Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 710386, 20 pages
doi:10.1155/2009/710386

Research Article

On the Correct Solvability of the Boundary-Value Problem for One Class Operator-Differential Equations of the Fourth Order with Complex Characteristics

Araz R. Aliev[1, 2] and Aydin A. Gasymov[1]

[1] *Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 F. Agaev street, Baku, AZ1141, Azerbaijan*
[2] *Baku State University, 23 Z. Khalilov street, Baku, AZ1148, Azerbaijan*

Correspondence should be addressed to Araz R. Aliev, alievaraz@yahoo.com

Received 19 February 2009; Revised 22 July 2009; Accepted 26 August 2009

Recommended by Ivan T. Kiguradze

Sufficient coefficient conditions for the correct and unique solvability of the boundary-value problem for one class of operator-differential equations of the fourth order with complex characteristics, which cover the equations arising in solving the problems of stability of plastic plates, are obtained in this paper. Exact values of the norms of operators of intermediate derivatives, which are involved in the perturbed part of the operator-differential equation under investigation, are found along with these in subspaces $W_2^4(R_+;H)$ in relation to the norms of the operator generated by the main part of this equation. It is noted that this problem has its own mathematical interest.

1. Introduction

It is well known that a number of problems in mechanics lead to studying the completeness of all or part of the eigenvectors and joint vectors of certain polynomial operator groups and the completeness of elementary solutions of the operator-differential equations corresponding to these groups (see, e.g., [1, 2], and their references). In this case, it is first necessary to investigate the correct solvability of Cauchy or boundary-value problems for these equations, and only after this it will be possible to proceed to the abovementioned problems. The present paper is dedicated to the problem of correct solvability of the boundary-value problem for one class of operator-differential equations of the fourth order, considered on a semiaxis.

Let H be a separable Hilbert space and A be a self-adjoint positively defined operator in H.

Let us consider the following operator-differential equation of the fourth order:

$$Q\left(\frac{d}{dt}\right)u(t) \equiv \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u(t) + \sum_{s=1}^{3} A_s \frac{d^{4-s}u(t)}{dt^{4-s}} = f(t), \quad t \in R_+ = [0; +\infty), \tag{1.1}$$

with the boundary conditions

$$\frac{d^k u(0)}{dt^k} = 0, \quad k = 0, 1, 2, \tag{1.2}$$

where $f(t) \in L_2(R_+; H)$, $u(t) \in W_2^4(R_+; H), A_s, s = 1, 2, 3$, are linear and generally unbounded operators in H. Under $L_2(R_+; H)$ and $W_2^4(R_+; H)$, the following Hilbert spaces can be described:

$$L_2(R_+; H) = \left\{ f(t) : \|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 dt \right)^{1/2} < +\infty \right\},$$

$$W_2^4(R_+; H) = \left\{ u(t) : \|u\|_{W_2^4(R_+;H)} = \left(\int_0^{+\infty} \left(\left\| \frac{d^4 u(t)}{dt^4} \right\|_H^2 + \left\| A^4 u(t) \right\|_H^2 \right) dt \right)^{1/2} < +\infty \right\} \tag{1.3}$$

(see [3–5]).

Definition 1.1. If the vector function $u(t) \in W_2^4(R_+; H)$ *satisfies* (1.1) *almost everywhere in* R_+, *then it is called a regular solution of* (1.1).

Definition 1.2. If for any $f(t) \in L_2(R_+; H)$, *there exists a regular solution of* (1.1) *which satisfies boundary condition* (1.2) *in the sense that*

$$\lim_{t \to 0} \left\| A^{7/2-k} \frac{d^k u(t)}{dt^k} \right\|_H = 0, \quad k = 0, 1, 2, \tag{1.4}$$

and the inequality

$$\|u\|_{W_2^4(R_+;H)} \leq \text{const} \, \|f\|_{L_2(R_+;H)} \tag{1.5}$$

holds, then it can be said that problem (1.1), (1.2) *is regularly solvable.*

Let us define the following subspaces of the space $W_2^4(R_+;H)$:

$$\overset{o}{W}{}_2^4(R_+;H) = \left\{ u(t) : u(t) \in W_2^4(R_+;H),\ \frac{d^j u(0)}{dt^j} = 0,\ j = 0,1,2,3 \right\},$$
$$\overset{o}{W}{}_2^4(R_+;H;\{k\}) = \left\{ u(t) : u(t) \in W_2^4(R_+;H),\ \frac{d^k u(0)}{dt^k} = 0,\ k = 0,1,2 \right\}. \tag{1.6}$$

It should be noted that the solvability theory for the Cauchy problem and the boundary-value problems for first- and second-order operator-differential equations have been studied in more detail elsewhere. In addition to books [6, 7], these problems have been considered also by Agmon and Nirenberg [8], Gasymov and Mirzoev [9], Kostyuchenko and Shkalikov [10], and in works in their bibliographies. Other papers in which issues of the solvability of various problems for operator-differential equations of higher order have been studied have appeared alongside these works, and sufficiently interesting results have been obtained. Among these papers are those by Gasymov [11, 12], Dubinskii [13], Mirzoev [14], Shakhmurov [15], Shkalikov [16], Aliev [17, 18], Agarwal et al. [19], Favini and Yakubov [20], the book by Yakubov [7], and other works listed in their bibliographies.

Sufficient coefficient conditions for regular solvability of the boundary-value problem stated in (1.1) and (1.2) are presented in this paper. To obtain these conditions, the main challenge is to find the exact values of the norms of operators of intermediate derivatives in subspaces $\overset{o}{W}{}_2^4(R_+;H)$, $\overset{o}{W}{}_2^4(R_+;H;\{k\})$, the norms of which are expressed by the main part of (1.1). This problem has its own mathematical interest (see, e.g., [21, 22], and works given in their bibliographies). Estimation of the norms of operators of intermediate derivatives, which are involved in the perturbed part of (1.1), is performed with the help of a factorization method for one class of polynomial operator groups of eighth order, depending on a real parameter. A similar approach has been presented in [9, 14], which makes it possible to formulate solvability theorems for the boundary-value problems, with conditions which can be easily checked.

It should be noted that if the main part of the equation has the operator in the form $(-d^2/dt^2 + A^2)^2$, then a biharmonic equation results, which is of mathematical interest not only theoretically, and also from a practical point of view. Many problems of elasticity theory (e.g., the theory of bending of thin elastic slabs [23]) can be reduced to studying the boundary-value problems for such equations. Much research has been performed to investigate the solvability of such problems, for example, that reported in [24]. Operator-differential equations, which are studied in the present paper, include the fourth-order equations which arise when solving the stability problems of plates made of plastic material (see [25, pages 185–196]). It is very difficult to solve such problems because the differential equation must be solved in a more complete form, that is, when the main part of the equation has terms containing $du(t)/dt$ and $d^3u(t)/dt^3$. As a result, the equation has more complex characteristics, and (1.1) is of this type.

Furthermore, let us denote by $\sigma(A)$ the spectrum of the operator A.

2. Auxiliary Results

First, let us study the main part of (1.1):

$$Q_0\left(\frac{d}{dt}; A\right)u(t) \equiv \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u(t) = f(t), \tag{2.1}$$

where $f(t) \in L_2(R_+; H)$.

The following theorem is true.

Theorem 2.1. *Operator $Q_0^{(k)}$, acting from the space $\overset{o}{W}{}_2^4(R_+; H; \{k\})$ to $L_2(R_+; H)$ in the following way:*

$$Q_0^{(k)} u(t) \equiv Q_0\left(\frac{d}{dt}; A\right)u(t), \quad u(t) \in \overset{o}{W}{}_2^4(R_+; H; \{k\}), \tag{2.2}$$

is an isomorphism between the spaces $\overset{o}{W}{}_2^4(R_+; H; \{k\})$ and $L_2(R_+; H)$.

