Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 376471, 14 pages
doi:10.1155/2010/376471

Research Article

Triple Positive Solutions for a Type of Second-Order Singular Boundary Problems

Jiemei Li[1,2] and Jinxiang Wang[1]

[1] *Department of Mathematics, Northwest Normal University, Lanzhou 730070, China*
[2] *The School of Mathematics, Physics Software Engineering, Lanzhou Jiaotong University, Lanzhou 730070, China*

Correspondence should be addressed to Jiemei Li, lijiemei81@126.com

Received 7 April 2010; Accepted 26 August 2010

Academic Editor: Wenming Zou

This paper deals with the existence of triple positive solutions for a type of second-order singular boundary problems with general differential operators. By using the Leggett-Williams fixed point theorem, we establish an existence criterion for at least three positive solutions with suitable growth conditions imposed on the nonlinear term.

1. Introduction

In this paper, we study the existence of triple positive solutions for the following second-order singular boundary value problems with general differential operators:

$$\begin{gathered} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $a \in C(0,1) \cap L^1(0,1)$, $b \in C((0,1),(-\infty,0))$, and $h \in C((0,1),[0,\infty))$ with

$$\int_0^1 t(1-t)|b(t)|dt < \infty, \qquad \int_0^1 t(1-t)h(t)dt < \infty. \tag{1.2}$$

It is easy to see that a, b, and h may be singular at $t = 0$ and/or 1.

When $a = b \equiv 0$ or $a \equiv 0$ and $b \not\equiv 0$, the two kinds of singular boundary value problems have been discussed extensively in the literature; see [1–10] and the references therein. Hence, the problem that we consider is more general and is different from those in previous work.

Furthermore, we will see in the later that the presence of $a(t)u'(t) + b(t)u(t)$ brings us three main difficulties:

(1) the Green's function cannot be explicitly expressed;

(2) the equivalence between BVP (1.1) and its associated integral equation has to be proved;

(3) the compactness of associated integral operator has to be verified.

We will overcome the above mentioned difficulties in Section 2. Also, although the Leggett-William fixed point theorem is used extensively in the study of triple positive differential equations, the method has not been used to study this type of second-order singular boundary value problem with general differential operators. We are concerned with solving these problems in this paper.

To state our main tool used in this paper, we give some definitions and notations.

Let E be a real Banach space with a cone P. A map $\alpha : P \to [0, +\infty)$ is said to be a nonnegative continuous concave functional on P if α is a continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \tag{1.3}$$

for all $x, y \in P$ and $t \in [0,1]$. Let a, b be two numbers such that $0 < a < b$ and α a nonnegative continuous concave functional on P. We define the following convex sets:

$$\begin{gathered} P_a = \{x \in P : \|x\| < a\}, \\ P(\alpha, a, b) = \{x \in P : a \leq \alpha(x),\ \|x\| \leq b\}. \end{gathered} \tag{1.4}$$

Theorem 1.1 (Leggett-Williams fixed point theorem). *Let $T : \overline{P}_c \to \overline{P}_c$ be completely continuous, and let α be a nonnegative continuous concave functional on P such that $\alpha \leq \|x\|$ for all $x \in \overline{P}_c$. Suppose that there exist $0 < d < a < b \leq c$ such that*

(i) *$\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Tx) > a$, for $x \in P(\alpha, a, b)$;*

(ii) *$\|Tx\| < d$, for $\|x\| < d$;*

(iii) *$\alpha(Tx) > a$, for $x \in P(\alpha, a, c)$ with $\|Tx\| > b$.*

Then T has at least three fixed points x_1, x_2, x_3 in $\overline{P}_c$ satisfying $\|x_1\| < d$, $a < \alpha(x_2)$, $\|x_3\| > d$ and $\alpha(x_3) < a$.

Remark 1.2. We note the existence of triple positive solutions of other kind of boundary value problems; see He and Ge [11], Zhao et al. [12], Zhang and Liu [13], Graef et al. [14], and the references therein.

The rest of the paper is organized as follows. In Section 2, we overcome the above-mentioned difficulties in this work. The main results are formulated and proved in Section 3. Finally, an example is presented to demonstrate the application of the main theorems in Section 4.

