Research Article

Multiplicity Results Using Bifurcation Techniques for a Class of Fourth-Order m-Point Boundary Value Problems

Yansheng Liu[1] and Donal O'Regan[2]

[1] *Department of Mathematics, Shandong Normal University, Jinan 250014, China*
[2] *Department of Mathematics, National University of Ireland, Galway, Ireland*

Correspondence should be addressed to Yansheng Liu, yanshliu@gmail.com

By using bifurcation techniques, this paper investigates the existence of nodal solutions for a class of fourth-order m-point boundary value problems. Our results improve those in the literature.

1. Introduction

Consider the following fourth order m-point boundary value problem (BVP, for short)

$$u^{(4)}(t) = f(u(t), u''(t)), \quad t \in (0,1)$$

$$u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$$

$$u'''(0) = 0, \qquad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i), \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given sign-changing continuous function, $m \geq 3$, $\eta_i \in (0,1)$, and $\alpha_i > 0$ for $i = 1, \ldots, m-2$ with

$$\sum_{i=1}^{m-2} \alpha_i < 1. \tag{1.2}$$

Multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. The existence of solutions of the second order multi-point boundary value problems has been studied by many authors and the methods used are the nonlinear alternative of Leray-Schauder, coincidence degree theory, fixed point theorems in cones and global bifurcation techniques (see [1–9], and the references therein). In [5], Ma investigated the existence and multiplicity of nodal solutions for

$$u''(t) + f(u(t)) = 0, \quad t \in (0,1);$$

$$u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{1.3}$$

when

$$\eta_i \in \mathbb{Q}(i = 1, 2, \ldots, m-2) \quad \text{with } 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \tag{1.4}$$

and $\alpha_i > 0$ for $i = 1, \ldots, m-2$ satisfying (1.2). He obtained some results on the spectrum of the linear operator corresponding to (1.1). It should be pointed out that the main tool used in [5] is results on bifurcation coming from the trivial solutions and we note no use was made of global results on bifurcation from infinity.

Recently [10] Wei and Pang studied the existence and multiplicity of nontrivial solutions for the fourth order m-point boundary value problems:

$$u^{(4)}(t) = f\left(u(t), u''(t)\right), \quad t \in (0,1)$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{1.5}$$

$$u''(0) = 0, \qquad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i),$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given sign-changing continuous function, $m \geq 3$, $\eta_i \in (0,1)$, and $\alpha_i > 0$ for $i = 1, \ldots, m-2$ satisfies (1.2).

Motivated by [5, 10], in this paper we consider the existence and multiplicity of nodal solutions for BVP (1.1). The method used here is Rabinowitz's global bifurcation theorem. To the best of our best knowledge, only [10] seems to have considered the existence of nontrivial or positive solutions of the nonlinear multi-point boundary value problems for fourth order differential equations. As in [5, 10] we suppose (1.2) is satisfied throughout.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to the existence of multiple solutions for BVP (1.1). To conclude this section we give some notation and state three lemmas, which will be used in Section 3. Following the notation of Rabinowitz, let E be a real Banach space and $L : E \to E$ be a compact linear map. If there exists $\mu \in \mathbb{R} = [0, +\infty)$ and $0 \neq v \in E$ such that $v = \mu L v$, μ is said to be a real characteristic

value of L. The set of real characteristic values of L will be denoted by $\sigma(L)$. The multiplicity of $\mu \in \sigma(L)$ is

$$\dim \bigcup_{j=1}^{\infty} N\left((I - \mu L)^j\right), \tag{1.6}$$

where $N(A)$ denotes the null space of A. Suppose that $H : \mathbb{R} \times E \to E$ is compact and $H(\lambda, u) = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals. Then

$$u = \lambda L u + H(\lambda, u) \tag{1.7}$$

possesses the line of trivial solutions $\Theta = \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$. It is well known that if $\mu \in \mathbb{R}$, a necessary condition for $(\mu, 0)$ to be a bifurcation point of (1.7) with respect to Θ is that $\mu \in \sigma(L)$. If μ is a simple characteristic value of L, let v denote the eigenvector of L corresponding to μ normalized so $\|v\| = 1$. By Σ we denote the closure of the set of nontrivial solutions of (1.7). A component of Σ is a maximal closed connected subset. It was shown in (Rabinowitz [11, Theorems 1.3, 1.25, 1.27]), the following.

Lemma 1.1. *If $\mu \in \sigma(L)$ is simple, then Σ contains a component C_μ which can be decomposed into two subcontinua C_μ^+, C_μ^- such that for some neighborhood B of $(\mu, 0)$,*

$$(\lambda, u) \in C_\mu^+\left(C_\mu^-\right) \cap B, \quad (\lambda, u) \neq (\mu, 0) \tag{1.8}$$

implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1)$, $\|w\| = o(|\alpha|)$ at $\alpha = 0$.

   Moreover, each of C_μ^+, C_μ^- either
   (i) meets infinity in Σ, or
   (ii) meets $(\widehat{\mu}, 0)$ where $\mu \neq \widehat{\mu} \in \sigma(L)$, or
   (iii) contains a pair of points (λ, u), $(\lambda, -u)$, $u \neq 0$.

   The following are global results for (1.7) on bifurcation from infinity (see, Rabinowitz [9, Theorem 1.6 and Corollary 1.8]).

Lemma 1.2. *Suppose L is compact and linear, $H(\lambda, u)$ is continuous on $\mathbb{R} \times E$, $H(\lambda, u) = o(\|u\|)$ at $u = \infty$ uniformly on bounded λ intervals, and $\|u\|^2 H(\lambda, u/\|u\|^2)$ is compact. If $\mu \in \sigma(L)$ is of odd multiplicity, then Σ possesses an unbounded component D_μ which meets (μ, ∞). Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap \sigma(L) = \{\mu\}$ and $\wp$ is a neighborhood of (μ, ∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*
   (i) $D_\mu \setminus \wp$ is bounded in $\mathbb{R} \times E$ in which case $D_\mu \setminus \wp$ meets $\Theta = \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ or
   (ii) $D_\mu \setminus \wp$ is unbounded.
   If (ii) occurs and $D_\mu \setminus \wp$ has a bounded projection on $\mathbb{R}$, then $D_\mu \setminus \wp$ meets $(\widehat{\mu}, \infty)$ where $\mu \neq \widehat{\mu} \in \sigma(L)$.

