Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 516260, 13 pages
doi:10.1155/2010/516260

Research Article

A New Conservative Difference Scheme for the General Rosenau-RLW Equation

Jin-Ming Zuo,[1] Yao-Ming Zhang,[1] Tian-De Zhang,[2] and Feng Chang[2]

[1] *School of Science, Shandong University of Technology, Zibo 255049, China*
[2] *School of Mathematics, Shandong University, Jinan 250100, China*

Correspondence should be addressed to Jin-Ming Zuo, zuojinming@sdut.edu.cn

Received 28 May 2010; Accepted 14 October 2010

Academic Editor: Colin Rogers

A new conservative finite difference scheme is presented for an initial-boundary value problem of the general Rosenau-RLW equation. Existence of its difference solutions are proved by Brouwer fixed point theorem. It is proved by the discrete energy method that the scheme is uniquely solvable, unconditionally stable, and second-order convergent. Numerical examples show the efficiency of the scheme.

1. Introduction

In this paper, we consider the following initial-boundary value problem of the general Rosenau-RLW equation:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + (u^p)_x = 0 \quad (x_l < x < x_r,\ 0 < t < T), \tag{1.1}$$

with an initial condition

$$u(x,0) = u_0(x) \quad (x_l \le x \le x_r), \tag{1.2}$$

and boundary conditions

$$u(x_l,t) = u(x_r,t) = 0, \quad u_{xx}(x_l,t) = u_{xx}(x_r,t) = 0 \quad (0 \le t \le T), \tag{1.3}$$

where $p \geq 2$ is a integer and $u_0(x)$ is a known smooth function. When $p = 2$, (1.1) is called as usual Rosenau-RLW equation. When $p = 3$, (1.1) is called as modified Rosenau-RLW (MRosenau-RLW) equation. The initial boundary value problem (1.1)–(1.3) possesses the following conservative quantities:

$$Q(t) = \frac{1}{2}\int_{x_l}^{x_r} u(x,t)dx = \frac{1}{2}\int_{x_l}^{x_r} u_0(x,t)dx = Q(0), \tag{1.4}$$

$$E(t) = \frac{1}{2}\left(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2\right) = \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|u_{0x}\|_{L^2}^2 + \|u_{0xx}\|_{L^2}^2\right) = E(0). \tag{1.5}$$

It is known the conservative scheme is better than the nonconservative ones. Zhang et al. [1] point out that the nonconservative scheme may easily show nonlinear blow up. In [2] Li and Vu-Quoc said "... in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation". In [3–11], some conservative finite difference schemes were used for a system of the generalized nonlinear Schrödinger equations, Regularized long wave (RLW) equations, Sine-Gordon equation, Klein-Gordon equation, Zakharov equations, Rosenau equation, respectively. Numerical results of all the schemes are very good. Hence, we propose a new conservative difference scheme for the general Rosenau-RLW equation, which simulates conservative laws (1.4) and (1.5) at the same time. The outline of the paper is as follows. In Section 2, a nonlinear difference scheme is proposed and corresponding convergence and stability of the scheme are proved. In Section 3, some numerical experiments are shown.

2. A Nonlinear-Implicit Conservative Scheme

In this section, we propose a nonlinear-implicit conservative scheme for the initial-boundary value problem (1.1)–(1.3) and give its numerical analysis.

2.1. The Nonlinear-Implicit Scheme and Its Conservative Law

For convenience, we introduce the following notations

$$x_j = x_r + jh, \quad t_n = n\tau, \quad j = 0,1,\ldots,J, \quad n = 0,1,\ldots, \quad \left[\frac{T}{\tau}\right] = N, \tag{2.1}$$

where $h = (x_r - x_l)/J$ and τ denote the spatial and temporal mesh sizes, $u_j^n \equiv u(x_j, t_n)$, $U_j^n \approx u(x_j, t_n)$, respectively,

$$\left(U_j^n\right)_t = \frac{U_j^{n+1} - U_j^n}{\tau}, \quad \left(U_j^n\right)_x = \frac{U_{j+1}^n - U_j^n}{h}, \quad \left(U_j^n\right)_{\overline{x}} = \frac{U_j^n - U_{j-1}^n}{h},$$
$$\left(U_j^n\right)_{\hat{x}} = \frac{1}{2}\left(\left(U_j^n\right)_x + \left(U_j^n\right)_{\overline{x}}\right), \quad U_j^{n+1/2} = \frac{1}{2}\left(U_j^{n+1} + U_j^n\right), \quad \langle U^n, V^n\rangle = h\sum_{j=1}^{J-1} U_j^n V_j^n,$$
$$\|U^n\|^2 = \langle U^n, U^n\rangle, \quad \|U^n\|_\infty = \max_{1\leq j\leq J}\left|U_j^n\right|, \tag{2.2}$$

and in the paper, C denotes a general positive constant, which may have different values in different occurrences.