2. Preliminaries and Lemmas

Throughout this paper, we assume the following:

(H1) $a \in C(0,1) \cap L^1(0,1)$;

(H2) $b \in C((0,1),(-\infty,0))$, and $\int_0^1 s(1-s)|b(s)|ds < \infty$;

(H3) $h : (0,1) \to [0,\infty)$ is continuous and does not vanish identically on any subinterval of $(0,1)$, and $\int_0^1 s(1-s)h(s)ds < \infty$;

(H4) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Lemma 2.1. *Suppose that (H1) and (H2) hold. Then*

(i) *the initial value problem*

$$\begin{aligned} &u''(t) + a(t)u'(t) + b(t)u(t) = 0, \quad t \in (0,1), \\ &u(0) = 0, \qquad u'(0) = 1, \end{aligned} \tag{2.1}$$

has a unique solution $\varphi \in AC[0,1] \cap C^1[0,1)$ *and* $\varphi' \in AC_{\text{loc}}[0,1)$;

(ii) *the initial value problem*

$$\begin{aligned} &u''(t) + a(t)u'(t) + b(t)u(t) = 0, \quad t \in (0,1), \\ &u(1) = 0, \qquad u'(1) = -1, \end{aligned} \tag{2.2}$$

has a unique solution $\psi \in AC[0,1] \cap C^1(0,1]$ *and* $\psi' \in AC_{\text{loc}}(0,1]$.

Proof. We only prove (i). (ii) can be treated in the same way.

Suppose that $\varphi \in AC[0,1] \cap C^1[0,1)$ and $\varphi' \in AC_{\text{loc}}[0,1)$ is a solution of (2.1), that is,

$$\begin{aligned} &\varphi''(t) + a(t)\varphi'(t) + b(t)\varphi(t) = 0, \quad t \in (0,1), \\ &\varphi(0) = 0, \qquad \varphi'(0) = 1. \end{aligned} \tag{2.3}$$

Let

$$p(t) = \exp\left(\int_0^t a(s)ds\right). \tag{2.4}$$

Multiplying both sides of (2.3) by $p(t)$, then

$$\left(p(t)\varphi'(t)\right)' = -p(t)b(t)\varphi(t), \quad t \in (0,1). \tag{2.5}$$

Since $\varphi \in C^1[0,1)$ and $\varphi' \in AC_{\text{loc}}[0,1)$, integrating (2.5) on $[0,t]$, $(0 \le t < 1)$, we have

$$p(t)\varphi'(t) = 1 - \int_0^t p(s)b(s)\varphi(s)ds. \tag{2.6}$$

Moreover, integrating (2.6) on $[0,t]$, $(0 \le t \le 1)$, we have

$$p(t)\varphi(t) = t + \int_0^t p(s)a(s)\varphi(s)ds - \int_0^t \int_0^s p(\tau)b(\tau)\varphi(\tau)d\tau\, ds. \tag{2.7}$$

Let

$$v(t) = \begin{cases} \dfrac{p(t)\varphi(t)}{t}, & t \in (0,1], \\ p(0)\varphi'(0), & t = 0. \end{cases} \tag{2.8}$$

Clearly, $v \in C[0,1]$, and (2.7) reduces to

$$v(t) = 1 + \frac{1}{t}\int_0^t sa(s)v(s)ds - \frac{1}{t}\int_0^t \int_0^s \tau b(\tau)v(\tau)d\tau\, ds. \tag{2.9}$$

By using Fubini's theorem, we have

$$\int_0^t \int_0^s \tau b(\tau)v(\tau)d\tau ds = \int_0^t (t-s)sb(s)v(s)ds. \tag{2.10}$$

Therefore,

$$v(t) = 1 + \frac{1}{t}\int_0^t sa(s)v(s)ds - \frac{1}{t}\int_0^t (t-s)sb(s)v(s)ds, \tag{2.11}$$

which implies that v is a solution of integral equation (2.11).

Conversely, if $v \in C[0,1]$ is a solution of (2.11) with $v(0) = 1$, by reversing the above argument we could deduce that the function $\varphi(t) = tv(t)/p(t)$ is a solution of (2.1) and satisfy $\varphi \in AC[0,1] \cap C^1[0,1)$ and $\varphi' \in AC_{\text{loc}}[0,1)$. Therefore, to prove that (2.1) has a unique solution, $\varphi \in AC[0,1] \cap C^1[0,1)$, and $\varphi' \in AC_{\text{loc}}[0,1)$ is equivalent to prove that (2.11) has a unique solution $v \in C[0,1]$.