Lemma 1.3. *Suppose the assumptions of Lemma 1.2 hold. If $\mu \in \sigma(L)$ is simple, then D_μ can be decomposed into two subcontinua D_μ^+, D_μ^- and there exists a neighborhood $\mathfrak{I} \subset \wp$ of (μ, ∞) such that $(\lambda, u) \in D_\mu^+(D_\mu^-) \cap \mathfrak{I}$ and $(\lambda, u) \neq (\mu, \infty)$ implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1)$, $\|w\| = o(|\alpha|)$ at $|\alpha| = \infty$.*

2. Preliminaries

Let $X = C[0,1]$ with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$, $Y = \{u \in C^1[0,1] : u'(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$ with the norm $\|u\|_1 = \max\{\|u\|, \|u'\|\}$, $Z = \{u \in C^2[0,1] : u'(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$ with the norm $\|u\|_2 = \max\{\|u\|, \|u'\|, \|u''\|\}$. Then X, Y, and Z are Banach spaces.

For any C^1 function u, if $u(t_0) = 0$, then t_0 is a simple zero of u if $u'(t_0) \neq 0$. For any integer $k \in \mathbb{N}$ and any $v \in \{\pm\}$, as in [6], define sets $T_k^v \subset Z$ consisting of the set of functions $u \in Z$ satisfying the following conditions:

  (i) $u'(0) = 0$, $vu(0) > 0$ and $u'(1) \neq 0$;

  (ii) u' has only simple zeros in $(0,1)$, and has exactly $k-1$ such zeros;

  (iii) u has a zero strictly between each two consecutive zeros of u'.

Note $T_k^- = -T_k^+$ and let $T_k = T_k^- \cup T_k^+$. It is easy to see that the sets T_k^- and T_k^+ are disjoint and open in Z. Moreover, if $u \in T_k^v$, then u has at least $k-2$ zeros in $(0,1)$, and at most $k-1$ zeros in $(0,1]$.

Let $E = \mathbb{R} \times Y$ under the product topology. As in [12], we add the points $\{(\lambda, \infty) : \lambda \in \mathbb{R}\}$ to the space E. Let $\Phi_k^+ = \mathbb{R} \times T_k^+$, $\Phi_k^- = \mathbb{R} \times T_k^-$, and $\Phi_k = \mathbb{R} \times T_k$.

We first convert BVP (1.1) into another form. Suppose $u(t)$ is a solution of BVP (1.1). Let $v(t) = -u''(t)$. Notice that

$$u''(t) + v(t) = 0, \quad t \in I;$$

$$u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \tag{2.1}$$

Thus $u(t)$ can be written as

$$u(t) = Lv(t), \tag{2.2}$$

where the operator L is defined by

$$Lv(t) := \int_0^1 H(t,s)v(s)ds, \quad \forall v \in Y, \tag{2.3}$$

where

$$H(t,s) = G(t,s) + \frac{\sum_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i},$$

$$G(t,s) = \begin{cases} 1-t, & 0 \leq s \leq t \leq 1; \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4}$$

Therefore we obtain the following equivalent form of (1.1)

$$v''(t) + f((Lv)(t), -v(t)) = 0, \quad t \in (0,1);$$

$$v'(0) = 0, \qquad v(1) = \sum_{i=1}^{m-2} \alpha_i v(\eta_i). \tag{2.5}$$

For the rest of this paper we always suppose that the initial value problem

$$v''(t) + f((Lv)(t), -v(t)) = 0, \quad t \in (0,1);$$

$$v(t_0) = v'(t_0) = 0 \tag{2.6}$$

has the unique trivial solution $v \equiv 0$ on $[0,1]$ for any $t_0 \in [0,1]$; in fact some suitable conditions such as a Lipschitz assumption or $f \in C^1$ guarantee this.

Define two operators on Y by

$$(Av)(t) := (LFv)(t), \qquad (Fv)(t) := f((Lv)(t), -v(t)), \quad t \in I, \ v \in Y. \tag{2.7}$$

Then it is easy to see the following lemma holds.

Lemma 2.1. *The linear operator L and operator A are both completely continuous from Y to Y and*

$$\|Lv\|_1 \leq M\|v\| \leq M\|v\|_1, \quad \forall v \in Y, \tag{2.8}$$

where $M = \max\{1, (1/8)(1 + \sum_{i=1}^{m-2} \alpha_i / (1 - \sum_{i=1}^{m-2} \alpha_i \eta_i))\}$.

Moreover, $u \in C^4[0,1]$ *is a solution of BVP* (1.1) *if and only if* $v = -u''$ *is a solution of the operator equation* $v = Av$.

Let the function $\Gamma(s)$ be defined by

$$\Gamma(s) = \cos s - \sum_{i=1}^{m-2} \alpha_i \cos \eta_i s, \quad s \in \mathbb{R}. \tag{2.9}$$

Then we have the following lemma.

Lemma 2.2. (i) *For each* $k \geq 1$, $\Gamma(s)$ *has exactly one zero* $s_k \in I_k := ((k-1)\pi, k\pi)$, *so*

$$s_1 < s_2 < \cdots < s_k \longrightarrow +\infty \quad (k \longrightarrow +\infty); \tag{2.10}$$

(ii) *the characteristic value of L is exactly given by* $\mu_k = s_k^2$, $k = 1, 2, \ldots$, *and the eigenfunction* ϕ_k *corresponding to* μ_k *is* $\phi_k(t) = \cos s_k t$;

(iii) *the algebraic multiplicity of each characteristic value* μ_k *of L is 1*;

(iv) $\phi_k \in T_k^+$ *for* $k = 1, 2, 3, \ldots$, *and* ϕ_1 *is strictly positive on* $(0,1)$.

Proof. From [5] and by a similar analysis as in the proof of [6, Lemma 3.3] we obtain (i) and (ii).

Now we assert (iii) holds. Suppose, on the contrary, there exists $y \in Y$ such that $(I - \mu_k L)y = \mu_k^{-1}\phi_k$. Then $y \in Z$ and

$$-y'' - s_k^2 y = \cos s_k t. \tag{2.11}$$

From $y'(0) = 0$ we know the general solution of this differential equation is

$$y = C \cos s_k t - \frac{1}{2s_k} t \sin s_k t. \tag{2.12}$$

From (i) and (ii) of this lemma, $C \cos s_k t$ satisfies the boundary condition. Thus

$$\cos s_k = \sum_{i=1}^{m-2} \alpha_i \cos \eta_i s_k, \qquad \sin s_k = \sum_{i=1}^{m-2} \alpha_i \eta_i \sin \eta_i s_k. \tag{2.13}$$

Then, by (1.2),

$$
\begin{aligned}
1 &= \left(\sum_{i=1}^{m-2} \alpha_i \cos \eta_i s_k \right)^2 + \left(\sum_{i=1}^{m-2} \alpha_i \eta_i \sin \eta_i s_k \right)^2 \\
&\leq \sum_{i,j=1}^{m-2} \alpha_i \alpha_j \left(|\cos \eta_i s_k \cos \eta_j s_k| + |\sin \eta_i s_k \sin \eta_j s_k| \right) \\
&\leq \left(\sum_{i=1}^{m-2} \alpha_i \right)^2 < 1,
\end{aligned}
\tag{2.14}
$$

a contradiction. Thus the algebraic multiplicity of each characteristic value μ_k of L is 1.

