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Research Article

Jost Solution and the Spectrum of the Discrete Dirac Systems

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We find polynomial-type Jost solution of the self-adjoint discrete Dirac systems. Then we investigate analytical properties and asymptotic behaviour of the Jost solution. Using the Weyl compact perturbation theorem, we prove that discrete Dirac system has the continuous spectrum filling the segment [-2,2]. We also study the eigenvalues of the Dirac system. In particular, we prove that the Dirac system has a finite number of simple real eigenvalues.

1. Introduction

Let us consider the boundary value problem (BVP) generated by the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \leq x < \infty \tag{1.1}$$

and the boundary condition

$$y(0) = 0, \tag{1.2}$$

where q is a real-valued function and $\lambda \in \mathbb{C}$ is a spectral parameter. The bounded solution of (1.1) satisfying the condition

$$\lim_{x \to \infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\} \tag{1.3}$$

will be denoted by $e(x,\lambda)$. The solution $e(x,\lambda)$ satisfies the integral equation

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty \frac{\sin\lambda(t-x)}{\lambda} q(t)e(t,\lambda)dt. \tag{1.4}$$

It has been shown that, under the condition

$$\int_0^\infty x\left|q(x)\right|dx < \infty, \tag{1.5}$$

the solution $e(x,\lambda)$ has the integral representation

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{1.6}$$

where the function $K(x,t)$ is defined by q. The function $e(x,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$, continuous $\overline{\mathbb{C}}_+$, and

$$e(x,\lambda) = e^{i\lambda x}[1+o(1)], \quad \lambda \in \overline{\mathbb{C}}_+,\ x \longrightarrow \infty \tag{1.7}$$

holds [1, chapter 3].

The functions $e(x,\lambda)$ and $e(\lambda) := e(0,\lambda)$ are called Jost solution and Jost function of the BVP (1.1) and (1.2), respectively. These functions play an important role in the solution of inverse problems of the quantum scattering theory [1–4]. In particular, the scattering date of the BVP (1.1) and (1.2) is defined in terms of Jost solution and Jost function. Let $i\lambda_k$, $k = 1,2,3,\ldots,n$, be the zeros of the Jost function, numbered in the order of increase of their moduli $(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n)$ and

$$m_k^{-1} := \left\{\int_0^\infty e^2(x,i\lambda_k)dx\right\}^{1/2}. \tag{1.8}$$

The functions

$$\begin{gathered} E(x,\lambda) := e(x,-\lambda) - s(\lambda)e(x,\lambda), \quad \lambda^2 \in (0,\infty), \\ E(x,i\lambda_k) := m_k e(x,i\lambda_k), \quad k = 1,2,\ldots,n, \end{gathered} \tag{1.9}$$

are bounded solutions of the BVP (1.1) and (1.2), where $s(\lambda) := e(-\lambda)/e(\lambda)$ is the scattering function [1–4]. Using (1.7), we get that

$$\begin{gathered} E(x,\lambda) = e^{-i\lambda x} - s(\lambda)e^{i\lambda x} + o(1), \quad \lambda^2 \in (0,\infty),\ x \longrightarrow \infty, \\ E(x,i\lambda_k) = m_k e^{-\lambda_k x}[1+o(1)], \quad k = 1,2,\ldots,n,\ x \longrightarrow \infty, \end{gathered} \tag{1.10}$$

hold. The collection of quantities $\{s(\lambda), \lambda \in \mathbb{R}; \lambda_k, m_k, k = 1,2,\ldots,n\}$ that specify to as the behaviour of the radial wave functions $E(x,\lambda)$ and $E(x,i\lambda_k)$ at infinity is called the scattering of the BVP (1.1) and (1.2).