Since $(u^p)_x = (2/(p+1))\sum_{i=0}^{p-1} u^i (u^{p-i})_x$, then the finite difference scheme for the problem (1.1)–(1.3) is written as follows:

$$\left(U_j^n\right)_t - \left(U_j^n\right)_{x\overline{x}t} + \left(U_j^n\right)_{xx\overline{x}\overline{x}t} + \left(U_j^{n+1/2}\right)_{\hat{x}} + \frac{2}{p+1}\sum_{i=0}^{p-1}\left(U_j^{n+1/2}\right)^i\left[\left(U_j^{n+1/2}\right)^{p-i}\right]_{\hat{x}} = 0,$$
$$j = 1,2,\ldots,J-1; \quad n = 1,2,\ldots,N, \tag{2.3}$$

$$U_j^0 = u_0(x_j), \quad j = 0,1,2,\ldots,J, \tag{2.4}$$

$$U_0^n = U_J^n = 0, \quad \left(U_0^n\right)_{x\overline{x}} = \left(U_J^n\right)_{x\overline{x}} = 0, \quad n = 1,2,\ldots,N. \tag{2.5}$$

Lemma 2.1 (see [12]). *For any two mesh functions, $U, V \in Z_h^0$, one has*

$$\begin{aligned}
\langle (U)_x, V\rangle &= -\langle U, (V)_{\overline{x}}\rangle,\\
\langle (U)_{\hat{x}}, V\rangle &= -\langle U, (V)_{\hat{x}}\rangle,\\
\langle V, (U)_{x\overline{x}}\rangle &= -\langle (V)_x, (U)_x\rangle,\\
\langle U, (U)_{x\overline{x}}\rangle &= -\langle (U)_x, (U)_x\rangle = -\|U_x\|^2.
\end{aligned} \tag{2.6}$$

Furthermore, if $(U_0^n)_{x\overline{x}} = (U_J^n)_{x\overline{x}} = 0$, then

$$\langle U, (U)_{xx\overline{x}\overline{x}}\rangle = \|U_{xx}\|^2. \tag{2.7}$$

Theorem 2.2. *Suppose that $u_0 \in H_0^2[x_l, x_r]$, then scheme* (2.3)–(2.5) *is conservative in the senses:*

$$Q^n = \frac{h}{2}\sum_{j=1}^{J-1} U_j^n = Q^{n-1} = \cdots = Q^0, \tag{2.8}$$

$$E^n = \frac{1}{2}\|U^n\|^2 + \frac{1}{2}\|U_x^n\|^2 + \frac{1}{2}\|U_{xx}^n\|^2 = E^{n-1} = \cdots = E^0. \tag{2.9}$$

Proof. Multiplying (2.3) with $h/2$, according to boundary condition (2.5), and then summing up for j from 1 to $J-1$, we have

$$\frac{h}{2}\sum_{j=1}^{J-1}\left(U_j^{n+1} - U_j^n\right) = 0. \tag{2.10}$$

Let

$$Q^n = \frac{h}{2}\sum_{j=1}^{J-1} U_j^n. \tag{2.11}$$

Then (2.8) is gotten from (2.10).

Computing the inner product of (2.3) with $U^{n+1/2}$, according to boundary condition (2.5) and Lemma 2.1, we obtain

$$\frac{1}{2}\|U^n\|_t^2 + \frac{1}{2}\|U_x^n\|_t^2 + \frac{1}{2}\|U_{xx}^n\|_t^2 + \left\langle \left(U^{n+1/2}\right)_{\hat{x}}, U^{n+1/2} \right\rangle + \left\langle \kappa\left(U^{n+1/2}, U^{n+1/2}\right), U^{n+1/2} \right\rangle = 0, \tag{2.12}$$

where

$$\begin{gathered} \kappa\left(U^{n+1/2}, U^{n+1/2}\right) = \frac{2}{p+1}\sum_{i=0}^{p-1}\left(U^{n+1/2}\right)^i \left[\left(U^{n+1/2}\right)^{p-i}\right]_{\hat{x}}, \\ U^{n+1/2} = \frac{1}{2}\left(U^{n+1} + U^n\right). \end{gathered} \tag{2.13}$$