To do this, we endow the following norm in $C[0,1]$:

$$\|v\|_* = \sup_{t\in[0,1]} \left| v(t) \exp\left(-\int_0^t [|a(s)| + s(1-s)|b(s)|]ds \right) \right|. \tag{2.12}$$

Let $K : C[0,1] \to C[0,1]$ be operator defined by

$$Kv(t) = 1 + \frac{1}{t}\int_0^t sa(s)v(s)ds - \frac{1}{t}\int_0^t s(t-s)b(s)v(s)ds. \tag{2.13}$$

Since

$$\left|\frac{1}{t}\int_0^t sa(s)v(s)ds\right| \le \int_0^t |v(s)a(s)|ds \longrightarrow 0, \quad t \longrightarrow 0,$$
$$\left|\frac{1}{t}\int_0^t s(t-s)b(s)v(s)ds\right| \le \int_0^t s(1-s)|b(s)v(s)|ds \longrightarrow 0, \quad t \longrightarrow 0, \tag{2.14}$$

then, K is well defined. Set

$$D(t) = \exp\left(-\int_0^t [|a(s)| + s(1-s)|b(s)|]ds\right). \tag{2.15}$$

Then, for any $z,\ w \in C[0,1]$,

$$\begin{aligned}
|Kz(t) - Kw(t)| &= \left|\frac{1}{t}\int_0^t [sa(s) - s(t-s)b(s)](z(s) - w(s))ds\right| \\
&\le \int_0^t [|a(s)| + s(1-s)|b(s)|]|z(s) - w(s)|D(s)\frac{1}{D(s)}ds \\
&\le \int_0^t [|a(s)| + s(1-s)|b(s)|]\frac{1}{D(s)}ds \cdot \|z - w\|_* \\
&= \left(\frac{1}{D(t)} - 1\right)\|z - w\|_*,
\end{aligned} \tag{2.16}$$

and subsequently,

$$D(t)|Kz(t) - Kw(t)| \le (1 - D(t)) \cdot \|z - w\|_*. \tag{2.17}$$

Thus,

$$\|K_z - K_w\|_* \le (1 - D(t)) \cdot \|z - w\|_*. \tag{2.18}$$

Since $0 \le 1 - D(t) < 1$, K has a unique fixed point $v \in C[0,1]$ by Banach contraction principle. That is, (2.11) has a unique solution $v \in C[0,1]$. □

Remark 2.2. Lemma 2.1 generalizes Theorem 2.1of [1], where $a \equiv 0$.

Lemma 2.3. *Suppose that (H1) and (H2) hold. Then*

(i) φ *is nondecreasing in* $[0,1]$;

(ii) ψ *is nonincreasing in* $[0,1]$.

Proof. We only prove (i). (ii) can be treated in the same way.

Suppose on the contrary that φ is not nondecreasing in $[0,1]$. Then there exists $\tau_0 \in (0,1)$ such that

$$\varphi(\tau_0) = \max\{\varphi(t) \mid t \in [0,1]\} > 0, \quad \varphi'(\tau_0) = 0, \quad \varphi''(\tau_0) \leq 0. \tag{2.19}$$

This together with the equation $\varphi''(t) + a(t)\varphi'(t) + b(t)\varphi(t) = 0$ implies that

$$\varphi''(\tau_0) = -b(\tau_0)\varphi(\tau_0) > 0, \tag{2.20}$$

which is a contradiction! □

Remark 2.4. From Lemmas 2.1 and 2.3, there exist positive constants C_1, C_2, D_1, and D_2 such that

$$D_1 t \leq \varphi(t) \leq C_1 t, \quad D_2(1-t) \leq \psi(t) \leq C_2(1-t), \quad t \in [0,1]. \tag{2.21}$$