Let us consider the self-adjoint system of differential equations of first order

$$\begin{aligned} y_2' + p(x)y_1 &= \lambda y_1, \\ -y_1' + q(x)y_2 &= \lambda y_2, \quad 0 \le x < \infty, \end{aligned} \tag{1.11}$$

where p and q are real-valued continuous functions. In the case $p(x) = V(x)+m$, $q(x) = V(x)-m$, where V is a potential function and m the mass of a particle, (1.11) is called stationary Dirac system in relativistic quantum theory [5, chapter 7]. Jost solution and the scattering theory of (1.11) have been investigated in [6].

Jost solutions of quadratic pencil of Schrödinger, Klein-Gordon, and q-Sturm-Liouville equations have been obtained in [7–9]. In [10–17], using the analytical properties of Jost functions, the spectral analysis of differential and difference equations has been investigated.

Discrete boundary value problems have been intensively studied in the last decade. The modelling of certain linear and nonlinear problems from economics, optimal control theory, and other areas of study has led to the rapid development of the theory of difference equations. Also the spectral analysis of the difference equations has been treated by various authors in connection with the classical moment problem (see the monographs of Agarwal [18], Agarwal and Wong [19], Kelley and Peterson [20], and the references therein). The spectral theory of the difference equations has also been applied to the solution of classes of nonlinear discrete Korteveg-de Vries equations and Toda lattices [21].

Now let us consider the discrete Dirac system

$$\begin{aligned} \Delta y_n^{(2)} + p_n y_n^{(1)} &= \lambda y_n^{(1)}, \\ -\nabla y_n^{(1)} + q_n y_n^{(2)} &= \lambda y_n^{(2)}, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \end{aligned} \tag{1.12}$$

with the boundary condition

$$y_0^{(1)} = 0, \tag{1.13}$$

where Δ is the forward difference operator: $\Delta u_n = u_{n+1} - u_n$ and ∇ is the backward difference operator: $\nabla u_n = u_n - u_{n-1}$; (p_n) and (q_n) are real sequences. It is evident that (1.12) is the discrete analogy of (1.11). Let L denote the operator generated in the Hilbert space $l_2(\mathbb{N}, \mathbb{C}^2)$ by the BVP (1.12) and (1.13). The operator L is self-adjoint, that is, $L = L^*$. In the following, we will assume that, the real sequences (p_n) and (q_n) satisfy

$$\sum_{n=1}^{\infty} n\left(|p_n| + |q_n|\right) < \infty. \tag{1.14}$$

In this paper, we find Jost solution of (1.12) and investigate analytical properties and asymptotic behaviour of the Jost solution. We also show that, $\sigma_c(L) = [-2, 2]$, where $\sigma_c(L)$ denotes the continuous spectrum of L, generated in $l_2(\mathbb{N}, \mathbb{C}^2)$ by (1.12) and (1.13).

We also prove that under the condition (1.14) the operator L has a finite number of simple real eigenvalues.

2. Jost Solution of (1.12)

If $p_n = q_n = 0$ for all $n \in \mathbb{N}$ and $\lambda = -iz - (iz)^{-1}$ from (1.12), we get

$$\begin{aligned} y_{n+1}^{(2)} - y_n^{(2)} &= \left[-iz - (iz)^{-1}\right] y_n^{(1)}, \\ y_{n-1}^{(1)} - y_n^{(1)} &= \left[-iz - (iz)^{-1}\right] y_n^{(2)}. \end{aligned} \tag{2.1}$$

It is clear that

$$e_n(z) = \begin{pmatrix} e_n^{(1)}(z) \\ e_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z \\ -i \end{pmatrix} z^{2n}, \quad n \in \mathbb{N}, \tag{2.2}$$

is a solution of (2.1). Now we find the solution $f_n(z) = \begin{pmatrix} f_n^{(1)} \\ f_n^{(2)} \end{pmatrix}$, $n \in \mathbb{N}$ of (1.12) for $\lambda = -iz - (iz)^{-1}$, satisfying the condition

$$f_n(z) = [I + o(1)] e_n(z), \quad |z| = 1,\ n \longrightarrow \infty, \tag{2.3}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 2.1. *Under the condition* (1.14) *for* $\lambda = -iz - (iz)^{-1}$ *and* $|z| = 1$, (1.12) *has the solution* $f_n(z) = \begin{pmatrix} f_n^{(1)} \\ f_n^{(2)} \end{pmatrix}$, $n \in \mathbb{N}$, *having the representation*