Proof. It holds that $Q_0^{(k)} u(t) = f(t)$ has a solution $u(t) \in \overset{o}{W}{}_2^4(R_+; H; \{k\})$ for any $f(t) \in L_2(R_+; H)$. In fact, the vector function

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-i\xi E + A)^{-3}(-i\xi E - A)^{-1}\left(\int_0^{+\infty} f(s)e^{-i\xi s}ds\right)e^{it\xi}d\xi, \quad t \in R, \tag{2.3}$$

satisfies the equation

$$\left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 v(t) = f(t) \tag{2.4}$$

in R_+ almost everywhere. Let us prove that $v(t) \in W_2^4(R; H)$ $(R = (-\infty; +\infty))$. As is made clear here, this means that

$$L_2(R; H) = \left\{ f(t) : \|f\|_{L_2(R;H)} = \left(\int_{-\infty}^{+\infty} \|f(t)\|_H^2 dt\right)^{1/2} < +\infty \right\},$$

$$W_2^4(R; H) = \left\{ u(t) : \|u\|_{W_2^4(R;H)} = \left(\int_{-\infty}^{+\infty}\left(\left\|\frac{d^4u(t)}{dt^4}\right\|_H^2 + \left\|A^4u(t)\right\|_H^2\right)dt\right)^{1/2} < +\infty \right\}. \tag{2.5}$$

From the Plancherel theorem, it follows that it is sufficient to show that $A^4\tilde{v}(\xi), \xi^4\tilde{v}(\xi) \in L_2(R;H)$, where $\tilde{v}(\xi)$ is the Fourier transform of the vector function $v(t)$. From the spectral theory of self-adjoint operators,

$$\begin{aligned}\left\|A^4\tilde{v}(\xi)\right\|_{L_2(R;H)} &= \left\|A^4(-i\xi E + A)^{-3}(-i\xi E - A)^{-1}\tilde{f}(\xi)\right\|_{L_2(R;H)} \\ &\le \left\|A^4(-i\xi E + A)^{-3}(-i\xi E - A)^{-1}\right\|_{H\to H}\left\|\tilde{f}(\xi)\right\|_{L_2(R;H)} \\ &\le \sup_{\sigma\in\sigma(A)}\left|\sigma^4(-i\xi + \sigma)^{-3}(-i\xi - \sigma)^{-1}\right|\left\|\tilde{f}(\xi)\right\|_{L_2(R;H)} \\ &= \text{const}\left\|\tilde{f}(\xi)\right\|_{L_2(R;H)} = \text{const}\left\|f\right\|_{L_2(R;H)}.\end{aligned} \tag{2.6}$$

Here $\tilde{f}(\xi)$ is the Fourier transform of the vector function $f(t)$. Analogously, it is possible to prove that $\xi^4\tilde{v}(\xi) \in L_2(R;H)$. Consequently, $v(t) \in W_2^4(R;H)$. Furthermore, let us denote by $u_1(t)$ the narrowing of the vector function $v(t)$ on $[0;+\infty)$. It is clear that $u_1(t) \in W_2^4(R_+;H)$. Now,

$$u(t) = u_1(t) + e^{-tA}\eta_0 + tAe^{-tA}\eta_1 + t^2A^2e^{-tA}\eta_2, \quad t \in R_+, \tag{2.7}$$

where the vectors $\eta_l \in D(A^{7/2-l})$, $l = \overline{0,2}$, and are defined by the condition $u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$. This is why the following system of equations can be obtained relatively to η_l, $l = \overline{0,2}$:

$$\begin{gathered}u_1(0) + \eta_0 = 0, \\ \frac{du_1(0)}{dt} - A\eta_0 + A\eta_1 = 0, \\ \frac{d^2u_1(0)}{dt^2} + A^2\eta_0 - 2A^2\eta_1 + 2A^2\eta_2 = 0.\end{gathered} \tag{2.8}$$

From this, it is possible to obtain the operator equation,

$$M(E)\eta = \zeta, \tag{2.9}$$

where

$$M(E) = \begin{pmatrix} E & 0 & 0 \\ -E & E & 0 \\ E & -2E & 2E \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \zeta = \begin{pmatrix} -u_1(0) \\ -A^{-1}\dfrac{du_1(0)}{dt} \\ -A^{-2}\dfrac{d^2u_1(0)}{dt^2} \end{pmatrix}. \tag{2.10}$$

Because $u_1(t) \in W_2^4(R_+;H)$, then from the theorem on trace [3–5, Chapter 1], it follows that all elements of the vector ζ belong to $D(A^{7/2})$. Continuing this process, it is apparent that the operator matrix $M(E)$ is boundedly invertible in $H^3 = \bigoplus_{p=1}^{3} H$. Therefore, all $\eta_l \in D(A^{7/2})$, $l = \overline{0,2}$. Consequently, $u(t) \in \overset{o}{W}{}_2^4(R_+; H; \{k\})$. In the same way, it can be established that the equation $Q_0^{(k)} u(t) = 0$ has only a trivial solution.

Operator $Q_0^{(k)}$ is bounded, because

$$
\begin{aligned}
\left\| Q_0^{(k)} u \right\|^2_{L_2(R_+;H)} &= \left\| \frac{d^4 u}{dt^4} \right\|^2_{L_2(R_+;H)} + 4 \left\| A \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} + 4 \left\| A^3 \frac{du}{dt} \right\|^2_{L_2(R_+;H)} + \left\| A^4 u \right\|^2_{L_2(R_+;H)} \\
&\quad + 4 \operatorname{Re} \left(\frac{d^4 u}{dt^4}, A \frac{d^3 u}{dt^3} \right)_{L_2(R_+;H)} - 4 \operatorname{Re} \left(\frac{d^4 u}{dt^4}, A^3 \frac{du}{dt} \right)_{L_2(R_+;H)} \\
&\quad - 2 \operatorname{Re} \left(\frac{d^4 u}{dt^4}, A^4 u \right)_{L_2(R_+;H)} - 8 \operatorname{Re} \left(A \frac{d^3 u}{dt^3}, A^3 \frac{du}{dt} \right)_{L_2(R_+;H)} \\
&\quad - 4 \operatorname{Re} \left(A \frac{d^3 u}{dt^3}, A^4 u \right)_{L_2(R_+;H)} + 4 \operatorname{Re} \left(A^3 \frac{du}{dt}, A^4 u \right)_{L_2(R_+;H)} \\
&\le 3 \left\| \frac{d^4 u}{dt^4} \right\|^2_{L_2(R_+;H)} + 6 \left\| A \frac{d^3 u}{dt^3} \right\|^2_{L_2(R_+;H)} + 6 \left\| A^2 \frac{d^2 u}{dt^2} \right\|^2_{L_2(R_+;H)} \\
&\quad + 4 \left\| A^3 \frac{du}{dt} \right\|^2_{L_2(R_+;H)} + \left\| A^4 u \right\|^2_{L_2(R_+;H)} \\
&\le \text{const} \, \|u\|^2_{W_2^4(R_+;H)},
\end{aligned}
\tag{2.11}
$$

because for $u(t) \in \overset{o}{W}{}_2^4(R_+; H; \{k\})$

$$
\begin{gathered}
2 \operatorname{Re} \left(\frac{d^4 u}{dt^4}, A^3 \frac{du}{dt} \right)_{L_2(R_+;H)} = 0, \qquad 2 \operatorname{Re} \left(\frac{d^4 u}{dt^4}, A^4 u \right)_{L_2(R_+;H)} = 2 \left\| A^2 \frac{d^2 u}{dt^2} \right\|^2_{L_2(R_+;H)}, \\
2 \operatorname{Re} \left(A \frac{d^3 u}{dt^3}, A^3 \frac{du}{dt} \right)_{L_2(R_+;H)} = -2 \left\| A^2 \frac{d^2 u}{dt^2} \right\|^2_{L_2(R_+;H)}, \qquad 2 \operatorname{Re} \left(A \frac{d^3 u}{dt^3}, A^4 u \right)_{L_2(R_+;H)} = 0, \\
2 \operatorname{Re} \left(A^3 \frac{du}{dt}, A^4 u \right)_{L_2(R_+;H)} = 0.
\end{gathered}
\tag{2.12}
$$

The theorem on intermediate derivatives [3–5, Chapter 1] can be used to obtain the last inequality, with the inequality

$$\left\| A^j \frac{d^{4-j}u}{dt^{4-j}} \right\|_{L_2(R_+;H)} \leq c_j \|u\|_{W_2^4(R_+;H)}, \quad j = \overline{0,4}, \tag{2.13}$$

assumed. Moreover, the Bunyakovsky-Schwartz and Young inequalities,

$$\begin{aligned} \operatorname{Re}\left(\frac{d^4u}{dt^4}, A\frac{d^3u}{dt^3} \right)_{L_2(R_+;H)} &\leq \left\| \frac{d^4u}{dt^4} \right\|_{L_2(R_+;H)} \left\| A\frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)} \\ &\leq \frac{1}{2}\left\| \frac{d^4u}{dt^4} \right\|^2_{L_2(R_+;H)} + \frac{1}{2}\left\| A\frac{d^3u}{dt^3} \right\|^2_{L_2(R_+;H)}, \end{aligned} \tag{2.14}$$

are used in the expression $\operatorname{Re}(d^4u/dt^4, A(d^3u/dt^3))_{L_2(R_+;H)}$.