In fact, since

$$\lim_{t\to 0}\frac{\varphi(t)}{t} = \varphi'(0) = 1 > 0, \tag{2.22}$$

we have that $\varphi(t)/t \in C[0,1]$ and $\varphi(t)/t \geq 0,\ t \in [0,1]$. Then, there exist constants $C_1 > 0$ and $D_1 \geq 0$, such that

$$D_1 \leq \frac{\varphi(t)}{t} \leq C_1, \quad t \in [0,1], \tag{2.23}$$

that is

$$D_1 t \leq \varphi(t) \leq C_1 t, \quad t \in [0,1]. \tag{2.24}$$

In the following, we will show that $D_1 > 0$. Suppose on the contrary, if there exist $t^* \in (0,1]$, such that

$$\frac{\varphi(t^*)}{t^*} = \min_{t\in[0,1]}\frac{\varphi(t)}{t} = D_1 = 0, \tag{2.25}$$

then, $\varphi(t^*) = 0$, which is a contradiction!

The other inequality can be treated in the same manner.

Lemma 2.5. *Suppose that (H1), (H2), and (H3) hold. Then*

$$\lim_{t\to 0}\varphi(t)\int_t^1 \psi(s)h(s)ds = 0, \qquad \lim_{t\to 1}\psi(t)\int_0^t \varphi(s)h(s)ds = 0. \tag{2.26}$$

Proof. We only prove the first equality; the other can be treated in the same way. From Remark 2.4 and (H3), we have

$$0 \leq \varphi(t) \int_t^1 \psi(s)h(s)ds \leq C_2\varphi(t) \int_t^1 (1-s)h(s)ds. \tag{2.27}$$

Lemma 2.1of [2] together with the facts that $\varphi \in C^1[0,1)$ and (H3) implies that

$$\lim_{t\to 0}\varphi(t) \int_t^1 (1-s)h(s)ds = 0. \tag{2.28}$$

Combining (2.27) and (2.28), we have

$$\lim_{t\to 0}\varphi(t) \int_t^1 \psi(s)h(s)ds = 0. \tag{2.29}$$

□

Lemma 2.6. *Suppose that (H1), (H2), and (H3) hold. Then the problem*

$$\begin{gathered} u''(t) + a(t)u'(t) + b(t)u(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0 \end{gathered} \tag{2.30}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)q(s)h(s)ds, \tag{2.31}$$

where

$$\begin{gathered} q(s) = \exp\left(\int_{1/2}^{s} a(\tau)d\tau\right), \\ G(t,s) = \frac{1}{\rho}\begin{cases} \varphi(s)\psi(t), & 0 \leq s \leq t \leq 1, \\ \varphi(t)\psi(s), & 0 \leq t \leq s \leq 1, \end{cases} \\ \rho = \varphi'\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{2}\right)\psi'\left(\frac{1}{2}\right). \end{gathered} \tag{2.32}$$

Moreover, $u(t) > 0$ *on* $(0,1)$.

Proof. By Lemma 2.3 and (2.32), we have

$$G(t,s) \leq G(s,s), \quad t,s \in [0,1]. \tag{2.33}$$

This together with Remark 2.4 implies that the right side of (2.31) iswell defined.

Now we check that the function

$$u(t) = \int_0^t \frac{1}{\rho}\varphi(s)\psi(t)q(s)h(s)ds + \int_t^1 \frac{1}{\rho}\psi(s)\varphi(t)q(s)h(s)ds \tag{2.34}$$

satisfies (2.30). In fact,

$$\begin{aligned} u'(t) &= \psi'(t)\int_0^t \frac{1}{\rho}\varphi(s)q(s)h(s)ds + \varphi'(t)\int_t^1 \frac{1}{\rho}\psi(s)q(s)h(s)ds, \\ u''(t) &= \psi''(t)\int_0^t \frac{1}{\rho}\varphi(s)q(s)h(s)ds + \psi'(t)\frac{1}{\rho}\varphi(t)q(t)h(t) \\ &\quad + \varphi''(t)\int_t^1 \frac{1}{\rho}\psi(s)q(s)h(s)ds - \varphi'(t)\frac{1}{\rho}\psi(t)q(t)h(t). \end{aligned} \tag{2.35}$$

