}$$

where $s = (1/r^* + 1/q)^{-1}$ *and* $r < q < \infty$. *Moreover*

$$\begin{aligned} \left\| p\left|\nabla p\right|^{r-2}\nabla p \right\|_{L^{r^*}(\Omega)} &\leq C\|\nabla p\|_{L^r(\Omega)}^{r-1}, \quad r > n, \\ \left\| p\left|\nabla p\right|^{r-2}\nabla p \right\|_{L^s(\Omega)} &\leq \|p\|_{L^q(\Omega)}\|\nabla p\|_{L^r(\Omega)}^{r-1}, \quad r = n. \end{aligned} \tag{2.14}$$

The existence proof will use the following general result of monotone operators [14, Corollary III.1.8, page 87] and [15, Proposition 17.2].

Proposition 2.2. *Let* $K \subset X$ *be a closed convex set* ($\neq \phi$), *and let* $\Lambda : K \to X'$ *be monotone, coercive, and weakly continuous on* K. *Then there exists*

$$u \in K : \langle \Lambda u, v - u \rangle \geq 0 \quad \textit{for any } v \in K. \tag{2.15}$$

The uniqueness proof is based on a supersolution argument (similar definition can be found in [15, Chapter 3]).

Definition 2.3. A function $u \in H^{1,r}_{loc}(\Omega)$ is a weak supersolution of the equation

$$-\operatorname{div}\left\{ \left(k(\theta)|\nabla u|^{r-2} + \beta(x)|\nabla u|^{r_0-2} \right)\nabla u \right\} = 0 \tag{2.16}$$

in Ω if

$$\int_\Omega \left(k(\theta)|\nabla u|^{r-2} + \beta(x)|\nabla u|^{r_0-2} \right)\nabla u \cdot \nabla\varphi \, dx \geq 0, \tag{2.17}$$

whenever $\varphi \in C_0^\infty(\Omega)$ is nonnegative.

3. A Dirichlet Boundary Value Problem

We study the following Dirichlet boundary value problem:

$$\begin{aligned} -\operatorname{div}\left\{ \left(k(\theta)\left|\nabla p\right|^{r-2} + \beta(x)\left|\nabla p\right|^{r_0-2} \right)\nabla p \right\} &= 0 \quad \text{in } \Omega, \\ p = p_0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Definition 3.1. We say that $p_\theta - p_0 \in H_0^{1,r}(\Omega)$ is a weak solution to (3.1) if

$$\int_\Omega \left(k(\theta)\left|\nabla p_\theta\right|^{r-2} + \beta(x)\left|\nabla p_\theta\right|^{r_0-2} \right)\nabla p_\theta \cdot \nabla\xi \, dx = 0 \tag{3.2}$$

for all $\xi \in H_0^{1,r}(\Omega)$ and a given $\theta \in H_0^{1,\sigma}(\Omega) + \theta_0$.

Theorem 3.2. *Assume that conditions* (2.1)–(2.6) *are satisfied. Then there exists a unique weak solution p_θ to the Dirichlet boundary value problem* (3.1) *in the sense of Definition 3.1. In addition, the solution p_θ satisfies the following properties.*

(1) *we have*

$$\left\|p_\theta\right\|_{H^{1,r}(\Omega)} \leq C, \tag{3.3}$$

where C is a constant independent of θ and p_θ;

(2) *if $\lim_{m\to\infty}\theta_m = \theta$ a.e. in Ω, then*

$$\lim_{m\to\infty} p_{\theta_m} = p_\theta \quad \text{strongly in } H^{1,r}(\Omega). \tag{3.4}$$

The idea behind the existence proof is related to [15, 16]. We will first consider the following Obstacle Problem.