As a result, $Q_0^{(k)}$ is bounded and acts mutually and uniquely from the space $\overset{o}{W}{}_2^4(R_+;H;\{k\})$ to the space $L_2(R_+;H)$. Then, taking into account the Banach theorem on the inverse operator, it can be established that the operator $Q_0^{(k)}$ carries out the isomorphism from the space $\overset{o}{W}{}_2^4(R_+;H;\{k\})$ to $L_2(R_+;H)$. Thus, the theorem is proved. □

Denoting by $Q_1^{(k)}$ the operator which acts from $\overset{o}{W}{}_2^4(R_+;H;\{k\})$ to $L_2(R_+;H)$ in the following way:

$$Q_1^{(k)}u(t) \equiv \sum_{s=1}^{3} A_s \frac{d^{4-s}u(t)}{dt^{4-s}}, \quad u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\}), \tag{2.15}$$

the following statement results.

Lemma 2.2. *Let A_sA^{-s}, $s = 1,2,3$, be bounded operators in H. Then the operator $Q_1^{(k)}$ is a bounded operator from $\overset{o}{W}{}_2^4(R_+;H;\{k\})$ to $L_2(R_+;H)$.*

Proof. Because for any vector function $u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$,

$$\left\| Q_1^{(k)}u \right\|_{L_2(R_+;H)} \leq \sum_{s=1}^{3} \left\| A_{4-s}A^{-4+s} \right\|_{H\to H} \left\| A^{4-s}\frac{d^su}{dt^s} \right\|_{L_2(R_+;H)}, \tag{2.16}$$

then, from the theorem on intermediate derivatives [3–5, Chapter 1], and from (2.16), we get

$$\left\| Q_1^{(k)} u \right\|_{L_2(R_+;H)} \leq \text{const} \, \|u\|_{W_2^4(R_+;H)}. \tag{2.17}$$

Thus, the lemma is proved. □

Now certain properties of polynomial operator groups will be investigated, which will have in the future a special role.

Let the following hold:

$$a_s = \frac{1}{256} s^s (4-s)^{4-s}, \quad s = 1,2,3. \tag{2.18}$$

Consider the following polynomial operator groups which depend on the parameter $\alpha \in [0; a_s^{-1})$, $s = 1,2,3$:

$$Q_s(\lambda;\alpha;A) = \left(\lambda^2 E - A^2\right)^4 - \alpha (i\lambda)^{2s} A^{8-2s}, \quad s = 1,2,3 \tag{2.19}$$

The following can then be established.

Lemma 2.3. *Let* $\alpha \in [0; a_s^{-1})$, $s = 1,2,3$. *Then the polynomial operator groups* $Q_s(\lambda;\alpha;A)$, $s = 1,2,3$, *are invertible on the imaginary axis and can be represented as follows:*

$$Q_s(\lambda;\alpha;A) = F_s(\lambda;\alpha;A) F_s(-\lambda;\alpha;A), \quad s = 1,2,3; \tag{2.20}$$

moreover,

$$\begin{aligned} F_s(\lambda;\alpha;A) &= \prod_{n=1}^{4} (\lambda E - \omega_{s,n}(\alpha) A) \\ &\equiv \lambda^4 E + d_{1,s}(\alpha)\lambda^3 A + d_{2,s}(\alpha)\lambda^2 A^2 + d_{3,s}(\alpha)\lambda A^3 + A^4, \end{aligned} \tag{2.21}$$

where $\operatorname{Re}\omega_{s,n}(\alpha) < 0$, $n = 1,2,3,4$, *and the numbers* $d_{1,s}(\alpha)$, $d_{2,s}(\alpha)$, $d_{3,s}(\alpha)$ *satisfy the following systems of equations:*

(1) *for* $k = 1$

$$\begin{aligned} -d_{1,1}^2(\alpha) + 2d_{2,1}(\alpha) + 4 &= 0, \\ d_{2,1}^2(\alpha) - 2d_{1,1}(\alpha) d_{3,1}(\alpha) - 4 &= 0, \\ -d_{3,1}^2(\alpha) + 2d_{2,1}(\alpha) + 4 &= \alpha; \end{aligned} \tag{2.22}$$

(2) *for* $k = 2$

$$
\begin{gathered}
2d_{2,2}(\alpha) - d_{1,2}^2(\alpha) + 4 = 0, \\
d_{2,2}^2(\alpha) - 2d_{1,2}(\alpha)d_{3,2}(\alpha) - 4 = -\alpha, \\
-d_{3,2}^2(\alpha) + 2d_{2,2}(\alpha) + 4 = 0;
\end{gathered}
\tag{2.23}
$$

(3) *for* $k = 3$

$$
\begin{gathered}
-d_{1,3}^2(\alpha) + 2d_{2,3}(\alpha) + 4 = \alpha, \\
d_{2,3}^2(\alpha) - 2d_{1,3}(\alpha)d_{3,3}(\alpha) - 4 = 0, \\
-d_{3,3}^2(\alpha) + 2d_{2,3}(\alpha) + 4 = 0.
\end{gathered}
\tag{2.24}
$$

Proof. Characteristic polynomials of the operator groups $Q_s(\lambda;\alpha;A)$, $s = 1,2,3$, are

$$
Q_s(\lambda;\alpha;\sigma) = \left(\lambda^2 - \sigma^2\right)^4 - \alpha(i\lambda)^{2s}\sigma^{8-2s}, \quad s = 1,2,3,
\tag{2.25}
$$

where $\sigma \in \sigma(A)$. Let $\lambda = i\xi$, $\xi \in R = (-\infty;+\infty)$. Then it is clear that for these characteristic polynomials, the following correlations are true:

$$
\begin{aligned}
Q_s(\lambda;\alpha;\sigma) &= Q_s(i\xi;\alpha;\sigma) \\
&= \sigma^8\left(\frac{\xi^2}{\sigma^2}+1\right)^4\left[1-\alpha\frac{\left(\xi^2/\sigma^2\right)^s}{\left(\xi^2/\sigma^2+1\right)^4}\right] \\
&\geq \sigma^8\left(\frac{\xi^2}{\sigma^2}+1\right)^4\left[1-\alpha\sup_{\xi^2/\sigma^2\geq 0}\frac{\left(\xi^2/\sigma^2\right)^s}{\left(\xi^2/\sigma^2+1\right)^4}\right], \quad s = 1,2,3.
\end{aligned}
\tag{2.26}
$$