According to

$$\begin{aligned} &\left\langle \left(U^{n+1/2}\right)_{\hat{x}}, U^{n+1/2} \right\rangle = 0, \\ \langle \kappa(U^{n+1/2}, U^{n+1/2}), U^{n+1/2} \rangle &= \frac{2}{p+1}\left\langle \sum_{i=0}^{p-1} (U^{n+1/2})^i \left[(U^{n+1/2})^{p-i}\right]_{\hat{x}}, U^{n+1/2} \right\rangle \\ &= -\frac{2}{p+1}\left\langle \sum_{i=0}^{p-1} \left[(U^{n+1/2})^{i+1}\right]_{\hat{x}}, (U^{n+1/2})^{p-i} \right\rangle \\ &= -\frac{2}{p+1}\left\langle \sum_{i=0}^{p-1} (U^{n+1/2})^i \left[(U^{n+1/2})^{p-i}\right]_{\hat{x}}, U^{n+1/2} \right\rangle, \end{aligned} \tag{2.14}$$

we have $\langle \kappa(U^{n+1/2}, U^{n+1/2}), U^{n+1/2} \rangle = 0$. It follows from (2.12) that

$$\frac{1}{2}\|U^n\|_t^2 + \frac{1}{2}\|U_x^n\|_t^2 + \frac{1}{2}\|U_{xx}^n\|_t^2 = 0. \tag{2.15}$$

Let

$$E^n = \frac{1}{2}\|U^n\|^2 + \frac{1}{2}\|U_x^n\|^2 + \frac{1}{2}\|U_{xx}^n\|^2. \tag{2.16}$$

Then (2.9) is gotten from (2.15). This completes the proof of Theorem 2.2. □

2.2. Existence and Prior Estimates of Difference Solution

To show the existence of the approximations U^n $(n = 1, 2, \ldots, N)$ for scheme (2.3)–(2.5), we introduce the following *Brouwer* fixed point theorem [13].

Lemma 2.3. *Let H be a finite-dimensional inner product space, $\|\cdot\|$ be the associated norm, and $g : H \to H$ be continuous. Assume, moreover, that there exist $\alpha > 0$, for all $z \in H$, $\|z\| = \alpha$, $\langle \omega(z), z\rangle > 0$. Then, there exists a $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \le \alpha$.*

Let $Z_h^0 = \{v = (v_j) \mid v_0 = v_J = (v_0)_{x\overline{x}} = (v_J)_{x\overline{x}} = 0,\ j = 0, 1, \ldots, J\}$, then have the following.

Theorem 2.4. *There exists $U^{n+1} \in Z_h^0$ which satisfies scheme* (2.3)–(2.5)*.*

Proof (by Brouwer fixed point theorem). It follows from the original problem (1.1)–(1.3) that U^0 satisfies scheme (2.3)–(2.5). Assume there exists $U^1, U^2, \ldots, U^n \in Z_h^0$ which satisfy scheme (2.3)–(2.5), as $n \le N - 1$, now we try to prove that $U^{n+1} \in Z_h^0$, satisfy scheme (2.3)–(2.5).

We define ω on Z_h^0 as follows:

$$\omega(v) = 2v - 2U^n - 2v_{x\overline{x}} + 2U_{x\overline{x}} + 2v_{xx\overline{x}\overline{x}} - 2U_{xx\overline{x}\overline{x}} + \tau v_{\hat{x}} + \tau\kappa(v, v), \tag{2.17}$$

where $\kappa(v, v) = (2/(p+1)) \sum_{i=0}^{p-1} v^i (v^{p-i})_{\hat{x}}$. Computing the inner product of (2.17) with v and considering $\langle \kappa(v, v), v\rangle = 0$ and $\langle v_{\hat{x}}, v\rangle = 0$, we obtain

$$\begin{aligned}
\langle \omega(v), v\rangle &= 2\|v\|^2 + 2\|v_x\|^2 + 2\|v_{xx}\|^2 - 2\langle U^n, v\rangle + 2\langle U^n_{x\overline{x}}, v\rangle - 2\langle U^n_{xx\overline{x}\overline{x}}, v\rangle \\
&\ge 2\|v\|^2 + 2\|v_x\|^2 + 2\|v_{xx}\|^2 - \left(\|U^n\|^2 + \|v\|^2\right) \\
&\quad - \left(\|U^n_x\|^2 + \|v_x\|^2\right) - \left(\|U^n_{xx}\|^2 + \|v_{xx}\|^2\right) \\
&= \|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 - \left(\|U^n\|^2 + \|U^n_x\|^2 + \|U^n_{xx}\|^2\right) \\
&\ge \|v\|^2 - \left(\|U^n\|^2 + \|U^n_x\|^2 + \|U^n_{xx}\|^2\right).
\end{aligned} \tag{2.18}$$

Hence, for all $v \in Z_h^0$, $\|v\|^2 = \|U^n\|^2 + \|U^n_x\|^2 + \|U^n_{xx}\|^2 + 1$ there exists $\langle \omega(v), v\rangle \ge 0$. It follows from Lemma 2.3 that exists $v^* \in Z_h^0$ which satisfies $\omega(v^*) = 0$. Let $U^{n+1} = 2v - U^n$, then it can be proved that $U^{n+1} \in Z_h^0$ is the solution of scheme (2.3)–(2.5). This completes the proof of Theorem 2.4. □

Next we will give some priori estimates of difference solutions. First the following two lemmas [14] are introduced:

Lemma 2.5 (discrete Sobolev's estimate). *For any discrete function $\{u_j^n \mid j = 0, 1, \ldots, J\}$ on the finite interval $\{x_l, x_r\}$, there is the inequality*

$$\|u^n\|_\infty \le \varepsilon\|u^n_x\| + C(\varepsilon)\|u^n\|, \tag{2.19}$$

where $\varepsilon, C(\varepsilon)$ are two constants independent of $\{u_j^n \mid j = 0, 1, \ldots, J\}$ and step length h.