$$f_n(z) = \begin{pmatrix} f_n^{(1)} \\ f_n^{(2)} \end{pmatrix} = \left[I + \sum_{m=1}^{\infty} K_{nm} z^{2m}\right] \begin{pmatrix} z \\ -i \end{pmatrix} z^{2n}, \quad n \in \mathbb{N}, \tag{2.4}$$

$$f_0^{(1)}(z) = z + \sum_{m=1}^{\infty} \left(K_{0m}^{11} z^{2m+1} - i K_{0m}^{12} z^{2m}\right), \tag{2.5}$$

where

$$K_{nm} = \begin{pmatrix} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{pmatrix}. \tag{2.6}$$

Proof. Substituting $f_n(z)$ defined by (2.4) and (2.5) into (1.12) and taking $\lambda = -iz - (iz)^{-1}$, $|z| = 1$, we get the following:

$$\begin{aligned}
&p_n z^{2n+1} - \sum_{m=1}^{\infty} K_{nm}^{12} z^{2m+2n-1} + i\sum_{m=1}^{\infty}\left(K_{nm}^{22} - p_n K_{nm}^{12} - K_{nm}^{11}\right) z^{2m+2n} \\
&\quad + \sum_{m=1}^{\infty}\left(p_n K_{nm}^{11} - K_{nm}^{21} + K_{nm}^{12}\right) z^{2m+2n+1} \\
&\quad = i\sum_{m=1}^{\infty}\left(K_{nm}^{11} - K_{n+1,m}^{22}\right) z^{2m+2n+2} + \sum_{m=1}^{\infty} K_{n+1,m}^{21} z^{2m+2n+3},
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&iq_n z^{2n} + i\sum_{m=1}^{\infty} K_{n-1,m}^{12} z^{2m+2n-2} - \sum_{m=1}^{\infty}\left(K_{n-1,m}^{11} - K_{nm}^{22}\right) z^{2m+2n-1} - i\sum_{m=1}^{\infty}\left(K_{nm}^{12} - q_n K_{nm}^{22} - K_{nm}^{21}\right) z^{2m+2n} \\
&\quad = \sum_{m=1}^{\infty}\left(q_n K_{nm}^{21} - K_{nm}^{11} + K_{nm}^{22}\right) z^{2m+2n+1} + i\sum_{m=1}^{\infty} K_{nm}^{21} z^{2m+2n+2}.
\end{aligned} \tag{2.8}$$

Using (2.7) and (2.8),

$$\begin{aligned}
K_{n1}^{12} &= -\sum_{k=n+1}^{\infty} (p_k + q_k), \\
K_{n1}^{11} &= \sum_{k=n+1}^{\infty} p_k K_{k1}^{12}, \\
K_{n1}^{22} &= K_{n-1,1}^{11} = \sum_{k=n}^{\infty} p_k K_{k1}^{12}, \\
K_{n1}^{21} &= K_{n1}^{12} + p_n K_{n1}^{11} + \sum_{k=n+1}^{\infty}\left[q_k K_{k1}^{22} + p_k K_{k1}^{11}\right], \\
K_{n2}^{12} &= -\sum_{k=n+1}^{\infty}\left[p_k K_{k1}^{11} + q_k K_{k1}^{22}\right], \\
K_{n2}^{11} &= -K_{n+1,1}^{22} + \sum_{k=n+1}^{\infty}\left[p_k K_{k2}^{12} - q_k K_{k1}^{21}\right], \\
K_{n2}^{22} &= -K_{n1}^{11} + \sum_{k=n}^{\infty}\left[p_k K_{k2}^{12} - q_{k+1} K_{k+1,1}^{21}\right], \\
K_{n2}^{21} &= K_{n2}^{12} + \sum_{k=n}^{\infty}\left[p_k K_{k2}^{11} + q_{k+1} K_{k+1,2}^{22}\right]
\end{aligned} \tag{2.9}$$