Because

$$
\sup_{\xi^2/\sigma^2\geq 0}\frac{\left(\xi^2/\sigma^2\right)^s}{\left(\xi^2/\sigma^2+1\right)^4} = a_s, \quad s = 1,2,3,
\tag{2.27}
$$

then

$$
Q_s(i\xi;\alpha;\sigma) > 0
\tag{2.28}
$$

for $\alpha \in [0;a_s^{-1})$, $s = 1,2,3$. From (2.28), it becomes clear that the polynomials $Q_s(\lambda;\alpha;\sigma)$ do not have roots on the imaginary axis for $\alpha \in [0;a_s^{-1})$, $s = 1,2,3$. Each of the characteristic polynomials $Q_s(\lambda;\alpha;\sigma)$ for $\sigma \in \sigma(A)$ has exactly four roots from the left semiplane. Because these polynomials are homogeneous with respect to the arguments λ and σ, they can be stated in the following form:

$$Q_s(\lambda;\alpha;\sigma) = F_s(\lambda;\alpha;\sigma)F_s(-\lambda;\alpha;\sigma), \quad s = 1,2,3, \tag{2.29}$$

where

$$\begin{aligned} F_s(\lambda;\alpha;\sigma) &= \prod_{n=1}^{4}(\lambda - \omega_{s,n}(\alpha)\sigma) \\ &\equiv \lambda^4 + d_{1,s}(\alpha)\lambda^3\sigma + d_{2,s}(\alpha)\lambda^2\sigma^2 + d_{3,s}(\alpha)\lambda\sigma^3 + \sigma^4, \end{aligned} \tag{2.30}$$

and moreover $\operatorname{Re}\omega_{s,n}(\alpha) < 0$, $n = 1,2,3,4$, and the numbers $d_{1,s}(\alpha), d_{2,s}(\alpha), d_{3,s}(\alpha)$ satisfy the systems of equations shown in Lemma 2.3, which are obtained from (2.29) in the process of comparing the coefficients for the same degrees. Then, from the spectral decomposition of operator A, the proof of the lemma can be obtained from (2.29). Thus, the lemma is proved. □

The next step is to prove the theorem, which will play an important role in future investigations and will show the special importance of the spectral properties of the polynomial operator groups $Q_s(\lambda;\alpha;A)$ and $F_s(\lambda;\alpha;A)$, $s = 1,2,3$.

Theorem 2.4. *Let $\alpha \in [0; a_s^{-1})$. Then for any $u(t) \in W_2^4(R_+;H)$, the following equality is true:*

$$\begin{aligned} &\left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|^2_{L_2(R_+;H)} - \alpha \left\| A^{4-s}\frac{d^s u}{dt^s} \right\|^2_{L_2(R_+;H)} \\ &\quad = \left\| F_s\left(\frac{d}{dt};\alpha;A\right)u \right\|^2_{L_2(R_+;H)} + \left(R_s(\alpha)\varphi,\varphi\right)_{H^4}, \end{aligned} \tag{2.31}$$

where

$$H^4 = \bigoplus_{p=1}^{4} H,$$

$$R_s(\alpha) = \begin{pmatrix} d_{3,s}(\alpha) - 2 & d_{2,s}(\alpha) & d_{1,s}(\alpha) + 2 & 2 \\ d_{2,s}(\alpha) & d_{2,s}(\alpha)d_{3,s}(\alpha) - d_{1,s}(\alpha) - 2 & d_{1,s}(\alpha)d_{3,s}(\alpha) + 2 & d_{3,s}(\alpha) + 2 \\ d_{1,s}(\alpha) + 2 & d_{1,s}(\alpha)d_{3,s}(\alpha) + 2 & d_{1,s}(\alpha)d_{2,s}(\alpha) - d_{3,s}(\alpha) - 2 & d_{2,s}(\alpha) \\ 2 & d_{3,s}(\alpha) + 2 & d_{2,s}(\alpha) & d_{1,s}(\alpha) - 2 \end{pmatrix},$$

$$\varphi = \left(\varphi_l = A^{4-l-1/2}\frac{d^l u(0)}{dt^l}\right)_{l=0}^{3}. \tag{2.32}$$

Proof. First define the space $D^4(R_+;H)$ as the set of infinitely differentiable functions with values in $D(A^4)$, having compact support in R_+. Because the space $D^4(R_+;H)$ is dense in $W_2^4(R_+;H)$ (see [3–5, Chapter 1]), it is sufficient to prove the theorem for the vector functions $u(t) \in D^4(R_+;H)$. Then

$$\begin{aligned}
&\left\| F_s\left(\frac{d}{dt};\alpha;A\right)u \right\|^2_{L_2(R_+;H)} \\
&\quad = \left\| \frac{d^4u}{dt^4} \right\|^2_{L_2(R_+;H)} + d^2_{1,s}(\alpha)\left\| A\frac{d^3u}{dt^3} \right\|^2_{L_2(R_+;H)} + d^2_{2,s}(\alpha)\left\| A^2\frac{d^2u}{dt^2} \right\|^2_{L_2(R_+;H)} \\
&\quad + d^2_{3,s}(\alpha)\left\| A^3\frac{du}{dt} \right\|^2_{L_2(R_+;H)} + \left\| A^4u \right\|^2_{L_2(R_+;H)} + 2d_{1,s}(\alpha)\operatorname{Re}\left(\frac{d^4u}{dt^4}, A\frac{d^3u}{dt^3}\right)_{L_2(R_+;H)} \\
&\quad + 2d_{2,s}(\alpha)\operatorname{Re}\left(\frac{d^4u}{dt^4}, A^2\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} + 2d_{3,s}(\alpha)\operatorname{Re}\left(\frac{d^4u}{dt^4}, A^3\frac{du}{dt}\right)_{L_2(R_+;H)} \\
&\quad + 2\operatorname{Re}\left(\frac{d^4u}{dt^4}, A^4u\right)_{L_2(R_+;H)} + 2d_{1,s}(\alpha)d_{2,s}(\alpha)\operatorname{Re}\left(A\frac{d^3u}{dt^3}, A^2\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} \\
&\quad + 2d_{1,s}(\alpha)d_{3,s}(\alpha)\operatorname{Re}\left(A\frac{d^3u}{dt^3}, A^3\frac{du}{dt}\right)_{L_2(R_+;H)} + 2d_{1,s}(\alpha)\operatorname{Re}\left(A\frac{d^3u}{dt^3}, A^4u\right)_{L_2(R_+;H)} \\
&\quad + 2d_{2,s}(\alpha)d_{3,s}(\alpha)\operatorname{Re}\left(A^2\frac{d^2u}{dt^2}, A^3\frac{du}{dt}\right)_{L_2(R_+;H)} \\
&\quad + 2d_{2,s}(\alpha)\operatorname{Re}\left(A^2\frac{d^2u}{dt^2}, A^4u\right)_{L_2(R_+;H)} + 2d_{3,s}(\alpha)\operatorname{Re}\left(A^3\frac{du}{dt}, A^4u\right)_{L_2(R_+;H)}.
\end{aligned} \tag{2.33}$$

After integration by parts,

$$\begin{aligned}
\left\| F_s\left(\frac{d}{dt};\alpha;A\right)u \right\|^2_{L_2(R_+;H)} &= \left\| \frac{d^4u}{dt^4} \right\|^2_{L_2(R_+;H)} + \left(d^2_{1,s}(\alpha) - 2d_{2,s}(\alpha)\right)\left\| A\frac{d^3u}{dt^3} \right\|^2_{L_2(R_+;H)} \\
&\quad + \left(d^2_{3,s}(\alpha) - 2d_{2,s}(\alpha)\right)\left\| A^3\frac{du}{dt} \right\|^2_{L_2(R_+;H)} + \left\| A^4u \right\|^2_{L_2(R_+;H)} \\
&\quad + \left(2 - 2d_{1,s}(\alpha)d_{3,s}(\alpha) + d^2_{2,s}(\alpha)\right)\left\| A^2\frac{d^2u}{dt^2} \right\|^2_{L_2(R_+;H)}
\end{aligned}$$

$$
\begin{aligned}
&- d_{1,s}(\alpha)\left\|\varphi_3\right\|^2 - 2d_{2,s}(\alpha)\operatorname{Re}\left(\varphi_3,\varphi_2\right)\\
&+ \left(d_{3,s}(\alpha) - d_{1,s}(\alpha)d_{2,s}(\alpha)\right)\left\|\varphi_2\right\|^2\\
&- 2d_{3,s}(\alpha)\operatorname{Re}\left(\varphi_3,\varphi_1\right) - 2\operatorname{Re}\left(\varphi_3,\varphi_0\right)\\
&+ \left(2 - 2d_{1,s}(\alpha)d_{3,s}(\alpha)\right)\operatorname{Re}\left(\varphi_2,\varphi_1\right) - 2d_{1,s}(\alpha)\operatorname{Re}\left(\varphi_2,\varphi_0\right)\\
&+ \left(d_{1,s}(\alpha) - d_{2,s}(\alpha)d_{3,s}(\alpha)\right)\left\|\varphi_1\right\|^2\\
&- 2d_{2,s}(\alpha)\operatorname{Re}\left(\varphi_1,\varphi_0\right) - d_{3,s}(\alpha)\left\|\varphi_0\right\|^2.
\end{aligned}
\tag{2.34}
$$