Finally, from $s_k \in ((k-1)\pi, k\pi)$ and $s_1 \in (0, \pi/2)$, it is easy to see that (iv) holds. $\qquad\square$

Lemma 2.3. *For $d = (d_1, d_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\}$, define a linear operator*

$$L_d v(t) = \left(d_1 L^2 + d_2 L \right) v(t), \quad \forall t \in I, \ v \in Y, \tag{2.15}$$

where L is defined as in (2.3). *Then the generalized eigenvalues of L_d are simple and are given by*

$$0 < \lambda_1(L_d) < \lambda_2(L_d) < \cdots < \lambda_k(L_d) \longrightarrow +\infty \quad (k \longrightarrow +\infty), \tag{2.16}$$

where

$$\lambda_k(L_d) = \frac{\mu_k^2}{d_1 + d_2 \mu_k}. \tag{2.17}$$

The generalized eigenfunction corresponding to $\lambda_k(L_d)$ is

$$\phi_k(t) = \cos s_k t, \tag{2.18}$$

where μ_k, s_k, ϕ_k are as in Lemma 2.2.

Proof. Suppose there exist λ and $v \neq 0$ such that $v = \lambda L_d v$. Set $u(t) = Lv(t)$. Then from (2.2)–(2.7) and (2.15) it is easy to see that $u \neq 0$ and

$$u^{(4)}(t) = \lambda\left(d_1 u(t) - d_2 u''(t)\right), \quad t \in (0,1);$$

$$u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i); \tag{2.19}$$

$$u'''(0) = 0, \qquad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i).$$

Denote $L^{-1}u = -u''$ for $u \in Z$. Then there exist two complex numbers r_1 and r_2 such that

$$u^{(4)}(t) - \lambda\left(d_1 u(t) - d_2 u''(t)\right) = \left(L^{-1} - r_2 I\right)\left(L^{-1} - r_1 I\right)u(t) = 0. \tag{2.20}$$

Now if there exists some $r_i (i = 1, 2)$ such that

$$\left(L^{-1} - r_i I\right)u(t) = 0, \tag{2.21}$$

then by Lemma 2.2 we know $r_i = s_k^2 = \mu_k$ for some $k \in \mathbb{N}$, and consequently

$$u(t) = \cos s_k t \tag{2.22}$$

is a nontrivial solution. Substituting (2.22) into (2.19), we have

$$\lambda = \frac{\mu_k^2}{d_1 + d_2 \mu_k}. \tag{2.23}$$

On the other hand, suppose, for example,

$$\left(L^{-1} - r_1 I\right)u(t) \neq 0, \qquad \left(L^{-1} - r_2 I\right)\left(L^{-1} - r_1 I\right)u(t) = 0. \tag{2.24}$$

Let $w(t) := (L^{-1} - r_1 I)u(t)$. Then $(L^{-1} - r_2 I)w(t) = 0$. Reasoning as previously mentioned, we have $r_2 = s_k^2$ for some $k \in \mathbb{N}$, and consequently $w(t) = a \cos s_k t (a \neq 0)$ is a nontrivial solution. Therefore,

$$\left(L^{-1} - r_1 I\right)u(t) = a \cos s_k t. \tag{2.25}$$

If $r_1 = s_k^2$, then the general solution of the differential equation (2.25), satisfying $u'(0) = 0$, is

$$u(t) = C \cos s_k t - \frac{a}{2s_k} t \sin s_k t, \tag{2.26}$$

which is similar to (2.12). Reasoning as in the proof of **Lemma 2.2** we can get a contradiction. Thus $r_1 \neq s_k^2$ and the general solution of (2.25), satisfying $u'(0) = 0$, is

$$u(t) = \tilde{u}(t) + \frac{a \cos s_k t}{s_k^2 - r_1}, \tag{2.27}$$

where $\tilde{u}(t)$ is the general solution of homogeneous differential equation corresponding to (2.25)

$$\left(L^{-1} - r_1 I\right)u(t) = 0. \tag{2.28}$$

Notice the term $a \cos s_k t / (s_k^2 - r_1)$ in (2.27) satisfies the boundary condition of (1.1) at $t = 1$, so $\tilde{u}(t)$ also satisfies

$$\tilde{u}'(0) = 0, \qquad \tilde{u}(1) = \sum_{i=1}^{m-2} \alpha_i \tilde{u}(\eta_i). \tag{2.29}$$

Therefore, by **Lemma 2.2** we know $\tilde{u}(t) = C \cos s_j t$ for some $j \in \mathbb{N}$, and consequently

$$r_1 = s_j^2 \neq s_k^2, \qquad u(t) = C \cos s_j t + \frac{a \cos s_k t}{s_k^2 - s_j^2}. \tag{2.30}$$

By substituting this into (2.19), we have

$$a\lambda(d_1 + d_2 \mu_k) = a\mu_k^2, \qquad C\lambda(d_1 + d_2 \mu_j) = C\mu_j^2. \tag{2.31}$$

Since $\mu_j \neq \mu_k$, if there exists some λ such that (2.31) holds, then

$$\frac{d_1 + d_2 \mu_k}{d_1 + d_2 \mu_j} = \frac{\mu_k^2}{\mu_j^2}, \tag{2.32}$$

which implies

$$d_1 d_2 \neq 0, \qquad d_1 \left(\frac{1}{\mu_k} + \frac{1}{\mu_j} \right) = -d_2, \tag{2.33}$$

a contradiction with $d_1 > 0$ and $d_2 > 0$.

Consequently, (2.24) does not hold. This together with (2.20)–(2.23) and Lemma 2.2 guarantee that the generalized eigenvalues of L_d are given by

$$0 < \lambda_1(L_d) < \lambda_2(L_d) < \cdots < \lambda_k(L_d) \longrightarrow +\infty \quad (k \longrightarrow +\infty), \tag{2.34}$$

where $\lambda_k(L_d) = \mu_k^2/(d_1 + d_2\mu_k)$. The generalized eigenfunction corresponding to $\lambda_k(L_d)$ is $\phi_k(t) = \cos s_k t$.

Now we are in a position to show the generalized eigenvalues of L_d are simple.

