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Research Article

Periodic Problem with a Potential Landesman Lazer Condition

Petr Tomiczek

Department of Mathematics, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic

Correspondence should be addressed to Petr Tomiczek, tomiczek@kma.zcu.cz

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We prove the existence of a solution to the periodic nonlinear second-order ordinary differential equation with damping $u''(x) + r(x)u'(x) + g(x, u(x)) = f(x)$, $u(0) = u(T)$, $u'(0) = u'(T)$. We suppose that $\int_0^T r(x)dx = 0$, the nonlinearity g satisfies the potential Landesman Lazer condition and prove that a critical point of a corresponding energy functional is a solution to this problem.

1. Introduction

Let us consider the nonlinear problem

$$\begin{gathered} u''(x) + r(x)u'(x) + g(x, u(x)) = f(x), \quad x \in [0, T], \\ u(0) = u(T), \quad u'(0) = u'(T), \end{gathered} \tag{1.1}$$

where $r \in L^1(0, T)$, the nonlinearity $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and $f \in L^1(0, T)$.

To state an existence result to (1.1) Amster [1] assumes that r is a nondecreasing function (see also [2]). He supposes that the nonlinearity g satisfies the growth condition $(g(x, s) - g(x, t))/(s - t) \le c_1$, $c_1 < \lambda_1$ for $x \in [0, T]$, $s, t \in \mathbb{R}$, $s \neq t$, where λ_1 is the first eigenvalue of the problem $-u'' = \lambda u$, $u(0) = u(T) = 0$ and there exist a^-, a^+ such that $g|_{[0,T]\times I_{a^+}} \ge \int_0^T p_1(x) f(x) dx / \|p_1\|_1 \ge g|_{[0,T]\times I_{a^-}}$. An interval I_a is centered in a with the radius $\delta_1|a| + \delta_2$ where $\delta_1 = \sqrt{\lambda_1} c_1 T/(\lambda_1 - c_1) < 1$, $0 < \delta_2$ and p_1 is a solution to the problem $p_1' - r p_1 = k_1$, $k_1 \in \mathbb{R}$ with $p_1(0) = p_1(T) = 1$.

In [3, 4] authors studied (1.1) with a constant friction term $r(x) = c$ and results with repulsive singularities were obtained in [5, 6].

In this paper we present new assumptions, we suppose that the friction term r has zero mean value

$$\int_0^T r(x)dx = 0, \tag{1.2}$$

the nonlinearity g is bounded by a L^1 function and satisfies the following potential Landesman-Lazer condition (see also [7, 8])

$$\int_0^T \left[R(x)^2 G_-(x)\right] dx < \int_0^T \left[R(x)^2 f(x)\right] dx < \int_0^T \left[R(x)^2 G_+(x)\right] dx, \tag{1.3}$$

where $G(x,s) = \int_0^s g(x,t)dt$, $G_+(x) = \lim\inf_{s\to+\infty} G(x,s)/s$, $G_-(x) = \lim\sup_{s\to-\infty}(G(x,s)/s)$ and $R(x) = e^{\int_0^x (1/2)r(\xi)d\xi}$.

To obtain our result we use variational approach even if the linearization of the periodic problem (1.1) is a non-self-adjoint operator.

2. Preliminaries

Notation. We will use the classical space $C^k(0,T)$ of functions whose kth derivative is continuous and the space $L^p(0,T)$ of measurable real-valued functions whose pth power of the absolute value is Lebesgue integrable. We denote H the Sobolev space of absolutely continuous functions $u : (0,T) \to \mathbb{R}$ such that $u' \in L^2(0,T)$ and $u(0) = u(T)$ with the norm $\|u\| = (\int_0^T u^2(x) + u'^2(x)dx)^{1/2}$. By a solution to (1.1) we mean a function $u \in C^1(0,T)$ such that u' is absolutely continuous, u satisfies the boundary conditions and (1.1) is satisfied a.e. in $(0,T)$.

We denote $R(x) = e^{\int_0^x (1/2)r(\xi)d\xi}$ and we study (1.1) by using variational methods. We investigate the functional $J : H \to \mathbb{R}$, which is defined by

$$J(u) = \frac{1}{2}\int_0^T \left[R^2 (u')^2\right] dx - \int_0^T \left[R^2 G(x,u) - R^2 f u\right] dx, \tag{2.1}$$

where

$$G(x,s) = \int_0^s g(x,t)\, dt. \tag{2.2}$$

We say that u is a critical point of J, if

$$\left\langle J'(u), v\right\rangle = 0 \quad \forall v \in H. \tag{2.3}$$

We see that every critical point $u \in H$ of the functional J satisfies

$$\int_0^T \left[R^2 u' v'\right] dx - \int_0^T \left[R^2 (g(x,u) - f) v\right] dx = 0 \tag{2.4}$$

for all $v \in H$.