Calculating $\|(d/dt - A)(d/dt + A)^3 u\|^2_{L_2(R_+;H)}$ analogously to $\|F_s(d/dt;\alpha;A)u\|^2_{L_2(R_+;H)}$,

$$
\begin{aligned}
\left\|\left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u\right\|^2_{L_2(R_+;H)} &= \left\|\frac{d^4u}{dt^4}\right\|^2_{L_2(R_+;H)} + 4\left\|A\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + 6\left\|A^2\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}\\
&\quad + 4\left\|A^3\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + \left\|A^4u\right\|^2_{L_2(R_+;H)} - 2\left\|\varphi_3\right\|^2\\
&\quad - 2\left\|\varphi_2\right\|^2 + 4\operatorname{Re}\left(\varphi_3,\varphi_1\right) + 2\operatorname{Re}\left(\varphi_3,\varphi_0\right)\\
&\quad + 6\operatorname{Re}\left(\varphi_2,\varphi_1\right) + 4\operatorname{Re}\left(\varphi_2,\varphi_0\right) - 2\left\|\varphi_1\right\|^2 - 2\left\|\varphi_0\right\|^2.
\end{aligned}
\tag{2.35}
$$

Substituting (2.35) into (2.34), from Lemma 2.3, (2.31) can be obtained. Thus, the theorem is proved. □

From Theorem 2.4, it follows that:

Corollary 2.5. *If* $u(t) \in \overset{o}{W}{}_2^4(R_+;H)$ *and* $\alpha \in [0; a_s^{-1})$, *then*

$$
\left\|\left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u\right\|^2_{L_2(R_+;H)} - \alpha\left\|A^{4-s}\frac{d^su}{dt^s}\right\|^2_{L_2(R_+;H)} = \left\|F_s\left(\frac{d}{dt};\alpha;A\right)u\right\|^2_{L_2(R_+;H)}. \tag{2.36}
$$

Note that from Theorem 2.1, $\|Q_0^{(k)}u\|_{L_2(R_+;H)}$ is the norm in the space $\overset{o}{W}{}_2^4(R_+;H;\{k\})$, which is equivalent to the initial norm $\|u\|_{W_2^4(R_+;H)}$. Because the operators of the intermediate derivatives

$$A^{4-s}\frac{d^s}{dt^s} : \overset{o}{W}{}_2^4(R_+; H; \{k\}) \longrightarrow L_2(R_+; H), \quad s = 1, 2, 3, \tag{2.37}$$

are continuous [3–5, Chapter 1], then the norms of these operators can be estimated using $\|Q_0^{(k)}u\|_{L_2(R_+;H)}$. It is also easy to demonstrate that the norms $\|u\|_{W_2^4(R_+;H)}$ and $\|(d/dt - A)(d/dt + A)^3 u\|_{L_2(R_+;H)}$ are equivalent in the space $\overset{o}{W}{}_2^4(R_+; H)$.

3. Norms of the Operators of Intermediate Derivatives

The rest of this paper will be related to the calculation of the following numbers:

$$\begin{aligned} m_{0,s} &= \sup_{0 \neq u \in \overset{o}{W}{}_2^4(R_+;H)} \left\| A^{4-s}\frac{d^s u}{dt^s} \right\|_{L_2(R_+;H)} \left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|_{L_2(R_+;H)}^{-1}, \\ m_{k,s} &= \sup_{0 \neq u \in \overset{o}{W}{}_2^4(R_+;H;\{k\})} \left\| A^{4-s}\frac{d^s u}{dt^s} \right\|_{L_2(R_+;H)} \left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|_{L_2(R_+;H)}^{-1}. \end{aligned} \tag{3.1}$$

First, let us calculate $m_{0,s}$.

Lemma 3.1. *It holds that* $m_{0,s} = a_s^{1/2}$, $s = 1, 2, 3$.

Proof. As (2.36) goes to the limit as $\alpha \longrightarrow a_s^{-1}$, it is apparent that for any vector function $u(t) \in \overset{o}{W}{}_2^4(R_+; H)$, the following inequality:

$$\left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|_{L_2(R_+;H)}^2 \geq a_s^{-1} \left\| A^{4-s}\frac{d^s u}{dt^s} \right\|_{L_2(R_+;H)}^2 \tag{3.2}$$

is true. Thus, $m_{0,s} \leq a_s^{1/2}$, $s = 1, 2, 3$. Furthermore, it is necessary to show that here the equalities $m_{0,s} = a_s^{1/2}$, $s = 1, 2, 3$ also hold. This can be done by taking an arbitrary number $\delta > 0$ and showing that there exists a vector function $u_\delta(t) \in \overset{o}{W}{}_2^4(R_+; H)$ such that the following holds functional:

$$\Lambda(u_\delta) \equiv \left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u_\delta \right\|_{L_2(R_+;H)}^2 - \left(a_s^{-1} + \delta\right) \left\| A^{4-s}\frac{d^s u_\delta}{dt^s} \right\|_{L_2(R_+;H)}^2 < 0. \tag{3.3}$$

Let the vector $\varsigma \in D(A^8)$ and $\|\varsigma\| = 1$, $h(t)$ be the numeral function; moreover, $h(t)\varsigma \in W_2^4(R;H)$. Then using the Parseval equality, it is possible to obtain

$$\begin{aligned}
\Lambda(h(t)\varsigma) &= \left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 h(t)\varsigma \right\|^2_{L_2(R;H)} - \left(a_s^{-1} + \delta\right)\left\| A^{4-s}\frac{d^s h(t)}{dt^s}\varsigma \right\|^2_{L_2(R;H)} \\
&= \int_{-\infty}^{+\infty} \left[(Q_0(-i\xi;A)\varsigma, Q_0(-i\xi;A)\varsigma)\left|\tilde{h}(\xi)\right|^2 - \left(a_s^{-1} + \delta\right)\xi^{2s}\left(A^{4-s}\varsigma, A^{4-s}\varsigma\right)\left|\tilde{h}(\xi)\right|^2 \right] d\xi \\
&= \int_{-\infty}^{+\infty} \left(Q_0(-i\xi;A)Q_0(-i\xi;A)\varsigma - \left(a_s^{-1} + \delta\right)\xi^{2s}A^{8-2s}\varsigma, \varsigma \right)\left|\tilde{h}(\xi)\right|^2 d\xi \\
&= \int_{-\infty}^{+\infty} \phi(\xi;\varsigma)\left|\tilde{h}(\xi)\right|^2 d\xi,
\end{aligned} \tag{3.4}$$

where $\phi(\xi;\varsigma) = (Q_s(i\xi;a_s^{-1} + \delta;A)\varsigma,\varsigma)$.

It will next be shown that $\phi(\xi;\varsigma)$ for a given vector ς has negative values in some interval $(\varepsilon_0;\varepsilon_1)$. If μ_0 is an eigenvalue of the operator $A(\mu_0 > 0)$, and if ς is its eigenvector, then it is obvious that

$$\phi(\xi;\varsigma) = \left(Q_s\left(i\xi;a_s^{-1} + \delta;A\right)\varsigma,\varsigma\right) = \left(Q_s\left(i\xi;a_s^{-1} + \delta;\mu_0\right)\varsigma,\varsigma\right), \tag{3.5}$$

and, as can be seen from the properties of the polynomial $Q_s(i\xi;\alpha;\mu_0)$, is negative for $\alpha = a_s^{-1} + \delta$ for sufficiently small $\delta > 0$. If $\mu_0 \in \sigma(A)$ is not the eigenvalue, then μ_0 is close to an eigenvalue, that is, there exists ς_δ such that $\|\varsigma_\delta\| = 1$ and

$$\phi(\xi;\varsigma_\delta) = Q_s\left(i\xi;a_s^{-1} + \delta;\mu_0\right)\varsigma_\delta + o(\delta) \quad \text{as } \delta \longrightarrow 0, \tag{3.6}$$

because in this case, for sufficiently small δ, the smallest value is negative for some ς_δ. Then there exists an interval $(\varepsilon_0;\varepsilon_1)$ such that $\phi(\xi;\varsigma) < \delta$ for $\xi \in (\varepsilon_0;\varepsilon_1)$.