Problem 3. Find a function p in K_{ψ,p_0} such that

$$\int_\Omega \left(k(\theta)\left|\nabla p\right|^{r-2} + \beta(x)\left|\nabla p\right|^{r_0-2}\right)\nabla p\, \nabla(\xi - p)\,dx \geq 0 \tag{3.5}$$

for all $\xi \in K_{\psi,p_0}$. Here

$$K_{\psi,p_0}(\Omega) = \left\{p \in H^{1,r}(\Omega) : p \geq \psi \text{ a.e. in } \Omega,\ p - p_0 \in H_0^{1,r}(\Omega)\right\}. \tag{3.6}$$

Lemma 3.3. *If K_{ψ,p_0} is nonempty, then there is a unique solution p to the Problem 3 in K_{ψ,p_0}.*

Proof of Lemma 3.3. Our proof will use Proposition 2.2.

Let $X = L^r(\Omega; R^n)$ and write

$$K = \left\{\nabla v : v \in K_{\psi,p_0}\right\}. \tag{3.7}$$

It follows from the proof in [15, Proposition 17.2] that $K \subset X$ is a closed convex set.

Next we define a mapping $\Lambda : K \to X'$ by

$$\langle \Lambda v, u\rangle = \int_\Omega \left(k(\theta)|v|^{r-2} + \beta(x)|v|^{r_0-2}\right)vu\,dx \quad \forall u \in X. \tag{3.8}$$

By Hölder's inequality,

$$\begin{aligned}|\langle \Lambda v, u\rangle| &\leq k_2\|v\|_{L^r(\Omega)}^{r-1}\|u\|_{L^r(\Omega)} + \beta_0\|v\|_{L^{r_0}(\Omega)}^{r_0-1}\|u\|_{L^{r_0}(\Omega)} \\ &\leq C\left(\|v\|_{L^r(\Omega)}^{r-1} + \|v\|_{L^{r_0}(\Omega)}^{r_0-1}\right)\|u\|_{L^r(\Omega)}.\end{aligned} \tag{3.9}$$

Here we used Assumption (2.6), that is, $1 < r_0 < r < \tau < \infty$. Therefore we have $\Lambda v \in X'$ whenever $v \in K$. Moreover, it follows from inequality (2.7) that Λ is monotone.

To show that Λ is coercive on K, fix $\varphi \in K$. Then

$$\begin{aligned}
&\langle \Lambda u - \Lambda\varphi, u - \varphi\rangle \\
&\quad = \int_\Omega \left[\left(k(\theta)|u|^{r-2} + \beta|u|^{r_0-2}\right)u - \left(k(\theta)|\varphi|^{r-2} + \beta|\varphi|^{r_0-2}\right)\varphi\right](u-\varphi)\,dx \\
&\quad = \int_\Omega k(\theta)\left(|u|^{r-2}u - |\varphi|^{r-2}\varphi]\right)(u-\varphi)\,dx + \int_\Omega \beta\left(|u|^{r_0-2}u - |\varphi|^{r_0-2}\varphi]\right)(u-\varphi)\,dx \\
&\quad \geq \int_\Omega k(\theta)\left(|u|^{r-2}u - |\varphi|^{r-2}\varphi]\right)(u-\varphi)\,dx \\
&\quad \geq k_1(\|u\|^r + \|\varphi\|^r) - k_2\left(\|u\|^{r-1}\|\varphi\| + \|\varphi\|^{r-1}\|u\|\right) \\
&\quad \geq k_1 2^{-r}\|u-\varphi\|^r - k_2 2^{r-1}\|\varphi\|\left(\|u-\varphi\|^{r-1} + \|\varphi\|^{r-1}\right) - k_2\|\varphi\|^{r-1}(\|\varphi - u\| + \|\varphi\|).
\end{aligned} \tag{3.10}$$

Inequality (2.8) is used to arrive at the last step. This implies that Λ is coercive on K.