hold, where $n \in \mathbb{N}$. For $m \geq 3$, we obtain

$$\begin{aligned}
K_{nm}^{12} &= K_{n+1,m-2}^{21} - \sum_{k=n+1}^{\infty}\left[q_k K_{k,m-1}^{22} + p_k K_{k,m-1}^{11}\right],\\
K_{nm}^{11} &= -K_{n+1,m-1}^{22} + \sum_{k=n+1}^{\infty}\left[p_k K_{km}^{12} - q_k K_{k,m-1}^{21}\right],\\
K_{nm}^{22} &= -K_{n,m-1}^{11} + \sum_{k=n}^{\infty}\left[p_k K_{km}^{12} - q_{k+1} K_{k+1,m-1}^{21}\right],\\
K_{nm}^{21} &= K_{nm}^{12} + p_n K_{nm}^{11} + \sum_{k=n+1}^{\infty}\left[q_k K_{km}^{22} + p_k K_{km}^{11}\right].
\end{aligned} \tag{2.10}$$

By the condition (1.14), the series in the definition of K_{nm}^{ij} $(i,j = 1,2)$ are absolutely convergent. Therefore, K_{nm}^{ij} $(i,j = 1,2)$ can, by uniquely be defined by p_n and q_n, that is, the system (1.12) for $\lambda = -iz - (iz)^{-1}$ and $|z| = 1$, has the solution $f_n(z)$ given by (2.4) and (2.5).

By induction, we easily obtain that

$$\left|K_{nm}^{ij}\right| \leq C \sum_{r=n+\lfloor m/2 \rfloor} \left(|p_r| + |q_r|\right), \tag{2.11}$$

where $\lfloor m/2 \rfloor$ is the integer part of $m/2$ and $C > 0$ is a constant. It follows from (2.4) and (2.11) that (2.3) holds. □

Theorem 2.2. *The solution $f_n(z)$ has an analytic continuation from $\{z : |z| = 1\}$ to $D := \{z : |z| < 1\} \setminus \{0\}$.*

Proof. From (1.14) and (2.11), we obtain that the series $\sum_{m=1}^{\infty} K_{nm} e^{imz}$ and $\sum_{m=1}^{\infty} m K_{nm} e^{imz}$ are uniformly convergent in D. This shows that the solution $f_n(z)$ has an analytic continuation from $\{z : |z| = 1\}$ to D. □

The functions $f_n(z)$ and $f_0^{(1)}(z)$ are called Jost solution and Jost function of the BVP (1.12) and (1.13), respectively. It follows from Theorem 2.2 that Jost solution and Jost function are analytic in D and continuous on $\overline{D} := \{z : |z| \leq 1\} \setminus \{0\}$.

Theorem 2.3. *The following asymptotics hold:*

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = [I + o(1)] \begin{pmatrix} z \\ -i \end{pmatrix} z^{2n}, \quad z \in \overline{D},\ n \longrightarrow \infty. \tag{2.12}$$

Proof. From (2.4), we get that

$$f_n^{(1)}(z) = z^{2n+1} + \left[\sum_{m=1}^{\infty} K_{nm}^{11} z^{2m} - i\sum_{m=1}^{\infty} K_{nm}^{12} z^{2m-1}\right] z^{2n+1}, \quad z \in D,$$
$$f_n^{(1)}(z) z^{-2n-1} = 1 + \sum_{m=1}^{\infty} K_{nm}^{11} z^{2m} - i\sum_{m=1}^{\infty} K_{nm}^{12} z^{2m-1}, \quad z \in D. \tag{2.13}$$