Lemma 2.6 (discrete Gronwall's inequality). *Suppose that the discrete function* $\{w^n \mid n = 0,1,\dots,N\}$ *satisfies the inequality*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n\tau, \tag{2.20}$$

where A, B *and* C_n $(n = 0,1,2,\dots,N)$ *are nonnegative constants. Then*

$$\max_{1\le n\le N}|w^n| \le \left(w^0 + \tau\sum_{l=1}^{N} C_l\right)e^{2(A+B)T}, \tag{2.21}$$

where τ *is sufficiently small, such that* $(A+B)\tau \le (N-1)/2N, (N>1)$.

Theorem 2.7. *Suppose that* $u_0 \in H_0^2[x_l, x_r]$, *then the following inequalities*

$$\|U^n\| \le C, \qquad \|U_x^n\| \le C, \qquad \|U^n\|_\infty \le C, \qquad \|U_{xx}^n\| \le C. \tag{2.22}$$

hold.

Proof. It is follows from (2.9) that

$$\|U^n\| \le C, \qquad \|U_x^n\| \le C, \qquad \|U_{xx}^n\| \le C. \tag{2.23}$$

According to Lemma 2.5, we obtain

$$\|U^n\|_\infty \le C. \tag{2.24}$$

This completes the proof of Theorem 2.7. □

Remark 2.8. Theorem 2.7 implies that scheme (2.3)–(2.5) is unconditionally stable.

2.3. Convergence and Uniqueness of Difference Solution

First, we consider the convergence of scheme (2.3)–(2.5). We define the truncation error as follows:

$$r_j^n = \left(u_j^n\right)_t - \left(u_j^n\right)_{x\bar{x}t} + \left(u_j^n\right)_{xx\bar{x}\bar{x}t} + \left(u_j^{n+1/2}\right)_{\hat{x}} + \frac{2}{p+1}\sum_{i=0}^{p-1}\left(u_j^{n+1/2}\right)^i\left[\left(u_j^{n+1/2}\right)^{p-i}\right]_{\hat{x}},$$
$$j = 1,2,\dots,J-1; \quad n = 1,2,\dots,N, \tag{2.25}$$

then from Taylor's expansion, we obtain the following.

Theorem 2.9. *Suppose that $u_0 \in H_0^2[x_l, x_r]$ and $u(x,t) \in C^{5,3}$, then the truncation errors of scheme* (2.3)–(2.5) *satisfy*

$$\left|r_j^n\right| = O\left(\tau^2 + h^2\right), \tag{2.26}$$

as $\tau \to 0$, $h \to 0$.

Theorem 2.10. *Suppose that the conditions of Theorem 2.9 are satisfied, then the solution of scheme* (2.3)–(2.5) *converges to the solution of problem* (1.1)–(1.3) *with order* $O(\tau^2 + h^2)$ *in the* L_∞ *norm.*

Proof. Subtracting (2.3) from (2.25) letting

$$e_j^n = u_j^n - U_j^n, \tag{2.27}$$

we obtain

$$r_j^n = \left(e_j^n\right)_t - \left(e_j^n\right)_{x\overline{x}t} + \left(e_j^n\right)_{xx\overline{x}\overline{x}t} + \left(e_j^{n+1/2}\right)_{\hat{x}} + \kappa\left(u_j^{n+1/2}, u_j^{n+1/2}\right) - \kappa\left(U_j^{n+1/2}, U_j^{n+1/2}\right). \tag{2.28}$$

Computing the inner product of (2.28) with $2e^{n+1/2}$, we obtain

$$\begin{aligned}\left\langle 2r^n, e^{n+1/2}\right\rangle &= \|e^n\|_t^2 + \|e_x^n\|_t^2 + \|e_{xx}^n\|_t^2 + 2\left\langle \left(e_j^{n+1/2}\right)_{\hat{x}}, e_j^{n+1/2}\right\rangle \\ &\quad + 2\left\langle \kappa\left(u_j^{n+1/2}, u_j^{n+1/2}\right) - \kappa\left(U_j^{n+1/2}, U_j^{n+1/2}\right), e^{n+1/2}\right\rangle.\end{aligned} \tag{2.29}$$