Finally, we show that Λ is weakly continuous on K. Let $u_i \in K$ be a sequence that converges to an element $u \in K$ in $L^r(\Omega)$. Select a subsequence u_{i_j} such that $u_{i_j} \to u$ a.e. in Ω. Then it follows that

$$k(\theta)\left|u_{i_j}\right|^{r-2}u_{i_j} + \beta\left|u_{i_j}\right|^{r_0-2}u_{i_j} \longrightarrow k(\theta)|u|^{r-2}u + \beta|u|^{r_0-2}u \tag{3.11}$$

a.e. in Ω. Moreover,

$$\begin{aligned}
\int_\Omega \left|k(\theta)\left|u_{i_j}\right|^{r-2}u_{i_j} + \beta\left|u_{i_j}\right|^{r_0-2}u_{i_j}\right|^{r/(r-1)}dx &\leq C\int_\Omega\left(\left|u_{i_j}\right|^r + \left|u_{i_j}\right|^{r\times(r_0-1)/(r-1)}\right)dx \\
&\leq C\left[\int_\Omega \left|u_{i_j}\right|^r dx + \left(\int_\Omega \left|u_{i_j}\right|^r dx\right)^{(r_0-1)/(r-1)}\right] \\
&\leq C.
\end{aligned} \tag{3.12}$$

Thus we have that

$$k(\theta)\left|u_{i_j}\right|^{r-2}u_{i_j} + \beta\left|u_{i_j}\right|^{r_0-2}u_{i_j} \rightharpoonup k(\theta)|u|^{r-2}u + \beta|u|^{r_0-2}u \tag{3.13}$$

weakly in $L^{r/(r-1)}(\Omega)$. Since the weak limit is independent of the choice of the subsequence, it follows that

$$k(\theta)|u_i|^{r-2}u_i + \beta|u_i|^{r_0-2}u_i \rightharpoonup k(\theta)|u|^{r-2}u + \beta|u|^{r_0-2}u \tag{3.14}$$

weakly in $L^{r/(r-1)}(\Omega)$. Hence Λ is weakly continuous on K. We may apply Proposition 2.2 to obtain the existence of p.

Our uniqueness proof is inspired by [15, Lemmas 3.11, 3.22, and Theorem 3.21]. Since $(k(\theta)|\nabla u|^{r-2}+\beta(x)|\nabla u|^{r_0-2})\nabla u$ does not satisfy condition (3.4) of $\mathscr{A}$ operator in [15], we need to prove the following lemma, which is equivalent to [15, Lemma 3.11]. Then uniqueness can follow immediately from [15, Lemma 3.22]. □

Lemma 3.4. *If $u \in H^{1,r}(\Omega)$ is a supersolution of* (2.16) *in Ω, then*

$$\int_{\Omega}\left(k(\theta)|\nabla u|^{r-2}+\beta(x)|\nabla u|^{r_0-2}\right)\nabla u\cdot\nabla\varphi\,dx \geq 0 \tag{3.15}$$

for all nonnegative $\varphi \in H_0^{1,r}(\Omega)$.

Proof. Let $\varphi \in H_0^{1,r}(\Omega)$ and choose nonnegative sequence $\phi_i \in C_0^{\infty}(\Omega)$ such that $\varphi_i \to \varphi$ in $H^{1,r}(\Omega)$. Equation (2.6) and Hölder inequality imply that

$$\begin{aligned}
&\left|\int_{\Omega}\left(k(\theta)|\nabla u|^{r-2}+\beta|\nabla u|^{r_0-2}\right)\nabla u\cdot\nabla\varphi\,dx-\int_{\Omega}\left(k(\theta)|\nabla u|^{r-2}+\beta|\nabla u|^{r_0-2}\right)\nabla u\cdot\nabla\varphi_i dx\right|\\
&\quad=\left|\int_{\Omega}k(\theta)|\nabla u|^{r-2}\nabla u\cdot\nabla(\varphi-\varphi_i)dx+\int_{\Omega}\beta|\nabla u|^{r_0-2}\nabla u\cdot\nabla(\varphi-\varphi_i)dx\right|\\
&\quad\leq k_2\|\nabla u\|_{L^r(\Omega)}^{r-1}\left\|\nabla(\varphi-\varphi_i)\right\|_{L^r(\Omega)}+\beta_0\|\nabla u\|_{L^{r_0}(\Omega)}^{r_0-1}\left\|\nabla(\varphi-\varphi_i)\right\|_{L^{r_0}(\Omega)}\\
&\quad\leq C\left(\|\nabla u\|_{L^r(\Omega)}^{r-1}+\beta_0\|\nabla u\|_{L^{r_0}(\Omega)}^{r_0-1}\right)\left\|\nabla(\varphi-\varphi_i)\right\|_{L^r(\Omega)}.
\end{aligned} \tag{3.16}$$