Clearly, from above we know for $\lambda_k := \lambda_k(L_d)$, $(I - \lambda_k L_d)\phi_k = 0$ and $\dim N(I - \lambda_k L_d) = 1$. Suppose there exists an $v \in C^2$ such that

$$(I - \lambda_k L_d)v = \frac{1}{\mu_k} \phi_k(t). \tag{2.35}$$

This together with (2.3) and (2.15) guarantee that $v \in Y$. If we let $u(t) = (Lv)(t)$ as above, then we have

$$u^{(4)}(t) - \lambda_k \left(d_1 u(t) - d_2 u''(t) \right) = \cos s_k t, \quad t \in (0,1), \tag{2.36}$$

$$u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i); \qquad u'''(0) = 0, \qquad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i). \tag{2.37}$$

Consider the following homogeneous equation corresponding to (2.36):

$$u^{(4)}(t) - \frac{\mu_k^2}{d_1 + d_2\mu_k} \left(d_1 u(t) - d_2 u''(t) \right) = 0. \tag{2.38}$$

The characteristic equation associated with (2.38) is

$$\lambda^4 - \frac{\mu_k^2}{d_1 + d_2\mu_k} \left(d_1 - d_2\lambda^2 \right) = 0. \tag{2.39}$$

Then there exists a real number η such that

$$\left(\lambda^2 + \mu_k \right) \left(\lambda^2 - \eta \right) = \lambda^4 - \frac{\mu_k^2}{d_1 + d_2\mu_k} \left(d_1 - d_2\lambda^2 \right) = 0. \tag{2.40}$$

Notice that $-\eta\mu_k = -d_1\mu_k^2/(d_1 + d_2\mu_k) < 0$ if $d_1 > 0$. So $\eta > 0$ if $d_1 > 0$, and $\eta = 0$ if $d_1 = 0$.

First we consider the case $d_1 > 0$. In this case the general solution of (2.38) is

$$c_1 e^{\sqrt{\eta}t} + c_2 e^{-\sqrt{\eta}t} + c_3 \cos s_k t + c_4 \sin s_k t. \tag{2.41}$$

After computation we obtain that the general solution of (2.36) is

$$u(t) = c_1 e^{\sqrt{\eta}t} + c_2 e^{-\sqrt{\eta}t} + c_3 \cos s_k t + c_4 \sin s_k t + at \sin s_k t, \tag{2.42}$$

where $a = -(d_1 + d_2\mu_k)/2s_k(2d_1\mu_k + d_2\mu_k^2)$. From boundary condition $u'(0) = u'''(0) = 0$ in (2.37) it follows that

$$\begin{aligned} \sqrt{\eta}(c_1 - c_2)s_k c_4 &= 0; \\ \eta\sqrt{\eta}(c_1 - c_2) - s_k^3 c_4 &= 0. \end{aligned} \tag{2.43}$$

By $\eta > 0$ and $\mu_k > 0$, we know $c_1 - c_2 = 0$ and $c_4 = 0$. Then (2.42) can be rewritten as

$$u(t) = c_1\left[e^{\sqrt{\eta}t} + e^{-\sqrt{\eta}t}\right] + c_3 \cos s_k t + at \sin s_k t. \tag{2.44}$$

Notice that the term $c_3 \cos s_k t$ satisfies (2.37). From the boundary condition

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \qquad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i),$$

$$(t \sin s_k t)'' = 2s_k \cos s_k t - s_k^2 t \sin s_k t \tag{2.45}$$

we have

$$c_1\left[e^{\sqrt{\eta}} + e^{-\sqrt{\eta}}\right] + a \sin s_k = \sum_{i=1}^{m-2} \alpha_i\left[c_1\left(e^{\sqrt{\eta}\eta_i} + e^{-\sqrt{\eta}\eta_i}\right) + a\eta_i \sin s_k \eta_i\right], \tag{2.46}$$

$$c_1\eta\left[e^{\sqrt{\eta}} + e^{-\sqrt{\eta}}\right] - as_k^2 \sin s_k = \sum_{i=1}^{m-2} \alpha_i\left[c_1\eta\left(e^{\sqrt{\eta}\eta_i} + e^{-\sqrt{\eta}\eta_i}\right) - a\eta_i s_k^2 \sin s_k \eta_i\right]. \tag{2.47}$$

Multiply (2.46) by s_k^2 and then add to (2.47) to obtain

$$c_1\left(\eta + s_k^2\right)\left[e^{\sqrt{\eta}} + e^{-\sqrt{\eta}}\right] = c_1\left(\eta + s_k^2\right)\sum_{i=1}^{m-2} \alpha_i\left(e^{\sqrt{\eta}\eta_i} + e^{-\sqrt{\eta}\eta_i}\right). \tag{2.48}$$

On the other hand, from (1.2) it can be seen that

$$e^{\sqrt{\eta}} + e^{-\sqrt{\eta}} > \sum_{i=1}^{m-2} \alpha_i\left(e^{\sqrt{\eta}\eta_i} + e^{-\sqrt{\eta}\eta_i}\right). \tag{2.49}$$

This together with (2.48) guarantee that $c_1 = 0$. Therefore, (2.42) reduces to

$$u(t) = c_3 \cos s_k t + at \sin s_k t. \tag{2.50}$$

Similar to (2.12), a contradiction can be derived.

Next consider the case $d_1 = 0$. Then $\eta = 0$ from above. In this case the general solution of (2.38) is

$$c_1 + c_2 t + c_3 \cos s_k t + c_4 \sin s_k t. \tag{2.51}$$

By a similar process, one can easily get a contradiction.

To sum up, the generalized eigenvalues of L_d are simple, and the proof of this lemma is complete. □

3. Main Results

We now list the following hypotheses for convenience.

(H1) There exists $a = (a_1, a_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\}$ such that

$$f(x,y) = a_1 x - a_2 y + o(|(x,y)|), \quad \text{as } |(x,y)| \longrightarrow 0, \tag{3.1}$$

where $(x,y) \in \mathbb{R} \times \mathbb{R}$, and $|(x,y)| := \max\{|x|, |y|\}$.

(H2) There exists $b = (b_1, b_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\}$ such that

$$f(x,y) = b_1 x - b_2 y + o(|(x,y)|), \quad \text{as } |(x,y)| \longrightarrow \infty. \tag{3.2}$$

(H3) There exists $R > 0$ such that

$$|f(x,y)| < \frac{R}{M}, \quad \text{for } (x,y) \in \{(x,y) : |x| \leq MR, |y| \leq R\}, \tag{3.3}$$

where M is defined as in Lemma 2.1.

(H4) There exist two constants $r_1 < 0 < r_2$ such that $f(x, -r_1) \geq 0$ and $f(x, -r_2) \leq 0$ for $x \in [-Mr, Mr]$, and $f(x, -y)$ satisfies a Lipschitz condition in y for $(x,y) \in [-Mr, Mr] \times [r_1, r_2]$, where $r = \max\{|r_1|, r_2\}$.

Now we are ready to give our main results.