From the conservative property (1.5), it can be proved by Lemma 2.5 that $\|u\|_{L_\infty} \le C$. Then by Theorem 2.7 we can estimate (2.29) as follows:

$$\begin{aligned}&\left\langle \kappa\left(u_j^{n+1/2}, u_j^{n+1/2}\right) - \kappa\left(U_j^{n+1/2}, U_j^{n+1/2}\right), e^{n+1/2}\right\rangle \\ &\quad = \frac{2}{p+1} h \sum_{j=1}^{J-1} \left\{ \sum_{i=0}^{p-1} \left(u_j^{n+1/2}\right)^i \left[\left(u_j^{n+1/2}\right)^{p-i}\right]_{\hat{x}} - \sum_{i=0}^{p-1} \left(U_j^{n+1/2}\right)^i \left[\left(u_j^{n+1/2}\right)^{p-i}\right]_{\hat{x}} \right\} \left(e_j^{n+1/2}\right) \\ &\quad = \frac{2}{p+1} h \sum_{j=1}^{J-1} \left\{ \sum_{i=0}^{p-1} \left(e_j^{n+1/2}\right)^i \sum_{r=0}^{i-1} \left[\left(u_j^{n+1/2}\right)^{i-1-r} \left(U_j^{n+1/2}\right)^r\right] \left[\left(u_j^{n+1/2}\right)^{p-i}\right]_{\hat{x}} \right. \\ &\qquad\qquad \left. - \sum_{i=0}^{p-1} \left(U_i^{n+1/2}\right)^i \left[\left(e_j^{n+1/2}\right) \sum_{r=0}^{p-i-1} \left[\left(u_j^{n+1/2}\right)^{p-i-1-r} \left(U_j^{n+1/2}\right)^r\right]\right] \right\} \left(e_j^{n+1/2}\right) \\ &\quad \le C\left(\|e^n\|^2 + \left\|e^{n+1}\right\|^2 + \|e_x^n\|^2 + \left\|e_x^{n+1}\right\|^2\right).\end{aligned} \tag{2.30}$$

According to the following inequality [11]

$$\|e_x^n\|^2 \le \frac{1}{2}\left(\|e^n\|^2 + \|e_{xx}^n\|^2\right), \qquad \left\|e_x^{n+1}\right\|^2 \le \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e_{xx}^{n+1}\right\|^2\right),$$
$$\left\langle \left(e_j^{n+1/2}\right)_{\hat{x}}, e_j^{n+1/2} \right\rangle = 0, \tag{2.31}$$
$$\left\langle 2r^n, e^{n+1/2} \right\rangle \le \|r^n\|^2 + \left(\|e^n\|^2 + \left\|e^{n+1}\right\|^2\right).$$

Substituting (2.30)–(2.31) into (2.29), we obtain

$$\|e^n\|_t^2 + \|e_x^n\|_t^2 + \|e_{xx}^n\|_t^2 \le \|r^n\|^2 + C\left(\|e^n\|^2 + \left\|e^{n+1}\right\|^2 + \|e_x^n\|^2 + \left\|e_x^{n+1}\right\|^2 + \|e_{xx}^n\|^2 + \left\|e_{xx}^{n+1}\right\|^2\right). \tag{2.32}$$

Let

$$B^n = \|e^n\|^2 + \|e_x^n\|^2 + \|e_{xx}^n\|^2, \tag{2.33}$$

then (2.32) can be rewritten as

$$B^n - B^{n-1} \le C\tau\left(\tau^2 + h^2\right)^2 + C\tau\left(B^n - B^{n-1}\right). \tag{2.34}$$

Choosing suitable τ which is small enough, we obtain by Lemma 2.6 that

$$B^n \le C\left(B^0 + \left(\tau^2 + h^2\right)^2\right). \tag{2.35}$$

From the discrete initial conditions, we know that e^0 is of second-order accuracy, then

$$B^0 = O\left(\tau^2 + h^2\right)^2. \tag{2.36}$$

Then we have

$$\|e^n\| \le O\left(\tau^2 + h^2\right), \qquad \|e_x^n\| \le O\left(\tau^2 + h^2\right), \qquad \|e_{xx}^n\| \le O\left(\tau^2 + h^2\right) \tag{2.37}$$

It follows from Lemma 2.5, we have $\|e^n\|_\infty \le O(\tau^2 + h^2)$. This completes the proof of Theorem 2.10. □

Table 1: The errors of numerical solutions at $t = 60$ with $\tau = h$ for $p = 2$.

h	$\|u^n - U^n\|$	$\|u^n - U^n\|_\infty$	$\|u^{n/4} - U^{n/4}\|/\|u^n - U^n\|$	$\|u^{n/4} - U^{n/4}\|_\infty/\|u^n - U^n\|_\infty$
0.4	$5.476\,327 \times 10^{-2}$	$1.958\,718 \times 10^{-2}$		
0.2	$1.385\,256 \times 10^{-2}$	$4.983\,761 \times 10^{-3}$	3.953 296	3.930 200
0.1	$3.474\,318 \times 10^{-3}$	$1.252\,185 \times 10^{-3}$	3.987 130	3.980 050
0.05	$8.691\,419 \times 10^{-4}$	$3.134\,571 \times 10^{-4}$	3.997 412	3.994 759
0.025	$2.059\,064 \times 10^{-4}$	$7.550\,730 \times 10^{-5}$	4.221 051	4.151 348