Because $\lim_{i\to\infty}\|\nabla(\varphi-\varphi_i)\|_{L^r(\Omega)}=0$, we obtain

$$\int_{\Omega}\left(k(\theta)|\nabla u|^{r-2}+\beta|\nabla u|^{r_0-2}\right)\nabla u\cdot\nabla\varphi\,dx=\lim_{i\to\infty}\int_{\Omega}\left(k(\theta)|\nabla u|^{r-2}+\beta|\nabla u|^{r_0-2}\right)\nabla u\cdot\nabla\varphi_i dx\geq 0 \tag{3.17}$$

and the lemma follows. □

Similar to [15, Corollary 17.3, page 335], one can also obtain the following Corollary.

Corollary 3.5. *Let Ω be bounded and $p_0 \in H^{1,r}(\Omega)$. There is a weak solution $p_\theta \in H_0^{1,r}(\Omega)+p_0$ to* (3.1) *in the sense of Definition 3.1.*

Proof of Theorem 3.2. The existence result is given in Corollary 3.5, and we now turn to proof of uniqueness. For a given θ, assume that there exists another solution p_θ^1. Then we have that

$$\begin{aligned}
\Delta := \int_{\Omega}\Big[&k\left(\theta\left|\nabla p_\theta\right|^{r-2}\nabla p_\theta-\left|\nabla p_\theta^1\right|^{r-2}\nabla p_\theta^1\right)\\
&+\beta\left(\left|\nabla p_\theta\right|^{r_0-2}\nabla p_\theta-\left|\nabla p_\theta^1\right|^{r_0-2}\nabla p_\theta^1\right)\Big]\cdot\nabla\xi\,dx=0
\end{aligned} \tag{3.18}$$

for all $\xi \in H_0^{1,r}(\Omega)$. If we take $\xi = p_\theta - p_\theta^1$ in above equation, from inequality (2.7), we have the following.

(i) when $r \geq 2$,

$$\begin{aligned} 0 &= \Delta \\ &\geq \int_\Omega k(\theta)\left(|\nabla p_\theta|^{r-2}\nabla p_\theta - \left|\nabla p_\theta^1\right|^{r-2}\nabla p_\theta^1\right)\cdot\left(\nabla p_\theta - \nabla p_\theta^1\right)\, dx \\ &\geq C\int_\Omega \left|\nabla p_\theta - \nabla p_\theta^1\right|^r dx, \end{aligned} \tag{3.19}$$

where C is a positive constant;

(ii) when $1 < r < 2$,

$$\begin{aligned} 0 &= \Delta \\ &\geq \int_\Omega k(\theta)\left(|\nabla p_\theta|^{r-2}\nabla p_\theta - \left|\nabla p_\theta^1\right|^{r-2}\nabla p_\theta^1\right)\cdot\left(\nabla p_\theta - \nabla p_\theta^1\right)\, dx \\ &\geq C\int_\Omega \left|\nabla p_\theta - \nabla p_\theta^1\right|^2\left(b + |\nabla p_\theta| + \left|\nabla p_\theta^1\right|\right)^{r-2} dx \\ &\geq C\left(\int_\Omega \left|\nabla p_\theta^1 - \nabla p_\theta\right|^r dx\right)^{2/r}\left(\int_\Omega \left(b + |\nabla p_\theta| + \left|\nabla p_\theta^1\right|\right)^r dx\right)^{(r-2)/r}. \end{aligned} \tag{3.20}$$

Here the Hölder inequality for $0 < t < 1$, namely,

$$\left|\int_\Omega fg\,dx\right| \geq \left(\int_\Omega |f|^t dx\right)^{1/t}\left(\int_\Omega |g|^{t^*} dx\right)^{1/t^*}, \quad t^* = \frac{t}{t-1} \tag{3.21}$$

is applied to the last inequality.