Theorem 3.1. *Suppose (H1)-(H2) hold. Suppose there exists two integers $i_0 \geq 0$ and $k > 0$ such that either*

$$\frac{\mu_{i_0+k}^2}{a_1 + a_2 \mu_{i_0+k}} < 1 < \frac{\mu_{i_0+1}^2}{b_1 + b_2 \mu_{i_0+1}} \tag{3.4}$$

or

$$\frac{\mu_{i_0+k}^2}{b_1 + b_2\mu_{i_0+k}} < 1 < \frac{\mu_{i_0+1}^2}{a_1 + a_2\mu_{i_0+1}} \tag{3.5}$$

holds. Then BVP (1.1) *has at least* $2k$ *nontrivial solutions.*

Theorem 3.2. *Suppose (H1) and (H2) hold and one of (H3) and (H4) hold. Suppose there exists two integers i_0 and j_0 such that*

$$\frac{\mu_{i_0}^2}{a_1 + a_2\mu_{i_0}} < 1, \qquad \frac{\mu_{j_0}^2}{b_1 + b_2\mu_{j_0}} < 1. \tag{3.6}$$

Then BVP(1.1) has at least $2(i_0 + j_0)$ solutions.

To set it up we first consider global results for the equation

$$v = \lambda Av, \tag{3.1$_\lambda$}$$

on Y, where $\lambda \in \mathbb{R}$, and the operator A is defined as in (2.7). Under the condition (H1), (3.1$_\lambda$) can be rewritten as

$$v = \lambda L_a v + H_a(\lambda, v), \tag{3.7}$$

here $H_a(\lambda, v) = \lambda Av - \lambda L_a v$, L_a is defined as in (2.12) (replacing d with a). Obviously, by (H1) and Lemma 2.1–2.3, it can be seen that $H_a(\lambda, v)$ is $o(\|v\|_1)$ for v near 0 uniformly on bounded λ intervals and L_a is a compact linear map on Y. A solution of (3.1$_\lambda$) is a pair $(\lambda, v) \in E$. By (H1), the known curve of solutions $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ will henceforth be referred to as the trivial solutions. The closure of the set on nontrivial solutions of (3.1$_\lambda$) will be denoted by Σ as in Lemma 1.1.

If $H_a(\lambda, v) \equiv 0$, then (3.7) becomes a linear system

$$v = \lambda L_a v. \tag{3.8}$$

By Lemma 2.3, (3.8) possesses an increasing sequence of simple eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots, \quad \text{with} \quad \lambda_k = \frac{\mu_k^2}{a_1 + a_2\mu_k} \quad \text{as } k \longrightarrow +\infty. \tag{3.9}$$

Any eigenfunction $\phi_k = \cos s_k t$ corresponding to λ_k is in T_k^+.

A similar analysis as in [6, Proposition 4.1] yields the following results.

Lemma 3.3. *Suppose that (λ, v) is a solution of (3.1$_\lambda$) and $v \not\equiv 0$. Then $v \in \cup_{i=1}^\infty T_i$.*

Lemma 3.4. *Assume that (H1) holds and λ_k is defined by (3.9). Then for each integer $k > 0$ and each $\nu = +$, or $-$, there exists a continua C_k^ν of solutions of (3.1_λ) in $\Phi_k^\nu \cup \{(\lambda_k, 0)\}$, which meets $\{(\lambda_k, 0)\}$ and ∞ in Σ.*

Proof. Consider (3.7) as a bifurcation problem from the trivial solution. From **Lemma 1.1** and condition (H1) it follows that for each positive integer $k \in \mathbb{N}$, Σ contains a component $C_k \subseteq E = \mathbb{R} \times Y$ which can be decomposed into two subcontinua C_k^+, C_k^- such that for some neighborhood B of $(\lambda_k, 0)$,

$$(\lambda, v) \in C_k^+ (C_k^-) \cap B, \qquad (\lambda, v) \neq (\lambda_k, 0) \tag{3.10}$$

imply $(\lambda, v) = (\lambda, \alpha \phi_k + w)$, where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \lambda_k| = o(1)$, $\|w\|_1 = o(|\alpha|)$ at $\alpha = 0$.

By (3.7) and the continuity of the operator $A : Y \to Z$, the set C_k^ν lies in $\mathbb{R} \times Z$ and the injection $C_k^\nu \to \mathbb{R} \times Z$ is continuous. Thus, C_k^ν is also a continuum in $\mathbb{R} \times Z$, and the above properties hold in $\mathbb{R} \times Z$.

Since T_k is open in Z and $\phi_k \in T_k^+$, we know

$$\frac{v}{\alpha} = \phi_k + \frac{w}{\alpha} \in T_k^+ \tag{3.11}$$

for $0 \neq \alpha$ sufficiently small. Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have

$$(\lambda, v) \in \Phi_k, \qquad \left(\frac{C_k}{\{(\lambda_k, 0)\}} \right) \cap B_\varepsilon \subset \Phi_k, \tag{3.12}$$

where B_ε is an open ball in $\mathbb{R} \times Z$ of radius ε centered at $(\lambda_k, 0)$. It follows from the proof of [6, Proposition 4.1] that

$$(\lambda, v) \in C_k \cap (\mathbb{R} \times \partial T_k) \Longrightarrow u = 0, \tag{3.13}$$

which means $C_k \setminus \{(\lambda_k, 0)\} \cap \partial \Phi_k = \emptyset$. Consequently, C_k lies in $\Phi_k \cup \{(\lambda_k, 0)\}$.

Similarly we have that C_k^ν lies in $\Phi_k^\nu \cup \{(\lambda_k, 0)\}$ ($\nu = +$ or $-$).

Next we show alternative (ii) of **Lemma 1.1** is impossible. If not, without loss of generality, assume that C_k^+ meets $(\lambda_i, 0)$ with $\lambda_k \neq \lambda_i \in \sigma(L_a)$. Then there exists a sequence $(\xi_m, z_m) \in C_k^+$ with $\xi_m \to \lambda_i$ and $z_m \to 0$ as $m \to +\infty$. Notice that

$$z_m = \xi_m L_a z_m + H(\xi_m, z_m). \tag{3.14}$$

Dividing this equation by $\|z_m\|_1$ and using **Lemma 2.1** and $H(\xi_m, z_m) = o(\|z_m\|_1)$ as $m \to +\infty$, we may assume without loss of generality that $z_m / \|z_m\|_1 \to \bar{z}$ as $m \to +\infty$. Thus from (3.14) it follows that

$$\bar{z} = \lambda_i L_a \bar{z}. \tag{3.15}$$

Since $\bar{z} \neq 0$, by Lemmas 2.2 and 2.3, $\bar{z}$ belongs to T_i^+ or T_i^-. By (3.1_λ) and the continuity of the operator $A : Y \to Z$, from $\|z_m - \bar{z}\|_1 \to 0$ it follows that $\|z_m - \bar{z}\|_2 \to 0$. Notice that T_i^+ and

T_i^- are open in Z. Therefore, $z_m \in T_i^+$ or T_i^- for m sufficiently large, which is a contradiction with $z_m \in T_k^+$ ($m \geq 1$), $i \neq k$. Hence alternative (ii) of Lemma 1.1 is not possible.