Table 2: The errors of numerical solutions at $t = 60$ with $\tau = h$ for $p = 3$.

h	$\|u^n - U^n\|$	$\|u^n - U^n\|_\infty$	$\|u^{n/4} - U^{n/4}\|/\|u^n - U^n\|$	$\|u^{n/4} - U^{n/4}\|_\infty/\|u^n - U^n\|_\infty$
0.4	$1.164\,674 \times 10^{-1}$	$4.251\,029 \times 10^{-2}$		
0.2	$2.940\,136 \times 10^{-2}$	$1.080\,424 \times 10^{-2}$	3.961 294	3.934 592
0.1	$7.357\,052 \times 10^{-3}$	$2.708\,996 \times 10^{-3}$	3.996 350	3.988 283
0.05	$1.837\,759 \times 10^{-3}$	$6.772\,212 \times 10^{-4}$	4.003 273	4.000 165
0.025	$4.283\,535 \times 10^{-4}$	$1.596\,208 \times 10^{-4}$	4.290 286	4.242 688

Theorem 2.11. *Scheme* (2.3)–(2.5) *is uniquely solvable.*

Proof. Assume that U^n and U'^n both satisfy scheme (2.3)–(2.5), let $W^n = U^n - U'^n$, we obtain

$$\begin{aligned}\left(W_j^n\right)_t - \left(W_j^n\right)_{x\bar{x}t} + \left(W_j^n\right)_{xx\bar{x}\bar{x}t} + \left(U_j^{n+1/2}\right)_{\hat{x}} - \left(U_j'^{n+1/2}\right)_{\hat{x}} \\ +\kappa\left(U_j^{n+1/2}, U_j^{n+1/2}\right) - \kappa\left(U_j'^{n+1/2}, U_j'^{n+1/2}\right) = 0, \\ W_j^0 = 0 \quad (j = 0, 1, \ldots, N).\end{aligned} \tag{2.38}$$

Similarly to the proof of Theorem 2.10, we have

$$\|W^n\|^2 + \|W_x^n\|^2 + \|W_{xx}^n\|^2 = 0. \tag{2.39}$$

This completes the proof of Theorem 2.11. □

Remark 2.12. All results above in this paper are correct for initial-boundary value problem of the general Rosenau-RLW equation with finite or infinite boundary.

3. Numerical Experiments

In order to test the correction of the numerical analysis in this paper, we consider the following initial-boundary value problems of the general Rosenau-RLW equation:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + (u^p)_x = 0 \quad (0 < t < T), \tag{3.1}$$

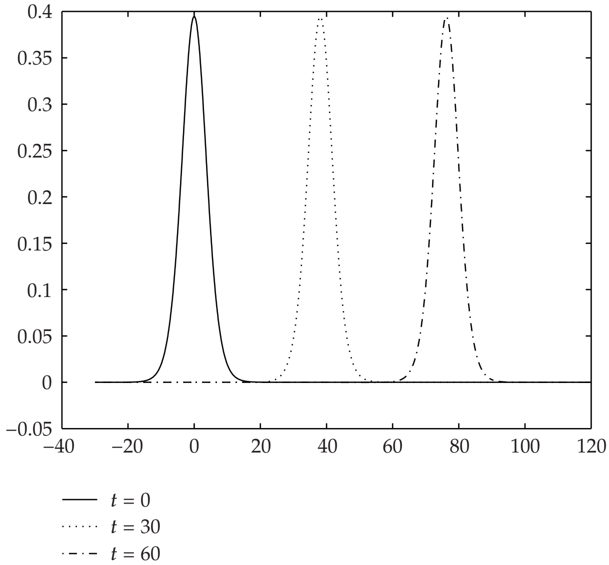

Figure 1: Exact solutions of $u(x,t)$ at $t = 0$ and numerical solutions computed by scheme (2.3)–(2.5) at $t = 30, 60$ for $p = 2$.

Table 3: The errors of numerical solutions at $t = 60$ with $\tau = h$ for $p = 6$.

h	$\|u^n - U^n\|$	$\|u^n - U^n\|_\infty$	$\|u^{n/4} - U^{n/4}\| / \|u^n - U^n\|$	$\|u^{n/4} - U^{n/4}\|_\infty / \|u^n - U^n\|_\infty$
0.4	$1.787\,127 \times 10^{-1}$	$6.353\,868 \times 10^{-2}$		
0.2	$4.598\,952 \times 10^{-2}$	$1.649\,585 \times 10^{-2}$	3.885 945	3.851 797
0.1	$1.156\,944 \times 10^{-2}$	$4.159\,339 \times 10^{-3}$	3.975 084	3.965 980
0.05	$2.892\,147 \times 10^{-3}$	$1.040\,878 \times 10^{-3}$	4.000 294	3.995 992
0.025	$6.585\,307 \times 10^{-4}$	$2.375\,782 \times 10^{-4}$	4.391 818	4.381 199

Table 4: Discrete mass Q^n and discrete energy E^n with $\tau = h = 0.1$ at various t for $p = 2$.