Poincaré's inequality implies that $p_\theta = p_\theta^1$ a.e. We complete the uniqueness proof.

Next we prove (3.3). Taking $\xi = p_\theta - p_0$ in (3.2), we have

$$\int_\Omega k(\theta)|\nabla p_\theta|^r dx \leq \int_\Omega k(\theta)|\nabla p_\theta|^{r-2}\nabla p_\theta \nabla p_0 dx + \int_\Omega \beta|\nabla p_\theta|^{r_0-2}\nabla p_\theta \nabla p_0 dx. \tag{3.22}$$

From (2.4), and the Hölder inequality, we obtain

$$\begin{aligned} k_1\int_\Omega |\nabla p_\theta|^r dx \leq{}& k_2\left(\int_\Omega |\nabla p_\theta|^r dx\right)^{(r-1)/r}\left(\int_\Omega |\nabla p_0|^r dx\right)^{1/r} \\ &+ \beta_0\left(\int_\Omega |\nabla p_\theta|^r dx\right)^{(r_0-1)/r}\left(\int_\Omega |\nabla p_0|^{r/(r-r_0+1)} dx\right)^{(r-r_0+1)/r}. \end{aligned} \tag{3.23}$$

Young's inequality with ε implies

$$k_1\int_\Omega |\nabla p_\theta|^r dx \le \varepsilon\int_\Omega |\nabla p_\theta|^r dx + C\left(\int_\Omega |\nabla p_0|^r dx + \int_\Omega |\nabla p_0|^{r/(r-r_0+1)} dx\right) \tag{3.24}$$

and (3.3) follows immediately from (2.3) and (2.6).

Finally, we prove (3.4). From weak solution definition (3.2), we know that

$$\begin{aligned}\int_\Omega &\left(k(\theta_m)|\nabla p_{\theta_m}|^{r-2} + \beta|\nabla p_{\theta_m}|^{r_0-2}\right)\nabla p_{\theta_m}\nabla\xi\, dx \\ &= \int_\Omega \left(k(\theta)|\nabla p_\theta|^{r-2} + \beta|\nabla p_\theta|^{r_0-2}\right)\nabla p_\theta \nabla\xi\, dx = 0.\end{aligned} \tag{3.25}$$

Setting $\xi = p_{\theta_m} - p_\theta$ and subtracting $\int_\Omega (k(\theta_m)|\nabla p_\theta|^{r-2} + \beta|\nabla p_\theta|^{r_0-2})\nabla p_\theta\nabla\xi\, dx$ from both sides, we obtain that

$$\begin{aligned}\int_\Omega &\Big[k(\theta_m)\left(|\nabla p_{\theta_m}|^{r-2}\nabla p_{\theta_m} - |\nabla p_\theta|^{r-2}\nabla p_\theta\right) + \beta\left(|\nabla p_{\theta_m}|^{r_0-2} p_{\theta_m} - |\nabla p_\theta|^{r_0-2}\nabla p_\theta\right)\Big]\nabla(p_{\theta_m} - p_\theta) dx \\ &= \int_\Omega (k(\theta) - k(\theta_m))|\nabla p_\theta|^{r-2}\nabla p_\theta \nabla(p_{\theta_m} - p_\theta) dx.\end{aligned} \tag{3.26}$$

Denote the right-hand side by Δ_1. Similar to arguments in the uniqueness proof, we arrive at the folloing:

(i) when $r \ge 2$,

$$C\int_\Omega |\nabla p_{\theta_m} - \nabla p_\theta|^r dx \le \Delta_1; \tag{3.27}$$

(ii) when $1 < r < 2$,

$$C\left(\int_\Omega |\nabla p_{\theta_m} - \nabla p_\theta|^r dx\right)^{2/r}\left(\int_\Omega (b + |\nabla p_\theta| + |\nabla p_{\theta_m}|)^r dx\right)^{(r-2)/r} \le \Delta_1. \tag{3.28}$$