Now consider the four times differentiable function $\tilde{h}(\xi)$, support of which comes from the interval $(\varepsilon_0;\varepsilon_1)$. Then from (3.4) and from the negativity of $\phi(\xi;\varsigma_\delta)$ in the interval $(\varepsilon_0;\varepsilon_1)$, it can be determined that

$$\Lambda(h(t)\varsigma_\delta) = \int_{\varepsilon_0}^{\varepsilon_1} \phi(\xi;\varsigma_\delta)\left|\tilde{h}(\xi)\right|^2 d\xi < 0. \tag{3.7}$$

Consequently, $m_{0,s} = a_s^{1/2}$, $s = 1,2,3$, and the lemma is proved. □

Because $\overset{o}{W}{}_2^4(R_+;H) \subset \overset{o}{W}{}_2^4(R_+;H;\{k\})$, then $m_{k,s} \geq m_{0,s} = a_s^{1/2}$, $s = 1,2,3$. It is necessary to note that, for any vector function $u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$ and $\alpha \in [0;a_s^{-1})$, the equality

$$\begin{aligned} &\left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|^2_{L_2(R_+;H)} - \alpha \left\| A^{4-s} \frac{d^s u}{dt^s} \right\|^2_{L_2(R_+;H)} \\ &\quad = \left\| F_s\left(\frac{d}{dt};\alpha;A\right)u \right\|^2_{L_2(R_+;H)} + \left(R_s(\alpha;k)\tilde{\varphi},\tilde{\varphi}\right)_H \end{aligned} \tag{3.8}$$

is true, where $R_s(\alpha;k) = d_{1,s}(\alpha) - 2$ is obtained from $R_s(\alpha)$ by removing the first three rows and columns, $\tilde{\varphi} = A^{1/2}(d^3u(0)/dt^3)$. The correctness of (3.8) follows directly from Theorem 2.4.

The following statement indicates when the numbers $m_{k,s}$, $s = 1,2,3$, can be equal to $a_s^{1/2}$, $s = 1,2,3$.

Lemma 3.2. *To establish the condition $m_{k,s} = a_s^{1/2}$, it is necessary and sufficient that $R_s(\alpha;k)$ be positive for any $\alpha \in [0;a_s^{-1})$.*

Proof. Necessity will be shown first. Let $m_{k,s} = a_s^{1/2}$. Then, from (3.8), for any vector function $u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$ and $\alpha \in [0;a_s^{-1})$,

$$\left\| F_s\left(\frac{d}{dt};\alpha;A\right)u \right\|^2_{L_2(R_+;H)} + \left(R_s(\alpha;k)\tilde{\varphi},\tilde{\varphi}\right)_H \geq \left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|^2_{L_2(R_+;H)} \left(1 - \alpha m_{k,s}^2\right) > 0. \tag{3.9}$$

Because the polynomial operator group $F_s(\lambda;\alpha;A)$ for $\alpha \in [0;a_s^{-1})$ has the form

$$F_s(\lambda;\alpha;A) = \prod_{n=1}^{4}(\lambda E - \omega_{s,n}(\alpha)A) \tag{3.10}$$

(see Lemma 2.3), where $\operatorname{Re}\omega_{s,n}(\alpha) < 0$, $n = 1,2,3,4$, then the Cauchy problem,

$$F_s\left(\frac{d}{dt};\alpha;A\right)u(t) = 0, \tag{3.11}$$

$$\frac{d^k u(0)}{dt^k} = 0, \quad k = 0,1,2, \tag{3.12}$$

$$\frac{d^3 u(0)}{dt^3} = A^{-1/2}\tilde{\varphi}, \quad \tilde{\varphi} \in H, \tag{3.13}$$

has a unique solution $u_\alpha(t) \in W_2^4(R_+;H)$, which can be presented in the form

$$u_\alpha(t) = e^{\omega_{s,1}(\alpha)tA}\psi_0 + e^{\omega_{s,2}(\alpha)tA}\psi_1 + e^{\omega_{s,3}(\alpha)tA}\psi_2 + e^{\omega_{s,4}(\alpha)tA}\psi_3, \tag{3.14}$$

where $\psi_0, \psi_1, \psi_2, \psi_3 \in D(A^{7/2})$ are uniquely determined from the conditions at zero in (3.12) and (3.13). As a result, writing inequality (3.9) for the vector function $u_\alpha(t)$, for $\alpha \in [0; a_s^{-1})(R_s(\alpha;k)\tilde{\varphi},\tilde{\varphi})_H > 0$. Necessity is thereby proved.

Now sufficiency must be proved. If for any $\alpha \in [0; a_s^{-1})$, $R_s(\alpha;k)$ is positive, then from (3.8), it follows that for all $u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$ and $\alpha \in [0; a_s^{-1})$,

$$\left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|^2_{L_2(R_+;H)} \geq \alpha \left\| A^{4-s}\frac{d^s u}{dt^s} \right\|^2_{L_2(R_+;H)}. \tag{3.15}$$

As this expression goes to the limit as $\alpha \to a_s^{-1}$, it can be observed that $m_{k,s} \leq a_s^{1/2}$, and from this, $m_{k,s} = a_s^{1/2}$. Sufficiency is thereby proved, and thus the lemma is completely proved.

It is interesting that for some s, it may occur that $m_{k,s} > a_s^{1/2}$. □

Lemma 3.3. *It holds that $m_{k,s} > a_s^{1/2}$ if and only if $R_s(\alpha;k) = 0$ has a solution in the interval $(0; a_s^{-1})$; moreover, this root is equal to $m_{k,s}^{-2}$.*

Proof. Let $m_{k,s} > a_s^{1/2}$, then $m_{k,s}^{-2} \in (0; a_s^{-1})$. From (3.8), for $\alpha \in (0; m_{k,s}^{-2})$,

$$\left\| F_s\left(\frac{d}{dt};\alpha;A\right)u \right\|^2_{L_2(R_+;H)} + (R_s(\alpha;k)\tilde{\varphi},\tilde{\varphi})_H \geq \left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 u \right\|^2_{L_2(R_+;H)} \left(1 - \alpha m_{k,s}^2\right) > 0. \tag{3.16}$$

Substituting the solution of (3.11)–(3.13) into the last inequality, the result is that $R_s(\alpha;k)$ is positive for $\alpha \in [0; m_{k,s}^{-2})$. From the definition of $m_{k,s}$, for $\alpha \in (m_{k,s}^{-2}; a_s^{-1})$, there exists a vector function $v_\alpha(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$ such that

$$\left\| \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)^3 v_\alpha \right\|^2_{L_2(R_+;H)} < \alpha \left\| A^{4-s}\frac{d^s v_\alpha}{dt^s} \right\|^2_{L_2(R_+;H)}. \tag{3.17}$$

From the last inequality in (3.8), it is possible to obtain

$$\left\| F_s\left(\frac{d}{dt};\alpha;A\right)v_\alpha \right\|^2_{L_2(R_+;H)} + (R_s(\alpha;k)\tilde{\varphi}_\alpha,\tilde{\varphi}_\alpha)_H < 0, \tag{3.18}$$

where

$$\tilde{\varphi}_\alpha = A^{1/2}\frac{d^3 v_\alpha(0)}{dt^3}. \tag{3.19}$$

Thus, there exists a vector $\tilde{\varphi}_\alpha \in H$ such that for $\alpha \in (m_{k,s}^{-2}; a_s^{-1})$, $(R_s(\alpha;k)\tilde{\varphi}_\alpha,\tilde{\varphi}_\alpha)_H < 0$. Because $R_s(\alpha;k)$ is a continuous function of the argument $\alpha \in [0; a_s^{-1})$, then $R_s(m_{k,s}^{-2};k) = 0$, and this means that $R_s(\alpha;k) = 0$ has a root in the interval $(0; a_s^{-1})$.