Therefore,

$$\begin{aligned} u''(t) + a(t)u'(t) + b(t)u(t) &= \frac{1}{\rho}q(t)h(t)\begin{vmatrix} \varphi(t) & \psi(t) \\ \varphi'(t) & \psi'(t) \end{vmatrix} \\ &= \frac{1}{\rho}q(t)h(t)\begin{vmatrix} \varphi\left(\frac{1}{2}\right) & \psi\left(\frac{1}{2}\right) \\ \varphi'\left(\frac{1}{2}\right) & \psi'\left(\frac{1}{2}\right) \end{vmatrix} \exp\left(-\int_{1/2}^t a(s)ds\right) \\ &= -h(t). \end{aligned} \tag{2.36}$$

Equation (2.34) and Lemma 2.5 imply that

$$u(0) = \lim_{t\to 0} u(t) = 0, \qquad u(1) = \lim_{t\to 1} u(t) = 0. \tag{2.37}$$

Since $G(t,s) > 0$ for $t, s \in (0,1)$, then

$$u(t) = \int_0^1 G(t,s)q(s)h(s)ds > 0, \quad t \in (0,1). \tag{2.38}$$

□

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,1]}|u(t)|, \tag{2.39}$$

and let P be a cone in $C[0,1]$ defined by

$$P = \{u \in C[0,1] : u(t) \geq 0,\ t \in [0,1]\}. \tag{2.40}$$

Lemma 2.7. *Suppose that (H1)–(H3) hold and u is a positive solution of* (2.30). *Then*

$$u(t) \geq \|u\|\gamma(t), \quad t \in [0,1], \tag{2.41}$$

where

$$\gamma(t) = \min_{t\in[0,1]}\left\{\frac{\varphi(t)}{\|\varphi\|}, \frac{\psi(t)}{\|\psi\|}\right\}. \tag{2.42}$$

Furthermore, for any $\delta \in (0,1/2)$, there exists corresponding $\gamma_\delta > 0$ such that

$$u(t) \geq \gamma_\delta\|u\|, \quad t \in [\delta, 1-\delta]. \tag{2.43}$$

Proof. In fact, if $0 < t \leq s < 1$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \geq \frac{\varphi(t)}{\|\varphi\|}, \tag{2.44}$$

and if $0 < s \leq t < 1$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\psi(t)}{\psi(s)} \geq \frac{\psi(t)}{\|\psi\|}. \tag{2.45}$$

Combining this and $G(t,s) \leq G(s,s)$, $t,s \in [0,1]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)q(s)h(s)ds \geq \gamma(t)\int_0^1 G(s,s)q(s)h(s)ds \\ &\geq \gamma(t)\|u\|, \quad t \in [0,1]. \end{aligned} \tag{2.46}$$

Take

$$\gamma_\delta = \min\{\gamma(t) : t \in [\delta, 1-\delta]\}. \tag{2.47}$$

Then Lemma 2.3 guarantees that $\gamma_\delta > 0$, and Lemma 2.7 guarantees that (2.43) holds. □

Remark 2.8. From Lemma 2.7 and Remark 2.4, we have

$$G(t,s) \geq \gamma(t)G(s,s) \geq \frac{D_1D_2}{\rho}\gamma(t)s(1-s), \quad t,s \in [0,1]. \tag{2.48}$$

Now, for any $u \in P$, we can define the operator $T : C[0,1] \to C[0,1]$ by

$$Tu(t) = \int_0^1 G(t,s)q(s)h(s)f(s,u(s))ds, \quad t \in [0,1]. \tag{2.49}$$

Lemma 2.9. *Let (H1)–(H4) hold. Then $T : P \to P$ is a completely continuous operator.*

Proof. From (H3) and (H4) and Lemma 2.6, it is easy to see that $T : P \to P$, and T is continuous by the Lebesgue's dominated convergence theorem.

Let $\mathfrak{B} \subseteq P$ be any bounded set. Then (H3) and (H4) imply that $T(\mathfrak{B})$ is a bounded set in P.