Finally it remains to show alternative (iii) of Lemma 1.1 is impossible. In fact, notice that $T_k^- = -T_k^+$, and $T_k^- \cap T_k^+ = \emptyset$. If $u \in T_k^+$, then $-u \in T_k^-$. This guarantees that C_k^ν does not contain a pair of points (λ, v), $(\lambda, -v)$, $v \neq 0$.

Therefore alternative (i) of Lemma 1.1 holds. This implies that for each integer $k \in \mathbb{N}$ and each $\nu = +$, or $-$, there exists a continua C_k^ν of solutions of (3.1_λ) in $\Phi_k^\nu \cup \{(\lambda_k, 0)\}$, which meets $\{(\lambda_k, 0)\}$ and ∞ in Σ. □

Under the condition (H2), (3.1_λ) can be rewritten as

$$v = \lambda L_b v + H_b(\lambda, v), \tag{3.16}$$

here $H_b(\lambda, v) = \lambda A v - \lambda L_b v$, L_b is defined as in (2.12) (replacing d with b).

Let $h(x, y) := f(x, y) - b_1 x + b_2 y$. Then from (H2) it follows that $\lim_{|(x,y)| \to \infty} h(x, y)/|(x, y)| = 0$. Define a function

$$\hat{h}(r) := \max\{|h(x, y)| : |(x, y)| \leq r\}. \tag{3.17}$$

Then $\hat{h}(r)$ is nondecreasing and

$$\lim_{r \to \infty} \frac{\hat{h}(r)}{r} = 0. \tag{3.18}$$

Obviously, by (3.18) and Lemma 2.1, it can be seen that $H_b(\lambda, v)$ is $o(\|v\|_1)$ for v near ∞ uniformly on bounded λ intervals and L_b is a compact linear map on Y.

Similar to (3.8), by Lemma 2.3, L_b possesses an increasing sequence of simple eigenvalues

$$0 < \bar{\lambda}_1 < \bar{\lambda}_2 < \cdots < \bar{\lambda}_k < \cdots, \quad \text{with} \quad \bar{\lambda}_k = \frac{\mu_k^2}{b_1 + b_2 \mu_k} \quad \text{as } k \longrightarrow +\infty. \tag{3.19}$$

Note $\phi_k = \cos s_k t$ is an eigenfunction corresponding to $\bar{\lambda}_k$. Obviously, it is in T_k^+.

Lemma 3.5. *Assume that (H1)-(H2) holds. Then for each integer $k > 0$ and each $\nu = +$, or $-$, there exists a continua D_k^ν of Σ in $\Phi_k^\nu \cup \{(\bar{\lambda}_k, \infty)\}$ coming from $\{(\bar{\lambda}_k, \infty)\}$, which meets $(\bar{\lambda}_k, 0)$ or has an unbounded projection on $\mathbb{R}$.*

Proof. From (2.7), (3.16), and (3.18) it follows that $H_b(\lambda, v)$ is continuous on E, $H_b(\lambda, v) = o(\|v\|_1)$ at $v = \infty$ uniformly on bounded λ intervals. Moreover, as in the proof of [12, Theorem 2.4], one can see that $\|v\|_1^2 H_b(\lambda, v/\|v\|_1^2)$ is compact. From Lemma 2.3 we know $\bar{\lambda}_k$ is a simple characteristic value of L_b for each integer $k \in \mathbb{N}$. Thus by Lemmas 1.2 and 1.3, Σ contains a component D_k which can be decomposed into two subcontinua D_k^+, D_k^- which meet $\{(\bar{\lambda}_k, \infty)\}$.

Now we show that for a smaller neighborhood $\mathfrak{I} \subset \wp$ of $(\bar{\lambda}_k, \infty)$, $(\lambda, v) \in D_k^+(D_k^-) \cap \mathfrak{I}$ and $(\lambda, v) \neq (\bar{\lambda}_k, \infty)$ imply that $v \in T_k^+$ (T_k^-). In fact, by Lemma 1.3 we already know that there

exists a neighborhood $\Im \subset \wp$ of $(\overline{\lambda}_k, \infty)$ satisfying $(\lambda, v) \in D_k^+(D_k^-) \cap \Im$ and $(\lambda, v) \neq (\overline{\lambda}_k, \infty)$ imply $(\lambda, v) = (\lambda, \alpha v_k + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \overline{\lambda}_k| = o(1)$, $\|w\|_1 = o(|\alpha|)$ at $|\alpha| = \infty$.

As in the proof of Lemma 3.4, D_k^ν is also a continuum in $\mathbb{R} \times Z$, and the above properties hold in $\mathbb{R} \times Z$. Since T_k^ν is open in Z and w/α is smaller compared to $\phi_k \in T_k^+$ near $\alpha = +\infty$, $\phi_k + w/\alpha$ and therefore $v = \alpha \phi_k + w \in T_k^+$ for α near $+\infty$. Here and in the following the same argument works if $+$ is replaced by $-$.

Therefore, $D_k^+ \cap \Im \subset (\mathbb{R} \times T_k^+) \cup (\overline{\lambda}_k, \infty)$. Now we have two cases to consider, that is, $D_k^+ \setminus \Im$ is bounded or unbounded. First suppose $D_k^+ \setminus \Im$ is bounded. Then there exists $(\lambda, v) \in \partial D_k^+$ with $v \in \partial T_k^+$. If $v \neq 0$, by Lemma 3.3 we know $v \in T_j^\nu$ for some positive integer $j \neq k$ and $\nu \in \{+, -\}$. As in the proof of Lemma 3.4, we get a contradiction, which means $v = 0$. Thus there exists a sequence $(\xi_m, z_m) \in D_k^+$ with $z_m \to v \equiv 0$ as $m \to +\infty$. This together with (H1) guarantee that (ξ_m, z_m) satisfies (3.14).

As in the proof of Lemma 3.4, we may assume without loss of generality that $z_m / \|z_m\|_1 \to \overline{z}$ and $\xi_m \to \xi$ as $m \to +\infty$. Then we have

$$\overline{z} = \xi L_a \overline{z}. \tag{3.20}$$

Since $\overline{z} \neq 0$, $\xi \neq 0$ is an eigenvalue of operator L_a. From this, (3.9), and (3.19) it follows that $\xi = \lambda_j$ for some positive integer j. Then by Lemma 2.3, $\overline{z}$ belongs to T_j^+ or T_j^-. Notice that $\|z_m - \overline{z}\|_1 \to 0$ and so $\|z_m - \overline{z}\|_2 \to 0$ as in the proof of Lemma 3.4. Thus $z_m \in T_j^+$ or T_j^- for m sufficiently large since T_j^+ and T_j^- are open. This together with $z_m \in T_k^+$ $(m \geq 1)$ guarantee that $k = j$. This means D_k^+ meets $(\lambda_k, 0)$ if $D_k^+ \setminus \Im$ is bounded.