	Q^n	E^n
10	1.897 658 262 960 01	0.533 175 231 580 85
20	1.897 658 268 873 21	0.533 175 231 872 51
30	1.897 658 262 993 93	0.533 175 231 177 25
40	1.897 658 265 568 93	0.533 175 231 478 09
50	1.897 658 260 975 87	0.533 175 231 776 18
60	1.897 658 265 384 88	0.533 175 231 074 05

Table 5: Discrete mass Q^n and discrete energy E^n with $\tau = h = 0.1$ at various t for $p = 3$.

	Q^n	E^n
10	2.672 608 675 265 30	1.113 462 678 852 70
20	2.672 608 676 236 58	1.113 462 678 465 22
30	2.672 608 674 147 13	1.113 462 678 083 94
40	2.672 608 672 639 88	1.113 462 678 711 58
50	2.672 608 672 874 71	1.113 462 678 330 06
60	2.672 608 679 729 44	1.113 462 678 958 71

Table 6: Discrete mass Q^n and discrete energy E^n with $\tau = h = 0.1$ at various t for $p = 6$.

	Q^n	E^n
10	3.988 663 320 390 89	1.917 613 014 656 71
20	3.988 663 260 854 26	1.917 613 014 739 89
30	3.988 663 167 685 49	1.917 613 014 820 83
40	3.988 663 194 506 97	1.917 613 014 927 44
50	3.988 663 973 359 89	1.917 613 014 009 71
60	3.988 663 621 972 59	1.917 613 014 679 17

with an initial condition

$$u(x,0) = u_0(x), \tag{3.2}$$

and boundary conditions

$$u(x_l,t) = u(x_r,t) = 0, \quad u_{xx}(x_l,t) = u_{xx}(x_r,t) = 0 \quad (0 \le t \le T), \tag{3.3}$$

where $u_0(x) = e^{\ln\{(p+3)(3p+1)(p+1)/[2(p^2+3)(p^2+4p+7)]\}/(p-1)}\mathrm{sech}^{4/(p-1)}((p-1/\sqrt{4p^2+8p+20})x)$. Then the exact solution of the initial value problem (3.1)-(3.2) is

$$u(x,t) = e^{\ln\{(p+3)(3p+1)(p+1)/[2(p^2+3)(p^2+4p+7)]\}/(p-1)}\mathrm{sech}^{4/(p-1)}\left[\frac{p-1}{\sqrt{4p^2+8p+20}}(x-ct)\right], \tag{3.4}$$

where $c = (p^4+4p^3+14p^2+20p+25)/(p^4+4p^3+10p^2+12p+21)$ is wave velocity.

It follows from (3.4) that the initial-boundary value problem (3.1)–(3.3) is consistent to the boundary value problem (3.3) for $-x_l \gg 0, x_r \gg 0$. In the following examples, we always choose $x_l = -30$, $x_r = 120$.

Tables 1, 2, and 3 give the errors in the sense of L_2-norm and L_∞-norm of the numerical solutions under various steps of τ and h at $t = 60$ for $p = 2, 3$ and 6. The three tables verify the second-order convergence and good stability of the numerical solutions. Tables 4, 5, and 6 shows the conservative law of discrete mass Q^n and discrete energy E^n computed by scheme (2.3)–(2.5) for $p = 2, 3$ and 6.

Figures 1, 2, and 3 plot the exact solutions at $t = 0$ and the numerical solutions computed by scheme (2.3)–(2.5) with $\tau = h = 0.1$ at $t = 30, 60$, which also show the accuracy of scheme (2.3)–(2.5).

Acknowledgments

The authors would like to express their sincere thanks to the referees for their valuable suggestions and comments. This paper is supported by the National Natural Science Foundation of China (nos. 10871117 and 10571110).

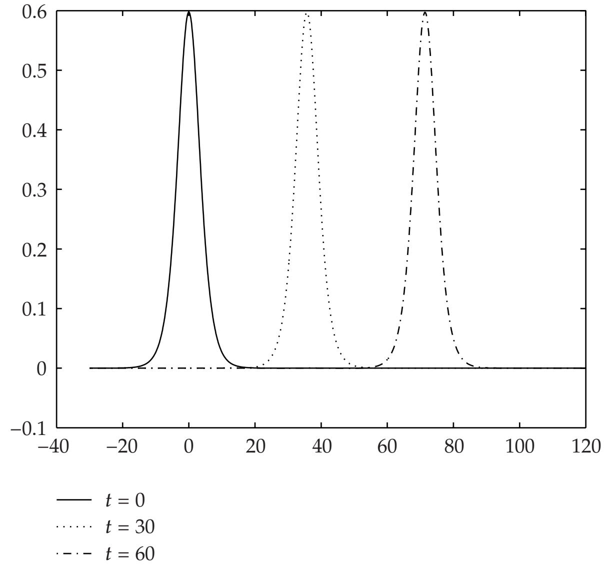

Figure 2: Exact solutions of $u(x,t)$ at $t = 0$ and numerical solutions computed by scheme (2.3)–(2.5) at $t = 30, 60$ for $p = 3$.