Since

$$\begin{aligned}(Tu)'(t) &= \psi'(t)\int_0^t \frac{1}{\rho}\varphi(s)q(s)h(s)f(s,u(s))ds \\ &\quad + \varphi'(t)\int_t^1 \frac{1}{\rho}\psi(s)q(s)h(s)f(s,u(s))ds,\end{aligned} \tag{2.50}$$

then this together with the similar proof of Lemma 2.1of [2] yields

$$(Tu)' \in L^1[0,1]. \tag{2.51}$$

From this fact, it is easy to verify that $T(\mathfrak{B})$ is equicontinuous. Therefore, by the Arzela-Ascoli theorem, $T : P \to P$ is a completely continuous operator. □

3. Main Result

Let $\alpha : P \to [0,\infty)$ be nonnegative continuous concave functional defined by

$$\alpha(u) = \min_{t\in[\delta,1-\delta]} u(t), \quad u \in P. \tag{3.1}$$

We notice that, for each $u \in P$, $\alpha(u) \le \|u\|$, and also that by Lemma 2.6, $u \in P$ is a solution of (1.1) if and only if u is a fixed point of the operator T.

For convenience we introduce the following notations. Let

$$\begin{aligned} m &= \rho \cdot \left(C_1C_2\int_0^1 s(1-s)q(s)h(s)ds\right)^{-1}, \\ \eta &= \frac{1}{\rho}D_1D_2 \min_{t\in[\delta,1-\delta]}\gamma(t)\int_\delta^{1-\delta} s(1-s)q(s)h(s)ds. \end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that (H1)–(H4) hold. Let $0 < a < b < b/\gamma_\delta \le c$, and suppose that f satisfies the following conditions:*

(S1) $f(t,u) < ma$, *for* $t \in [0,1]$, $u \in [0,a]$;

(S2) $f(t,u) > b/\eta$, *for* $t \in [\delta, 1-\delta]$, $u \in [b, b/\gamma_\delta]$;

(S3) $f(t,u) < mc$, *for* $t \in [0,1]$, $u \in [0,c]$.

Then the boundary value problem (1.1) *has at least three positive solutions u_1, u_2, u_3 in $\overline{P_c}$ satisfying $\|u_1\| < a$, $b < \alpha(u_2)$, $\|u_3\| > a$, and $\alpha(u_3) < b$.*

Proof. From Lemma 2.9, $T : P \longrightarrow P$ is a completely continuous operator. If $u \in \overline{P_c}$, then $\|u\| \le c$, and assumption (S3) implies that $f(t,u) < mc$, $t \in [0,1]$. Therefore

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]}|Tu(t)| = \max_{t\in[0,1]}\int_0^1 G(t,s)q(s)h(s)f(s,u(s))ds \\ &\le \frac{C_1C_2}{\rho}mc\int_0^1 s(1-s)q(s)h(s)ds \\ &\le c. \end{aligned} \tag{3.3}$$

Hence, $T : \overline{P_c} \longrightarrow \overline{P_c}$. In the same way, if $u \in \overline{P_a}$, then $T : \overline{P_a} \longrightarrow \overline{P_a}$. Therefore, condition (ii) of Leggett-williams fixed-point theorem holds.

To check condition (i) of Leggett-Williams fixed-point theorem, choose $u(t) = b/\gamma_\delta$, $t \in [0,1]$. It is easy to see that $u \in P(\alpha, b, b/\gamma_\delta)$ and $\alpha(u) = \alpha(b/\gamma_\delta) > b$. so,

$$\left\{u \in P\left(\alpha, b, \frac{b}{\gamma_\delta}\right) : \alpha(u) > b\right\} \ne \emptyset. \tag{3.4}$$

Hence, if $u \in P(\alpha, b, b/\gamma_\delta)$, then $b \le u(t) \le b/\gamma_\delta$, $t \in [\delta, 1-\delta]$. From assumption (S2) and Remark 2.8, we have

$$\begin{aligned} \min_{t\in[\delta,1-\delta]} Tu(t) &= \min_{t\in[\delta,1-\delta]}\int_0^1 G(t,s)q(s)h(s)f(s,u(s))ds \\ &\ge \min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} G(t,s)q(s)h(s)f(s,u(s))ds \\ &> \frac{b}{\eta}\cdot\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} G(t,s)q(s)h(s)ds \\ &> \frac{bD_1D_2}{\eta\rho}\min_{t\in[\delta,1-\delta]}\gamma(t)\int_\delta^{1-\delta} s(1-s)q(s)h(s)ds \\ &= b. \end{aligned} \tag{3.5}$$