Egorov's Theorem implies that for all $\epsilon > 0$, there is a closed subset Ω_ϵ of Ω such that $|\Omega\backslash\Omega_\epsilon| < \epsilon$ and $k(\theta_m) \to k(\theta)$ uniformly on Ω_ϵ. Application of the absolute continuity of the Lebesgue

Integral implies

$$\begin{aligned}
\Delta_1 &\leq \int_{\Omega_\epsilon} + \int_{\Omega\setminus\Omega_\epsilon} |k(\theta_m) - k(\theta)| \left|\nabla p_\theta\right|^{r-1} \left|\nabla\left(p_{\theta_m} - p_\theta\right)\right| dx \\
&\leq \varepsilon \left[\left(\int_\Omega \left|\nabla p_\theta\right|^r dx\right)^{(r-1)/r} + 2k_2\right] \left(\int_\Omega \left|\nabla\left(p_{\theta_m} - p_\theta\right)\right|^r dx\right)^{1/r} \\
&\longrightarrow 0 \quad \text{as } \theta_m \longrightarrow \theta.
\end{aligned} \tag{3.29}$$

Theorem 3.2 is proved. □

4. Nonlinear Elliptic Dirichlet System

Definition 4.1. We say that $\{\theta, p\}$ is a weak solution to Problem 1 if

$$\theta - \theta_0 \in H_0^{1,\sigma}(\Omega), \qquad p - p_0 \in H_0^{1,r}(\Omega), \tag{4.1}$$

and for all $v \in C_0^\infty(\Omega)$

$$-\int_\Omega \nabla\theta \cdot \nabla v \, dx = \int_\Omega \left(k(\theta)\left|\nabla p\right|^r + q\right) v \, dx, \tag{4.2}$$

and for all $\xi \in H_0^{1,r}(\Omega)$

$$\int_\Omega \left(k(\theta)\left|\nabla p\right|^{r-2} + \beta(x)\left|\nabla p\right|^{r_0-2}\right) \nabla p \cdot \nabla\xi \, dx = 0. \tag{4.3}$$

Theorem 4.2. *Assume that* (2.1)–(2.6) *hold. Then there exists a weak solution to Problem 1 in the sense of Definition 4.1.*

We shall bound the critical growth, $|\nabla p|^r$, on the right-hand side of (4.2).

Lemma 4.3. *Suppose that θ and p satisfy*

$$\theta - \theta_0 \in H_0^{1,\sigma}(\Omega), \qquad p - p_0 \in H_0^{1,r}(\Omega), \tag{4.4}$$

and (4.3). *Then, under the conditions of Theorem 4.2, for all* $v \in C^1(\overline{\Omega})$

$$\begin{aligned}\int_\Omega k(\theta)|\nabla p|^r v dx = &\int_\Omega k(\theta)|\nabla p|^{r-2}\nabla p \cdot \nabla p_0 v\, dx \\ &- \int_\Omega k(\theta)|\nabla p|^{r-2}\nabla p(p-p_0)\cdot \nabla v\, dx \\ &- \int_\Omega \beta|\nabla p|^{r_0-2}\nabla p \cdot \nabla(p-p_0)v\, dx \\ &- \int_\Omega \beta|\nabla p|^{r_0-2}\nabla p(p-p_0)\cdot \nabla v\, dx.\end{aligned} \tag{4.5}$$

Moreover, there exists a polynomial F *that is independent of* θ *and* p *such that*

$$\int_\Omega k(\theta)|\nabla p|^r v\, dx \le F\left(\|p\|_{H^{1,r}(\Omega)}\right)\|v\|_{H^{1,\sigma^*}(\Omega)}. \tag{4.6}$$

Proof. We first show (4.5). Letting $\xi = v(p-p_0)$ in (4.3), we obtain

$$\begin{aligned}&\int_\Omega k(\theta)|\nabla p|^{r-2}\nabla p\cdot[v\ \nabla(p-p_0) + (p-p_0)\nabla v]dx \\ &\quad + \int_\Omega \beta|\nabla p|^{r-2}\nabla p\cdot[v\ \nabla(p-p_0)+(p-p_0)\nabla v]dx = 0.\end{aligned} \tag{4.7}$$

After some straightforward computations this yields exactly (4.5).