Next suppose $D_k^+ \setminus \Im$ is unbounded. In this case we show $D_k^+ \setminus \Im$ has an unbounded projection on $\mathbb{R}$. If not, then there exists a sequence $(\zeta_m, y_m) \in D_k^+ \setminus \Im$ with $\zeta_m \to \overline{\zeta}$ and $\|y_m\|_1 \to +\infty$ as $m \to +\infty$. Let $x_m := y_m / \|y_m\|_1$, $m \geq 1$. From the fact that

$$y_m = \zeta_m L_b y_m + H_b(\zeta_m, y_m) \tag{3.21}$$

it follows that

$$x_m = \zeta_m L_b x_m + \frac{H_b(\zeta_m, y_m)}{\|y_m\|_1}. \tag{3.22}$$

Notice that $L_b : Y \to Y$ is completely continuous. We may assume that there exists $w \in Y$ with $\|w\|_1 = 1$ such that $\|x_m - w\|_1 \to 0$ as $m \to +\infty$.

Letting $m \to +\infty$ in (3.22) and noticing $H_b(\zeta_m, y_m)/\|y_m\|_1 \to 0$ as $m \to +\infty$ one obtains

$$w = \overline{\zeta} L_b w. \tag{3.23}$$

Since $w \neq 0$, $\overline{\zeta} \neq 0$ is an eigenvalue of operator L_b, that is, $\overline{\zeta} = \overline{\lambda}_{k_0}$ for some positive integer $k_0 \neq k$. Then by Lemma 2.3 w belongs to $T_{k_0}^+$ or $T_{k_0}^-$. Notice the fact that $\|x_m - w\|_1 \to 0$ and so $\|x_m - w\|_2 \to 0$ as in the proof of Lemma 3.4. Thus $x_m \in T_{k_0}^+$ or $T_{k_0}^-$ for m sufficiently large since $T_{k_0}^+$ and $T_{k_0}^-$ are open. This is a contradiction with $x_m \in T_k^+$ $(m \geq 1)$. Thus $D_k^+ \setminus \Im$ has an unbounded projection on $\mathbb{R}$. $\qquad\square$

Proof of Theorem 3.1. Suppose first that

$$\frac{\mu_{i_0+k}^2}{a_1 + a_2\mu_{i_0+k}} < 1 < \frac{\mu_{i_0+1}^2}{b_1 + b_2\mu_{i_0+1}}. \tag{3.24}$$

Using the notation of (3.9) and (3.19), this means $\lambda_{i_0+k} < 1 < \overline{\lambda}_{i_0+1}$ and so from Lemma 2.3 we know

$$\lambda_{i_0+1} < \lambda_{i_0+2} < \cdots < \lambda_{i_0+k} < 1 < \overline{\lambda}_{i_0+1} < \overline{\lambda}_{i_0+2} < \cdots < \overline{\lambda}_{i_0+k}. \tag{3.25}$$

Consider (3.7) as a bifurcation problem from the trivial solution. We need only show that

$$C_{i_0+j}^{\nu} \bigcap (\{1\} \times Y) \neq \emptyset, \quad j = 1, 2, \ldots, k; \ \nu = +, -. \tag{3.26}$$

Suppose, on the contrary and without loss of generality, that

$$C_{i_0+i}^{+} \bigcap (\{1\} \times Y) = \emptyset, \quad \text{for some } i, \ 1 \leq i \leq k. \tag{3.27}$$

By Lemma 3.4 we know that $C_{i_0+i}^{+}$ joins $(\lambda_{i_0+i}, 0)$ to infinity in Σ and $(\lambda, v) = (0, 0)$ is the unique solution of (3.1_λ) (in which $\lambda = 0$) in E. This together with $\lambda_{i_0+i} < 1$ guarantee that there exists a sequence $\{(\zeta_m, y_m)\} \subset C_{i_0+i}^{+}$ such that $\zeta_m \in (0, 1)$ and $\|y_m\|_1 \to \infty$ as $m \to +\infty$. We may assume that $\zeta_m \to \overline{\zeta} \in [0, 1]$ as $m \to +\infty$. Let $x_m := y_m / \|y_m\|_1$, $m \geq 1$, then (3.22) holds. Similarly, we may assume that there exists $w \in Y$ with $\|w\|_1 = 1$ such that $\|x_m - w\|_1 \to 0$ as $m \to +\infty$ and (3.23) holds. From the proof of Lemma 3.5 one can see $\overline{\zeta} = \overline{\lambda}_{i_0+i}$, which contradicts $\overline{\lambda}_{i_0+i} > 1$. Thus (3.27) is not true, which means (3.26) holds.

Next suppose that

$$\frac{\mu_{i_0+k}^2}{b_1 + b_2\mu_{i_0+k}} < 1 < \frac{\mu_{i_0+1}^2}{a_1 + a_2\mu_{i_0+1}}. \tag{3.28}$$

This means

$$\overline{\lambda}_{i_0+1} < \overline{\lambda}_{i_0+2} < \cdots < \overline{\lambda}_{i_0+k} < 1 < \lambda_{i_0+1} < \lambda_{i_0+2} < \cdots < \lambda_{i_0+k}. \tag{3.29}$$

Consider (3.16) as a bifurcation problem from infinity. As above we need only to prove that

$$D_{i_0+j}^{\nu} \bigcap (\{1\} \times Y) \neq \emptyset, \quad j = 1, 2, \ldots, k; \ \nu = +, -. \tag{3.30}$$

From Lemma 3.5, we know that $D_{i_0+j}^{\nu}$ comes from $\{(\overline{\lambda}_{i_0+j}, \infty)\}$, meets $(\lambda_{i_0+j}, 0)$ or has an unbounded projection on $\mathbb{R}$. If it meets $(\lambda_{i_0+j}, 0)$, then the connectedness of $D_{i_0+j}^{\nu}$ and $\lambda_{i_0+j} > 1$

guarantees that (3.30) is satisfied. On the other hand, if $D^\nu_{i_0+j}$ has an unbounded projection on $\mathbb{R}$, notice that $(\lambda, v) = (0, 0)$ is the unique solution of (3.1_λ) (in which $\lambda = 0$) in E, so (3.30) also holds. □

Proof of Theorem 3.2. First suppose that (H3) holds. Then there exists $\varepsilon > 0$ such that

$$(1 + \varepsilon)|f(x, y)| < \frac{R}{M}, \quad \text{for } (x, y) \in \{(x, y) : |x| \leq MR, \ |y| \leq R\}. \tag{3.31}$$