Inversely, if $R_s(\alpha; k) = 0$ has a root in the interval $(0; a_s^{-1})$, then this means that for any $\alpha \in [0; a_s^{-1})$, the number $R_s(\alpha; k)$ cannot be positive. This is why, from Lemma 3.2, $m_{k,s} > a_s^{1/2}$. Denoting the root of $R_s(\alpha; k) = 0$ by $\mu_{k,s}$, it can be seen that $m_{k,s}^{-2} \le \mu_{k,s}$, because from the proof of the lemma, it was obtained that for $\alpha \in [0; m_{k,s}^{-2})$, $R_s(\alpha; k)$ is positive. Moreover, because $R_s(m_{k,s}^{-2}; k) = 0$, it can be determined that $m_{k,s}^{-2} = \mu_{k,s}$. The lemma is thereby proved. □

By generalizing the last two lemmas, the following theorem can be derived.

Theorem 3.4. *The following equality is true:*

$$m_{k,s} = \begin{cases} a_s^{1/2} & \text{for } R_s(\gamma; k) \neq 0,\ \gamma \in (0; a_s^{-1}), \\ \mu_{k,s}^{-1/2} & \text{otherwise.} \end{cases} \tag{3.20}$$

Remark 3.5. In the same way, it is possible to determine the results for boundary-value problems of the form (1.1), (1.2) *for k having any three values from the collection* $\{0; 1; 2; 3\}$.

By considering concretely the cases s, the following statement results.

Theorem 3.6. $m_{k,1} = 1/\sqrt{3}$; $m_{k,2} = 1/2\sqrt{3}$; $m_{k,3} = a_3^{1/2}$.

Proof. Taking into account the abovementioned procedure for finding the numbers $m_{k,s}$, it is necessary to solve the systems from the proof of Lemma 2.3 together with the equation $R_s(\alpha; k) = 0$.

In the case $s = 1$, it can be determined that $d_{1,1}(\alpha) = 2 \Rightarrow d_{2,1}(\alpha) = 0 \Rightarrow d_{3,1}(\alpha) = -1 \Rightarrow \alpha = 3 \in (0; a_1^{-1})$. This is why $m_{k,1} = 1/\sqrt{3}$. To find the number $m_{k,2}$, it is necessary to solve the system from Lemma 2.3 for $s = 2$ together with the equation $d_{1,2}(\alpha) - 2 = 0$. In this case, $d_{1,2}(\alpha) = 2$, $d_{2,2}(\alpha) = 0$, $d_{3,2}(\alpha) = \pm 2$, and consequently $\alpha = 12 \in (0; a_2^{-1})$ and $\alpha = -4 \notin (0; a_2^{-1})$. As a result, $m_{k,2} = 1/2\sqrt{3}$. In the case $s = 3$, it is found that $d_{1,3}(\alpha) = 2$. Then, from the corresponding system, it can be obtained that $2d_{2,3}(\alpha) = \alpha$ and $d_{2,3}^4(\alpha) - 8d_{2,3}^2(\alpha) - 32d_{2,3}(\alpha) - 48 = 0$ or $(d_{2,3}(\alpha)+2)(d_{2,3}^3(\alpha) - 2d_{2,3}^2(\alpha) - 4d_{2,3}(\alpha) - 24) = 0$. It is clear that $d_{2,3}(\alpha) = -2 \Rightarrow d_{3,3}(\alpha) = 0 \Rightarrow \alpha = -4 \notin (0; a_3^{-1})$. From the other side, if in the equation $d_{2,3}^3(\alpha) - 2d_{2,3}^2(\alpha) - 4d_{2,3}(\alpha) - 24 = 0$, it is assumed that $2d_{2,3}(\alpha) = \alpha$, then the result is that $\alpha^3 - 4\alpha^2 - 16\alpha - 192 = 0$, which has only one real root, $\alpha = (1/3)(\sqrt[3]{-2944 + 2\sqrt{2113423}} - \sqrt[3]{2944 + 2\sqrt{2113423}}) \notin (0; a_3^{-1})$. Therefore, $m_{k,3} = a_3^{1/2}$, and the theorem is proved. □

4. Solvability Conditions for the Boundary-Value Problem (1.1), (1.2)

The results obtained make it possible to determine sufficient coefficient conditions of regular solvability for the boundary-value problem (1.1), (1.2). In particular, the following main theorem is true.

Theorem 4.1. *Let the operators A_sA^{-s}, $s = 1,2,3$, be bounded in H so that the inequality*

$$m_{k,1}\left\|A_1A^{-1}\right\|_{H\to H} + m_{k,2}\left\|A_2A^{-2}\right\|_{H\to H} + m_{k,3}\left\|A_3A^{-3}\right\|_{H\to H} < 1 \tag{4.1}$$

is satisfied, where the numbers $m_{k,s}$, $s = 1,2,3$, are as defined in Theorem 3.6. Then the boundary-value problem (1.1), (1.2) *is regularly solvable.*

Proof. The boundary-value problem (1.1), (1.2) can be presented in the form of the operator equation $Q_0^{(k)}u(t) + Q_1^{(k)}u(t) = f(t)$, where $f(t) \in L_2(R_+;H)$, $u(t) \in \overset{o}{W}{}_2^4(R_+;H;\{k\})$. From Theorem 2.1, it follows that the operator $Q_0^{(k)}$ has a bounded inverse operator $Q_0^{(k)^{-1}}$ which acts from the space $L_2(R_+;H)$ into the space $\overset{o}{W}{}_2^4(R_+;H;\{k\})$. Then, after substitution of $u(t) = Q_0^{(k)^{-1}}v(t)$, where $v(t) \in L_2(R_+;H)$, the equation $(E + Q_1^{(k)}Q_0^{(k)^{-1}})v(t) = f(t)$ results. Now it must be shown that whenever the conditions of the theorem are met, the norm of the operator $Q_1^{(k)}Q_0^{(k)^{-1}}$ is less than one. Assuming Theorem 3.6, the following can be obtained:

$$\begin{aligned}
&\left\|Q_1^{(k)}Q_0^{(k)^{-1}}v\right\|_{L_2(R_+;H)} \\
&= \left\|Q_1^{(k)}u\right\|_{L_2(R_+;H)} \\
&\le \left\|A_1\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)} + \left\|A_2\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} + \left\|A_3\frac{du}{dt}\right\|_{L_2(R_+;H)} \\
&\le \left\|A_1A^{-1}\right\|_{H\to H}\left\|A\frac{d^3u}{dt^3}\right\|_{L_2(R_+;H)} + \left\|A_2A^{-2}\right\|_{H\to H}\left\|A^2\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} \\
&\quad + \left\|A_3A^{-3}\right\|_{H\to H}\left\|A^3\frac{du}{dt}\right\|_{L_2(R_+;H)} \\
&\le \left(m_{k,1}\left\|A_1A^{-1}\right\|_{H\to H} + m_{k,2}\left\|A_2A^{-2}\right\|_{H\to H} + m_{k,3}\left\|A_3A^{-3}\right\|_{H\to H}\right)\left\|Q_0^{(k)}u\right\|_{L_2(R_+;H)} \\
&= \left(m_{k,1}\left\|A_1A^{-1}\right\|_{H\to H} + m_{k,2}\left\|A_2A^{-2}\right\|_{H\to H} + m_{k,3}\left\|A_3A^{-3}\right\|_{H\to H}\right)\|v\|_{L_2(R_+;H)}.
\end{aligned} \tag{4.2}$$

As a result:

$$\begin{aligned}
\left\|Q_1^{(k)}Q_0^{(k)^{-1}}\right\|_{L_2(R_+;H)\to L_2(R_+;H)} &\le m_{k,1}\left\|A_1A^{-1}\right\|_{H\to H} + m_{k,2}\left\|A_2A^{-2}\right\|_{H\to H} \\
&\quad + m_{k,3}\left\|A_3A^{-3}\right\|_{H\to H} < 1.
\end{aligned} \tag{4.3}$$

Then, in this case, the operator $E + Q_1^{(k)}Q_0^{(k)^{-1}}$ has an inverse in the space $L_2(R_+;H)$, and it is possible to determine $u(t)$ from the following formula:

$$u(t) = Q_0^{(k)^{-1}}\left(E + Q_1^{(k)}Q_0^{(k)^{-1}}\right)^{-1}f(t). \tag{4.4}$$

Moreover,

$$\begin{aligned}\|u\|_{W_2^4(R_+;H)} &\leq \left\|Q_0^{(k)^{-1}}\right\|_{L_2(R_+;H)\to W_2^4(R_+;H)} \left\|\left(E + Q_1^{(k)} Q_0^{(k)^{-1}}\right)^{-1}\right\|_{L_2(R_+;H)\to L_2(R_+;H)} \|f\|_{L_2(R_+;H)} \\ &\leq \text{const}\, \|f\|_{L_2(R_+;H)}.\end{aligned} \tag{4.5}$$

Thus, the theorem is proved. □

Remark 4.2. The conditions of regular solvability obtained here for the boundary-value problem (1.1), (1.2) are not improvable in terms of the operator coefficients of (1.1).