We now show (4.6). We denote the four terms on the right-hand side of equation (4.5) by I, II, III, and IV, respectively. Under the conditions of Lemma 4.3, we have

$$|\nabla p|^{r-2}\nabla p \in L^{r^*}(\Omega), \qquad \nabla p_0 \in L^\tau(\Omega), \qquad r^* = \frac{r}{r-1}. \tag{4.8}$$

Part (iii) of Lemma 2.1 and Sobolev's imbedding theorems indicate

$$\begin{aligned}|\mathrm{I}| &\le k_2\|\nabla p\|^{r-1}_{L^r(\Omega)}\|\nabla p_0\|_{L^\tau(\Omega)}\|v\|_{L^{\zeta^*}(\Omega)} \\ &\le C\|\nabla p\|^{r-1}_{L^r(\Omega)}\|\nabla p_0\|_{L^\tau(\Omega)}\|v\|_{H^{1,\sigma^*}(\Omega)} \\ &\le C\|\nabla p\|^{r-1}_{L^r(\Omega)}\|v\|_{H^{1,\sigma^*}(\Omega)},\end{aligned} \tag{4.9}$$

where $\zeta^* = \tau r/(\tau - r)$ satisfies $(r-1)/r + 1/\tau + 1/\zeta^* = 1$.

According to Sobolev's imbedding theorems, the integrability of $(p-p_0)$ depends on N. We estimate II in three different cases.

Case 1 ($1 < r < N$).

$$\begin{aligned}|\mathrm{II}| &\leq C\left\|p-p_0\right\|_{L^{Nr/(N-r)}(\Omega)}\left\|\nabla p\right\|_{L^r(\Omega)}^{r-1}\|\nabla v\|_{L^N(\Omega)} \\ &\leq C\left(\|p\|_{H^{1,r}(\Omega)}^{r}+\|p\|_{H^{1,r}(\Omega)}^{r-1}\right)\|v\|_{H^{1,N}(\Omega)}.\end{aligned} \tag{4.10}$$

Case 2 ($r = N$).

$$\begin{aligned}|\mathrm{II}| &\leq C\left\|p-p_0\right\|_{L^{q}(\Omega)}\left\|\nabla p\right\|_{L^r(\Omega)}^{r-1}\|\nabla v\|_{L^{qr/(q-r)}(\Omega)} \quad r<q<\infty \\ &\leq C\left(\|p\|_{H^{1,r}(\Omega)}^{r}+\|p\|_{H^{1,r}(\Omega)}^{r-1}\right)\|v\|_{H^{1,qr/(q-r)}}(\Omega).\end{aligned} \tag{4.11}$$

Case 3 ($r > N$). $p - p_0$ is a bounded continuous function, so

$$|\mathrm{II}| \leq C\left\|\nabla p\right\|_{L^r(\Omega)}^{r-1}\|\nabla v\|_{L^r(\Omega)}. \tag{4.12}$$

We next estimate III:

$$\begin{aligned}|\mathrm{III}| &\leq \left|\int_\Omega \beta\left|\nabla p\right|^{r_0} v\,dx\right| + \left|\int_\Omega \beta\left|\nabla p\right|^{r_0-2}\nabla p\cdot\nabla p_0 v\,dx\right| \\ &\leq \beta_0\left\|\nabla p\right\|_{L^r(\Omega)}^{r_0-1}\|v\|_{L^{r/(r-r_0)}(\Omega)} + C\left\|\nabla p\right\|_{L^r(\Omega)}^{r-1}\|v\|_{H^{1,\sigma^*}(\Omega)}.\end{aligned} \tag{4.13}$$

The estimate of the first term used Hölder inequality and Sobolev's imbedding theorems. The argument of the second estimate is similar to that of I.