Using (2.11) and (2.13), we obtain

$$\begin{aligned}\left|f_n^{(1)}(z) z^{-2n-1}\right| &\le 1 + \sum_{m=1}^{\infty}\left|K_{nm}^{11}\right| + \sum_{m=1}^{\infty}\left|K_{nm}^{12}\right| \\ &\le 1 + c\sum_{m=1}^{\infty}\sum_{k=n+[|m/2|]}\left(|p_k| + |q_k|\right) \le 1 + c\sum_{k=n+1}^{\infty}\sum_{m=1}^{k-n}\left(|p_k| + |q_k|\right) \\ &\le 1 + c\sum_{k=n+1}^{\infty}(k-n)\left(|p_k| + |q_k|\right) \le 1 + c\sum_{k=n+1}^{\infty} k\left(|p_k| + |q_k|\right).\end{aligned} \tag{2.14}$$

So we have

$$f_n^{(1)}(z) = z^{2n+1}[1 + o(1)], \quad z \in \overline{D},\ n \longrightarrow \infty, \tag{2.15}$$

by (2.14). In a manner similar to (2.15), we get

$$f_n^{(2)}(z) = -iz^{2n}[1 + o(1)], \quad z \in \overline{D},\ n \longrightarrow \infty. \tag{2.16}$$

From (2.15) and (2.16), we obtain (2.12). □

3. Continuous and Discrete Spectrum of the BVP (1.12) and (1.13)

Let $\ell_2(\mathbb{N}, C^2)$ denote the Hilbert space of all complex vector sequences

$$y = \left\{\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}\right\}_{n\in\mathbb{N}} \tag{3.1}$$

with the norm

$$\|y\|^2 = \sum_{n=1}^{\infty}\left(\left|y_n^{(1)}\right|^2 + \left|y_n^{(2)}\right|^2\right). \tag{3.2}$$

Theorem 3.1. $\sigma_c(L) = [-2, 2]$.

Proof. Let L_0 denote the operator generated in $\ell_2(\mathbb{N}, C^2)$ by the BVP

$$\begin{aligned} \Delta y_n^{(2)} &= \lambda y_n^{(1)}, \\ -\nabla y_n^{(1)} &= \lambda y_n^{(2)}, \\ y_0^{(1)} &= 0. \end{aligned} \tag{3.3}$$

We also define the operator P in $\ell_2(\mathbb{N}, C^2)$ by the following:

$$P\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} := \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix}\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}. \tag{3.4}$$

It is clear that $P = P^*$ and

$$L = L_0 + P, \tag{3.5}$$

where L denotes the operator generated in $\ell_2(\mathbb{N}, C^2)$ by the BVP (1.12) and (1.13). It follows from (1.14) that the operator P is compact in $\ell_2(\mathbb{N}, C^2)$. We easily prove that

$$\sigma_c(L_0) = [-2, 2]. \tag{3.6}$$

Using the Weyl theorem [22] of a compact perturbation, we obtain

$$\sigma_c(L) = \sigma_c(L_0) = [-2, 2]. \tag{3.7}$$

□

Since the operator L is selfadjoint, the eigenvalues of L are real. From the definition of the eigenvalues, we get that

$$\sigma_d(L) = \left\{\lambda : \lambda = -iz - (iz)^{-1}, iz \in (-1, 0) \cup (0, 1),\ f_0^{(1)}(z) = 0\right\}, \tag{3.8}$$

where $\sigma_d(L)$ denotes the set of all eigenvalues of L.

Definition 3.2. The multiplicity of a zero of the function $f_0^{(1)}$ is called the multiplicity of the corresponding eigenvalue of L.

Theorem 3.3. *Under the condition* (1.14)*, the operator L has a finite number of simple real eigenvalues.*

Proof. To prove the theorem, we have to show that the function $f_0^{(1)}$ has a finite number of simple zeros.