Finally, we assert that if $u \in P(\alpha, b, c)$ and $\|Tu\| > b/\gamma_\delta$, then $\alpha(Tu) > b$. To see this, suppose that $u \in P(\alpha, b, c)$ and $\|Tu\| > b/\gamma_\delta$, then

$$\begin{aligned} \min_{t\in[\delta,1-\delta]} Tu(t) &= \min_{t\in[\delta,1-\delta]}\int_0^1 G(t,s)q(s)h(s)f(s,u(s))ds \\ &\ge \gamma_\delta\int_0^1 G(s,s)q(s)h(s)f(s,u(s))ds \\ &\ge \gamma_\delta\max_{t\in[0,1]}\int_0^1 G(s,s)q(s)h(s)f(s,u(s))ds \\ &= \gamma_\delta\|Tu\| > b. \end{aligned} \tag{3.6}$$

To sum up, all the conditions of Leggett-williams fixed-point theorem are satisfied. Therefore, T has at least three fixed points, that is, problem (1.1) has at least three positive solutions u_1, u_2, u_3 in $\overline{P}_c$ satisfying $\|u_1\| < a,\ b < \alpha(u_2),\ \|u_3\| > a$ and $\alpha(u_3) < b$. □

Theorem 3.2. *Assume that (H1)–(H4) hold. Let* $0 < a_1 < b_1 < b_1/\gamma_\delta < a_2 < b_2 < b_2/\gamma_\delta < \cdots < a_n,\ n \in \mathbb{N}$, *and suppose that* f *satisfies the following conditions:*

(A1) $f(t,u) < ma_i$, *for* $t \in [0,1],\ u \in [0,a_i],\ 1 \le i \le n$;

(A2) $f(t,u) > b_i/\eta$, *for* $t \in [\delta, 1-\delta],\ u \in [b_i, b_i/\gamma_\delta],\ 1 \le i \le n-1$.

Then the boundary value problem (1.1) *has at least* $2n-1$ *positive solutions.*

Proof. When $n = 1$, it follows from condition (A1) that $T : \overline{P}_{a_1} \to \overline{P}_{a_1}$, which means that T has at least one fixed point $u_1 \in \overline{P}_{a_1}$ by the Schauder fixed-point Theorem. When $n = 2$, it is clear that Theorem 3.1 holds (with $c_1 = a_2$). Then we can obtain at least three positive solutions u_1, u_2, and u_3 satisfying $\|u_1\| \le a_1,\ \psi(u_2) > b_1$ and $\|u_3\| > a_1$ with $\psi(u_3) < b_1$. Following this way, we finish the proof by the induction method. □

4. Example

Consider the following boundary value problem:

$$\begin{gathered} u''(t) - 2u'(t) - 3u(t) + h(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{4.1}$$

where

$$\begin{gathered} h(t) = \frac{1}{t(1-t)}, \\ f(u) = \begin{cases} \dfrac{u}{4}, & u \in [0,1], \\ \dfrac{41u}{4} - 10, & u \in [1,2], \\ \dfrac{21}{2}, & u \in [2,30], \\ \dfrac{(u-9)(1+2u)}{2(31+u)}, & u \ge 30. \end{cases} \end{gathered} \tag{4.2}$$

Then, by computation, we have

$$\begin{gathered} \varphi(t) = \frac{1}{4}\left(e^{t} - e^{-3t}\right), \quad \psi(t) = \frac{1}{4}\left(e^{(1-t)} - e^{-3(1-t)}\right), \\ \rho = \frac{1}{8e^3}\left(e^2 - 1\right)\left(e^2 + 3\right), \quad q(t) = e^{-2t+1}. \end{gathered} \tag{4.3}$$

Furthermore, for $t \in [0,1]$,

$$\frac{\sqrt{3}}{3}t \le \varphi(t) \le t, \quad \frac{\sqrt{3}}{3}(1-t) \le \psi(t) \le 1-t. \tag{4.4}$$

In fact, let $\alpha(t) = e^t - e^{-3t} - (4\sqrt{3}/3)t$, then $\alpha(0) = 0$, and $\alpha'(t) = e^t + 3e^{-3t} - 4\sqrt{3}/3$. It is easy to compute that

$$\min_{t\in[0,1]} \left(e^t + 3e^{-3t}\right) = \frac{4\sqrt{3}}{3}. \tag{4.5}$$

Then, $\alpha(t) \ge 0,\ t \in [0,1]$, that is

$$\varphi(t) \ge \frac{\sqrt{3}}{3}t, \quad t \in [0,1]. \tag{4.6}$$

The other inequalities in (4.4) can be proved by the same method.