Recall that $1 < r_0 < r$. Similar to II, we estimate IV in three different cases.

Case 1 ($1 < r < N$).

$$|\mathrm{IV}| \leq C\left\|p-p_0\right\|_{L^{Nr/(N-r)}(\Omega)}\left\|\nabla p\right\|_{L^r(\Omega)}^{r_0-1}\|\nabla v\|_{L^{Nr/(N(r-r_0)+r)}(\Omega)}. \tag{4.14}$$

Since $Nr/(N(r-r_0)+r) < N$, we have

$$|\mathrm{IV}| \leq C\left(\|p\|_{H^{1,r}(\Omega)}^{r_0}+\|p\|_{H^{1,r}(\Omega)}^{r_0-1}\right)\|v\|_{H^{1,N}(\Omega)}. \tag{4.15}$$

Case 2 ($r = N$).

$$|\mathrm{IV}| \leq C\left\|p-p_0\right\|_{L^{q}(\Omega)}\left\|\nabla p\right\|_{L^r(\Omega)}^{r_0-1}\|\nabla v\|_{L^{qr/(q(r-r_0)+q-r)}(\Omega)} \quad r<q<\infty. \tag{4.16}$$

Since $qr/(q(r-r_0)+q-r) < qr/(q-r)$, we have

$$|\mathrm{IV}| \leq C\left(\|p\|_{H^{1,r}(\Omega)}^{r_0}+\|p\|_{H^{1,r}(\Omega)}^{r_0-1}\right)\|v\|_{H^{1,qr/(q-r)}(\Omega)}. \tag{4.17}$$

Case 3 $(r > N)$.

$$|\mathrm{IV}| \le C \left\| \nabla p \right\|_{L^r(\Omega)}^{r_0 - 1} \| \nabla v \|_{L^r(\Omega)}. \tag{4.18}$$

These estimates lead to

$$|\mathrm{I}| + |\mathrm{II}| + |\mathrm{III}| + |\mathrm{IV}| \le F\left(\| p \|_{H^{1,r}(\Omega)} \right) \| v \|_{H^{1,\sigma^*}(\Omega)} \tag{4.19}$$

for some polynomial F. □

Proof of Theorem 4.2. Using Theorem 3.2, let $z \in H_0^{1,\sigma}(\Omega) + \theta_0$, then for (3.2) there exists a unique solution p_z satisfying

$$\| p_z \|_{H^{1,r}(\Omega)} \le C. \tag{4.20}$$

Moreover, if $\lim_{m \to \infty} z_m = z$ a.e. in Ω, then

$$\lim_{m \to \infty} p_{z_m} = p_z \quad \text{strongly in } H^{1,r}(\Omega). \tag{4.21}$$

Next, using Lemma 4.3, we can define a linear functional $F_z \in (H^{1,\sigma^*}(\Omega))^*$ determined by

$$\begin{aligned} \langle F_z, v \rangle = & \int_\Omega k(\theta) \left| \nabla p_z \right|^{r-2} \nabla p_z \cdot \nabla p_0 v \, dx \\ & - \int_\Omega k(\theta) \left| \nabla p_z \right|^{r-2} \nabla p_z (p_z - p_0) \cdot \nabla v \, dx \\ & - \int_\Omega \beta \left| \nabla p_z \right|^{r_0 - 2} \nabla p_z \cdot \nabla (p_z - p_0) v \, dx \\ & - \int_\Omega \beta \left| \nabla p_z \right|^{r_0 - 2} \nabla p_z (p_z - p_0) \cdot \nabla v \, dx, \end{aligned} \tag{4.22}$$

for all $v \in H^{1,\sigma^*}(\Omega)$. By virtue of (4.6), F_z is well defined, and there exists a constant $C > 0$ independent of z such that

$$|\langle F_z, v \rangle| \le C \| v \|_{H^{1,\sigma^*}(\Omega)}. \tag{4.23}$$

We notice that (4.2) is the same as [11, equation (1.6)]. Therefore, arguments after [11, equation (3.19)] can be used to complete the proof of Theorem 4.2. □

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