Let z_0 be one of the zeros of $f_0^{(1)}$. Now we show that

$$\left.\frac{d}{dz}f_0^{(1)}(z)\right|_{z=z_0} \neq 0. \tag{3.9}$$

Let $f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix}$ be the Jost solution of (1.12) that is,

$$\begin{gathered} f_{n+1}^{(2)}(z) - f_n^{(2)}(z) + p_n f_n^{(1)}(z) = \left[-iz - (iz)^{-1}\right] f_n^{(1)}(z), \\ f_{n-1}^{(1)}(z) - f_n^{(1)}(z) + q_n f_n^{(2)}(z) = \left[-iz - (iz)^{-1}\right] f_n^{(2)}(z). \end{gathered} \tag{3.10}$$

Differentiating (3.10) with respect to z, we have

$$\begin{gathered} \frac{d}{dz}f_{n+1}^{(2)}(z) - \frac{d}{dz}f_n^{(2)}(z) + p_n\frac{d}{dz}f_n^{(1)}(z) = \left[-iz - (iz)^{-1}\right]\frac{d}{dz}f_n^{(1)}(z) - i\left(1 - z^{-2}\right)f_n^{(1)}(z), \\ \frac{d}{dz}f_{n-1}^{(1)}(z) - \frac{d}{dz}f_n^{(1)}(z) + q_n\frac{d}{dz}f_n^{(2)}(z) = \left[-iz - (iz)^{-1}\right]\frac{d}{dz}f_n^{(2)}(z) - i\left(1 - z^{-2}\right)f_n^{(2)}(z). \end{gathered} \tag{3.11}$$

Using (3.10) and (3.11), we obtain

$$\begin{aligned} &\left[\frac{df_n^{(1)}(z)}{dz}f_{n+1}^{(2)}(z) - f_n^{(1)}(z)\frac{df_{n+1}^{(2)}(z)}{dz}\right] - \left[\frac{df_{n-1}^{(1)}(z)}{dz}f_n^{(2)}(z) - f_{n-1}^{(1)}(z)\frac{df_n^{(2)}(z)}{dz}\right] \\ &\quad = i\left(1 - z^{-2}\right)\left\{\left[f_n^{(1)}(z)\right]^2 + \left[f_n^{(2)}(z)\right]^2\right\} \end{aligned} \tag{3.12}$$

or

$$-\frac{df_0^{(1)}(z)}{dz}f_1^{(2)}(z) = i\left(1 - z^{-2}\right)\sum_{n=1}^{\infty}\left\{\left[f_n^{(1)}(z)\right]^2 + \left[f_n^{(2)}(z)\right]^2\right\}. \tag{3.13}$$

It follows from (3.13) that

$$\left.\frac{d}{dz}f_0^{(1)}(z)\right|_{z=z_0} \neq 0, \tag{3.14}$$

that is, all zeros of $f_0^{(1)}$ are simple.

Let δ denote the infimum of distances between two neighboring zeros of $f_0^{(1)}$. We show that $\delta > 0$. Otherwise, we can take a sequence of zeros z_k and w_k of the function $f_0^{(1)}$, such that

$$\lim_{k\to\infty}(z_k - w_k) = 0. \tag{3.15}$$

It follows from (2.4) that, for large $p \in \mathbb{N}$,

$$\sum_{n=p}^{\infty}\left[f_n^{(1)}(z_k)\overline{f_n^{(1)}(w_k)} + f_n^{(2)}(z_k)\overline{f_n^{(2)}(w_k)}\right] \geq M \tag{3.16}$$

holds, where $M > 0$.

From the equation

$$\sum_{n=1}^{\infty}\left[f_n^{(1)}(z_k)\overline{f_n^{(1)}(w_k)} + f_n^{(2)}(z_k)\overline{f_n^{(2)}(w_k)}\right] = 0, \tag{3.17}$$

we get

$$\liminf_{k\to\infty}\sum_{n=p}^{\infty}\left[f_n^{(1)}(z_k)\overline{f_n^{(1)}(w_k)} + f_n^{(2)}(z_k)\overline{f_n^{(2)}(w_k)}\right] \leq 0. \tag{3.18}$$

There is a contradiction comparing (3.16) and (3.18). So $\delta > 0$ and $f_0^{(1)}$ function has only a finite number of zeros. □

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