Following is an example in which the conditions of Theorem 4.1 are verified. Consider the following problem on the semi-axis $R_+ \times [0;\pi]$:

$$\begin{gathered}\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^3 u(t,x) + p_1(x)\frac{\partial^5 u(t,x)}{\partial x^2 \partial t^3} + p_2(x)\frac{\partial^6 u(t,x)}{\partial x^4 \partial t^2} + p_3(x)\frac{\partial^7 u(t,x)}{\partial x^6 \partial t} = f(t,x) \\ \frac{\partial^k u(0,x)}{\partial t^k} = 0, \quad k = 0,1,2, \quad \frac{\partial^{2i} u(t,0)}{\partial x^{2i}} = \frac{\partial^{2i} u(t,\pi)}{\partial x^{2i}} = 0, \quad i = 0,1,2,3, \frac{a}{b}\end{gathered} \tag{4.6}$$

where $p_s(x),\ s = 1,2,3$, are bounded on segment $[0;\pi]$ functions, $f(t,x) \in L_2(R_+; L_2[0;\pi])$, which is a partial case of problem (1.1), (1.2). On the condition that

$$m_{k,1} \sup_{0\leq x\leq \pi} |p_1(x)| + m_{k,2} \sup_{0\leq x\leq \pi} |p_2(x)| + m_{k,3} \sup_{0\leq x\leq \pi} |p_3(x)| < 1, \tag{4.7}$$

the given problem has a unique solution in the space $W_{t,x,2}^{4,8}(R_+; L_2[0;\pi])$.

Acknowledgment

The authors have dedicated this paper, in gratitude for useful consultation in their work, as a sign of deep respect to the memory of the Academician of the Azerbaijan National Academy of Sciences, Professor M. G. Gasymov.

References

[1] I. I. Vorovich and V. E. Kovalchuk, "On the basis properties of one system of homogeneous solutions," *Prikladnaya Matematika i Mekhanika [Journal of Applied Mathematics and Mechanics]*, vol. 31, no. 5, pp. 861–869, 1967.

[2] Y. A. Ustinov and V. I. Yudovich, "On the completeness of the system of elementary solutions of biharmonic equation on a semi-strip," *Prikladnaya Matematika i Mekhanika [Journal of Applied Mathematics and Mechanics]*, vol. 37, no. 4, pp. 706–714, 1973.

[3] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Dunod, Paris, France, 1968.

[4] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Mir, Moscow, Russia, 1971.

[5] J. L. Lions and E. Magenes, *Non-HomogeneousBoundary Value Problems and Applications*, Springer, Berlin, Germany, 1972.
[6] S. G. Krein, *Linear Differential Equations in a Banach Space*, Nauka, Moscow, Russia, 1967.
[7] S. Y. Yakubov, *Linear Differential-Operator Equations and Their Applications*, Elm, Baku, Azerbaijan, 1985.
[8] S. Agmon and L. Nirenberg, "Properties of solutions of ordinary differential equations in Banach space," *Communications on Pure and Applied Mathematics*, vol. 16, pp. 121–239, 1963.
[9] M. G. Gasymov and S. S. Mirzoev, "Solvability of boundary value problems for second-order operator-differential equations of elliptic type," *Differentsial'nye Uravneniya [Differential Equations]*, vol. 28, no. 4, pp. 651–661, 1992.
[10] A. G. Kostyuchenko and A. A. Shkalikov, "Self-adjoint quadratic operator pencils and elliptic problems," *Funktsional'nyi Analiz i Ego Prilozheniya [Functional Analysis and Its Applications]*, vol. 17, no. 2, pp. 38–61, 1983.
[11] M. G. Gasymov, "The multiple completeness of part of the eigen- and associated vectors of polynomial operator bundles," *Izvestija Akademii Nauk Armjanskoĭ SSR. Serija Matematika*, vol. 6, no. 2-3, pp. 131–147, 1971.
[12] M. G. Gasymov, "On the theory of polynomial operator pencils," *Doklady Akademii Nauk SSSR [Soviet Mathematics. Doklady]*, vol. 199, no. 4, pp. 747–750, 1971.
[13] Y. A. Dubinskii, "On some differential-operator equations of arbitrary order," *Matematicheskii Sbornik [Mathematics of the USSR-Sbornik]*, vol. 90(132), no. 1, pp. 3–22, 1973.
[14] S. S. Mirzoev, "Conditions for the well-defined solvability of boundary-value problems for operator differential equations," *Doklady Akademii Nauk SSSR [Soviet Mathematics. Doklady]*, vol. 273, no. 2, pp. 292–295, 1983.
[15] V. B. Shakhmurov, "Coercive boundary value problems for strongly degenerate operator-differential equations," *Doklady Akademii Nauk SSSR*, vol. 290, no. 3, pp. 553–556, 1986.
[16] A. A. Shkalikov, "Elliptic equations in a Hilbert space and related spectral problems," *Trudy Seminara Imeni I. G. Petrovskogo*, no. 14, pp. 140–224, 1989.
[17] A. R. Aliev, "Boundary-value problems for a class of operator differential equations of high order with variable coefficients," *Matematicheskie Zametki [Mathematical Notes]*, vol. 74, no. 6, pp. 803–814, 2003.
[18] A. R. Aliev, "On the boundary value problem for a class of operator-differential equations of odd order with variable coefficients," *Doklady Akademii Nauk [Doklady Mathematics]*, vol. 421, no. 2, pp. 151–153, 2008.
[19] R. P. Agarwal, M. Bohner, and V. B. Shakhmurov, "Maximal regular boundary value problems in Banach-valued weighted space," *Boundary Value Problems*, vol. 2005, no. 1, pp. 9–42, 2005.
[20] A. Favini and Y. Yakubov, "Higher order ordinary differential-operator equations on the whole axis in UMD Banach spaces," *Differential and Integral Equations*, vol. 21, no. 5-6, pp. 497–512, 2008.
[21] N. P. Kuptsov, "Exact constants in inequalities between norms of functions and of their derivatives," *Matematicheskie Zametki [Mathematical Notes]*, vol. 41, no. 3, pp. 313–319, 1987.
[22] G. A. Kalyabin, "Some problems for Sobolev spaces on the half-line," *Trudy Matematicheskogo Instituta Imeni V. A. Steklova [Proceedings of the Steklov Institute of Mathematics]*, vol. 255, pp. 161–169, 2006.
[23] G. A. Grinberg, "On the method proposed by P.F. Papkovich for solving the plane problem of elasticity theory for the rectangular domain and the problems of bending of a rectangular thin slab with two fixed selvages, and on some of its generalizations," *Prikladnaya Matematika i Mekhanika [Applied Mathematics and Mechanics]*, vol. 37, no. 2, pp. 221–228, 1973.
[24] R. Z. Gumbataliyev, "On generalized solutions of one class of operator-differential equations of the fourth order," *Izvestiya Akademii Nauk Azerbajdzhana. Seriya Fiziko-Tekhnicheskikh i Matematicheskikh Nauk*, vol. 18, no. 2, pp. 18–21, 1998.
[25] G. A. Teters, *Complex Loading and Stability of the Covers from Polymeric Materials*, Zinatne Press, Riga, Latvia, 1969.