Let (λ, v) be a solution of (3.1_λ) such that $0 \leq \lambda < 1 + \varepsilon$ and $\|v\|_1 \leq R$. Then by (2.7), (3.1_λ), (3.31) and Lemma 2.1 it is easy to see

$$\|v\|_1 = \lambda\|Av\|_1 = \lambda\|LFv\|_1 \leq \lambda M\|Fv\| = M\max_{t \in [0,1]} |\lambda f((Lv)(t), -v(t))| < M\frac{R}{M} = R. \tag{3.32}$$

Therefore,

$$\Sigma \cap \left([0, 1 + \varepsilon] \times \partial\overline{B}_R\right) = \emptyset. \tag{3.33}$$

This together with (3.32) and Lemmas 3.4 and 3.5 implies that

$$C^\nu_k \cap \left([0, 1 + \varepsilon] \times \overline{B}_R\right) \subset [0, 1 + \varepsilon] \times B_R, \quad k = 1, 2, \ldots, i_0; \tag{3.34}$$

$$D^\nu_j \cap \left([0, 1 + \varepsilon] \times \partial\overline{B}_R\right) = \emptyset, \quad j = 1, 2, \ldots, j_0; \tag{3.35}$$

where $B_R = \{v \in Y \mid \|v\|_1 < R\}$ and $\overline{B}_R = \{v \in Y \mid \|v\|_1 \leq R\}$, C^ν_k and D^ν_j are obtained from Lemmas 3.4 and 3.5, respectively.

Since C^ν_k is a unbounded component of solutions of (3.1_λ) joining $(\lambda_k, 0)$ in E, it follows from (3.33) and (3.34) that C^ν_k crosses the hyperplane $\{1\} \times Y$ with $(1, v^\nu)$ such that $\|v^\nu\|_1 < R$ ($\nu = +$ or $-$, $k = 1, 2, \ldots, i_0$). This means BVP (2.5) has $2i_0$ nontrivial solutions $\{v^\nu_i\}^{i_0}_1$ in which v^+_1 and v^-_1 are positive and negative solutions, respectively.

On the other hand, by (3.33), (3.35), and Lemma 3.5 one can obtain

$$D^\nu_j \cap \left(\{1\} \times \left(Y \setminus \overline{B}_R\right)\right) \neq \emptyset, \quad j = 1, 2, \ldots, j_0. \tag{3.36}$$

This means BVP (2.5) has $2j_0$ nontrivial solutions $\{w^\nu_i\}^{j_0}_1$ in which w^+_1 and w^-_1 are positive and negative solutions, respectively.

Now it remains to show this theorem holds when the condition (H4) is satisfied.

From the above we need only to prove that

(i) for $(\lambda, v) \in C^\nu_i$ ($\nu = +$ or $-$, $i = 1, 2, \ldots, i_0$),

$$r_1 < v(t) < r_2, \quad t \in [0, 1]. \tag{3.37}$$

(ii) for $(\lambda, v) \in D_j^\nu$ ($\nu = +$ or $-$, $j = 1, 2, \ldots, j_0$), we have that either

$$\max_{t \in [0,1]} v(t) > r_2, \quad t \in [0,1] \tag{3.38}$$

or

$$\min_{t \in [0,1]} v(t) < r_1, \quad t \in [0,1]. \tag{3.39}$$

In fact, like in [13], suppose on the contrary that there exists $(\lambda, v) \in C_i^\nu \bigcup D_j^\nu$ such that either

$$\max\{v(t) : t \in [0,1]\} = r_2 \tag{3.40}$$

or

$$\min\{v(t) : t \in [0,1]\} = r_1 \tag{3.41}$$

for some i, j.

First consider the case $\max\{v(t) : t \in [0,1]\} = r_2$. Then there exists $\bar{t} \in [0,1]$ such that $v(\bar{t}) = r_2$. Let

$$\begin{aligned}
\tau_0 &=: \inf\left\{t \in \left[0, \bar{t}\right] : v(s) \geq \quad \text{for } s \in \left[t, \bar{t}\right]\right\}, \\
\tau_1 &=: \sup\left\{t \in \left[\bar{t}, 1\right] : v(s) \geq 0 \quad \text{for } s \in \left[\bar{t}, t\right]\right\}.
\end{aligned} \tag{3.42}$$

Then

$$\max\{v(t) : t \in [\tau_0, \tau_1]\} = r_2, \tag{3.43}$$

$$0 \leq v(t) \leq r_2, \quad t \in [\tau_0, \tau_1]. \tag{3.44}$$

Therefore, $v(t)$ is a solution of the following equation

$$-v''(t) = \lambda f((Lv)(t), -v(t)), \quad t \in (\tau_0, \tau_1) \tag{3.45}$$

with $v(\tau_0) = v(\tau_1) = 0$ if $0 < \tau_0 < \tau_1 < 1$ and $v'(\tau_0) = 0$ if $\tau_0 = 0$.

By (H4), there exists $\overline{M} \geq 0$ such that $f(x, -y) + \overline{M}y$ is strictly increasing in y for $(x, y) \in [r_1, r_2] \times [-Mr, Mr]$, where $r = \max\{|r_1|, r_2\}$. Then

$$-v'' + \lambda \overline{M}v = \lambda\left(f((Lv)(t), -v(t)) + \overline{M}v\right), \quad t \in (\tau_0, \tau_1). \tag{3.46}$$

Using (H4) and Lemma 2.1 again, we can obtain

$$
\begin{aligned}
&- (r_2 - v(t))'' + \lambda \overline{M}(r_2 - v(t)) \\
&\quad = -\lambda \Big[f((Lv)(t), -v(t)) + \overline{M}v(t) - \overline{M}r_2 \Big] \\
&\quad = -\lambda \Big[f((Lv)(t), -v(t)) + \overline{M}v(t) - \Big(f((Lv)(t), -r_2) + \overline{M}r_2 \Big) \Big] - \lambda f((Lv)(t), -r_2) \\
&\quad \geq 0, \quad t \in (\tau_0, \tau_1)
\end{aligned}
\tag{3.47}
$$

and if $\tau_1 = 1$, by (1.2) we know $v(1) < r_2$. Therefore,

$$
\begin{aligned}
r_2 - v(\tau_0) > 0, \qquad r_2 - v(\tau_1) > 0 \quad &\text{if } 0 < \tau_0 < \tau_1 < 1; \\
(r_2 - v(\tau_0))' = 0 \quad &\text{if } \tau_0 = 0; \\
r_2 - v(\tau_1) > 0 \quad &\text{if } \tau_1 = 1.
\end{aligned}
\tag{3.48}
$$

This together with (3.47) and the maximum principle [14] imply that $r_2 - v(t) > 0$ in $[\tau_0, \tau_1]$, which contradicts (3.43).

The proof in the case $\min\{v(t) : t \in [0,1]\} = r_1$ is similar, so we omit it. $\square$

Acknowledgments

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