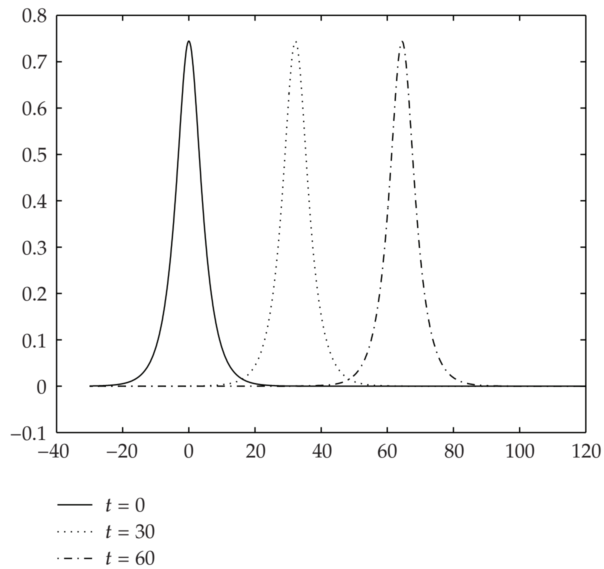

Figure 3: Exact solutions of $u(x,t)$ at $t = 0$ and numerical solutions computed by scheme (2.3)–(2.5) at $t = 30, 60$ for $p = 6$.

References

[1] F. Zhang, V. M. Pérez-García, and L. Vázquez, "Numerical simulation of nonlinear Schrödinger systems: a new conservative scheme," *Applied Mathematics and Computation*, vol. 71, no. 2-3, pp. 165–177, 1995.

[2] S. Li and L. Vu-Quoc, "Finite difference calculus invariant structure of a class of algorithms for the nonlinear Klein-Gordon equation," *SIAM Journal on Numerical Analysis*, vol. 32, no. 6, pp. 1839–1875, 1995.

[3] Q. Chang and L. Xu, "A numerical method for a system of generalized nonlinear Schrödinger equations," *Journal of Computational Mathematics*, vol. 4, no. 3, pp. 191–199, 1986.
[4] Q. Chang, E. Jia, and W. Sun, "Difference schemes for solving the generalized nonlinear Schrödinger equation," *Journal of Computational Physics*, vol. 148, no. 2, pp. 397–415, 1999.
[5] T.-C. Wang and L.-M. Zhang, "Analysis of some new conservative schemes for nonlinear Schrödinger equation with wave operator," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1780–1794, 2006.
[6] T. Wang, B. Guo, and L. Zhang, "New conservative difference schemes for a coupled nonlinear Schrödinger system," *Applied Mathematics and Computation*, vol. 217, no. 4, pp. 1604–1619, 2010.
[7] L. Zhang, "A finite difference scheme for generalized regularized long-wave equation," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 962–972, 2005.
[8] F. Zhang and L. Vázquez, "Two energy conserving numerical schemes for the sine-Gordon equation," *Applied Mathematics and Computation*, vol. 45, no. 1, pp. 17–30, 1991.
[9] Y. S. Wong, Q. Chang, and L. Gong, "An initial-boundary value problem of a nonlinear Klein-Gordon equation," *Applied Mathematics and Computation*, vol. 84, no. 1, pp. 77–93, 1997.
[10] Q. S. Chang, B. L. Guo, and H. Jiang, "Finite difference method for generalized Zakharov equations," *Mathematics of Computation*, vol. 64, no. 210, pp. 537–553, 1995.
[11] J. S. Hu and K. L. Zheng, "Two conservative difference schemes for the generalized Rosenau equation," *Boundary Value Problems*, vol. 2010, Article ID 543503, 18 pages, 2010.
[12] B. Hu, Y. Xu, and J. Hu, "Crank-Nicolson finite difference scheme for the Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 311–316, 2008.
[13] F. E. Browder, "Existence and uniqueness theorems for solutions of nonlinear boundary value problems," in *Proceedings of Symposia in Applied Mathematics, vol. 17*, pp. 24–49, American Mathematical Society, Providence, RI, USA, 1965.
[14] Y. Zhou, *Applications of Discrete Functional Analysis to the Finite Difference Method*, International Academic Publishers, Beijing, China, 1991.