Thus, we can choose that $C_1 = C_2 = 1$, $D_1 = D_2 = \sqrt{3}/3$ and $\delta = \ln\sqrt{2}$. By computation, we have

$$\begin{gathered}\gamma(t) = \frac{4e^3}{e^4-1}\min\{\varphi(t), \psi(t)\}, \\ \gamma_\delta = \min_{t\in[\delta,1-\delta]}\gamma(t) = \frac{3\sqrt{2}e^3}{4\left(e^4-1\right)} \approx 0.39747\ldots.\end{gathered} \tag{4.7}$$

Let $a = 1/4,\ b = 2$, and $c = 20$. Then, we can compute

$$\begin{gathered}m = \frac{e^3+3}{4e^2} \approx 0.78107\cdots, \\ \eta = \frac{\sqrt{2}e^5\left(e^2-4\right)}{\left(e^2-1\right)\left(e^4-1\right)\left(e^2+3\right)} \approx 0.19994\cdots.\end{gathered} \tag{4.8}$$

Consequently,

$$\begin{gathered}f(u) \le \frac{1}{4}\cdot\frac{1}{4} < \frac{1}{4}m = ma, \quad u \in \left[0, \frac{1}{4}\right], \\ f(u) = \frac{21}{2} > \frac{2}{\eta} = \frac{b}{\eta}, \quad u \in \left[2, \frac{2}{\gamma_\delta}\right] \subset [2,10], \\ f(u) \le \frac{21}{2} < 20m, \quad u \in [0,20].\end{gathered} \tag{4.9}$$

Therefore, all the conditions of Theorem 3.1 are satisfied, then problem (4.1) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\| \leq \frac{1}{4}, \qquad \psi(u_2) > 2, \qquad \|u_3\| > \frac{1}{4} \quad \text{with } \psi(u_3) < 2. \tag{4.10}$$

Acknowledgment

The first author was partially supported by NNSF of China (10901075).

References

[1] D. O'Regan, "Singular Dirichlet boundary value problems. I. Superlinear and nonresonant case," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 29, no. 2, pp. 221–245, 1997.

[2] H. Asakawa, "Nonresonant singular two-point boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 44, no. 6, pp. 791–809, 2001.

[3] R. Dalmasso, "Positive solutions of singular boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 27, no. 6, pp. 645–652, 1996.

[4] D. R. Dunninger and H. Y. Wang, "Multiplicity of positive radial solutions for an elliptic system on an annulus," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 42, no. 5, pp. 803–811, 2000.

[5] K. S. Ha and Y.-H. Lee, "Existence of multiple positive solutions of singular boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 28, no. 8, pp. 1429–1438, 1997.

[6] P. Habets and F. Zanolin, "Upper and lower solutions for a generalized Emden-Fowler equation," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 3, pp. 684–700, 1994.

[7] C.-G. Kim and Y.-H. Lee, "Existence and multiplicity results for nonlinear boundary value problems," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2870–2886, 2008.

[8] F. H. Wong, "Existence of positive solutions of singular boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 21, no. 5, pp. 397–406, 1993.

[9] X. Xu and J. P. Ma, "A note on singular nonlinear boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 293, no. 1, pp. 108–124, 2004.

[10] X. J. Yang, "Positive solutions for nonlinear singular boundary value problems," *Applied Mathematics and Computation*, vol. 130, no. 2-3, pp. 225–234, 2002.

[11] X. M. He and W. G. Ge, "Triple solutions for second-order three-point boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 268, no. 1, pp. 256–265, 2002.

[12] D. X. Zhao, H. Z. Wang, and W. G. Ge, "Existence of triple positive solutions to a class of p-Laplacian boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 972–983, 2007.

[13] B. G. Zhang and X. Y. Liu, "Existence of multiple symmetric positive solutions of higher order Lidstone problems," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 2, pp. 672–689, 2003.

[14] J. R. Graef, J. Henderson, P. J. Y. Wong, and B. Yang, "Three solutions of an nth order three-point focal type boundary value problem," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 10, pp. 3386–3404, 2008.