Now we prove that any critical point of the functional J is a solution to (1.1) mentioned above.

Lemma 2.1. *Let the condition* (1.2) *be satisfied. Then any critical point of the functional J is a solution to* (1.1).

Proof. Setting $v = 1$ in (2.4) we obtain

$$\int_0^T \left[R^2 (g(x,u) - f)\right] dx = 0. \tag{2.5}$$

We denote

$$\Phi(x) = \int_0^x \left[R(t)^2 (g(t,u(t)) - f(t))\right] dt \tag{2.6}$$

then previous equality (2.5) implies $\Phi(0) = \Phi(T) = 0$ and by parts in (2.4) we have

$$\int_0^T \left[\left(R^2 u' + \Phi\right) v'\right] dx = 0 \tag{2.7}$$

for all $v \in H$. Hence there exists a constant c_u such that

$$R^2 u' + \Phi = c_u \tag{2.8}$$

on $[0,T]$. The condition (1.2) implies $R(0) = R(T) = 1$ and from (2.8) we get $u'(0) = R^2(0)u'(0) = -\Phi(0) + c_u = -\Phi(T) + c_u = u'(T)$. Using $(R^2)' = R^2 r$ and differentiating equality (2.8) with respect to x we obtain

$$R^2 \left(u'' + r u' + g(x,u) - f\right) = 0. \tag{2.9}$$

Thus u is a solution to (1.1). □

We say that J satisfies the *Palais-Smale condition* (PS) if every sequence (u_n) for which $J(u_n)$ is bounded in H and $J'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

To prove the existence of a critical point of the functional J we use the Saddle Point Theorem which is proved in Rabinowitz [9] (see also [10]).

Theorem 2.2 (Saddle Point Theorem). *Let* $H = \widehat{H} \oplus \widetilde{H}$, $\dim \widehat{H} < \infty$ *and* $\dim \widetilde{H} = \infty$. *Let* $J : H \to \mathbb{R}$ *be a functional such that* $J \in C^1(H, \mathbb{R})$ *and*

(a) *there exists a bounded neighborhood* D *of* 0 *in* $\widehat{H}$ *and a constant* α *such that* $J/\partial D \leq \alpha$,

(b) *there is a constant* $\beta > \alpha$ *such that* $J/\widetilde{H} \geq \beta$,

(c) J *satisfies the Palais-Smale condition (PS).*

Then, the functional J *has a critical point in* H.

3. Main Result

We define

$$G_+(x) = \liminf_{s \to +\infty} \frac{G(x,s)}{s}, \qquad G_-(x) = \limsup_{s \to -\infty} \frac{G(x,s)}{s}. \tag{3.1}$$

Assume that the following potential Landesman-Lazer type condition holds:

$$\int_0^T \left[R(x)^2 G_-(x)\right] dx < \int_0^T \left[R(x)^2 f(x)\right] dx < \int_0^T \left[R(x)^2 G_+(x)\right] dx. \tag{3.2}$$

We also suppose that there exists a function $q(x) \in L^1(0,T)$ such that

$$\left|g(x,s)\right| \leq q(x), \quad x \in [0,T], \ s \in \mathbb{R}. \tag{3.3}$$

Theorem 3.1. *Under the assumptions* (1.2), (3.2), (3.3), *problem* (1.1) *has at least one solution.*

Proof. We verify that the functional J satisfies assumptions of the Saddle Point Theorem 2.2 on H, then J has a critical point u and due to Lemma 2.1 u is the solution to (1.1).

It is easy to see that $J \in C^1(H, \mathbb{R})$. Let $\widetilde{H} = \{u \in H : \int_0^T u(x)dx = 0\}$ then $H = \mathbb{R} \oplus \widetilde{H}$ and $\dim(\widetilde{H}) = \infty$.

In order to check assumption (a), we prove

$$\lim_{|s| \to \infty} J(s) = -\infty \tag{3.4}$$

by contradiction. Then, assume on the contrary there is a sequence of numbers $(s_n) \subset \mathbb{R}$ such that $|s_n| \to \infty$ and a constant c_1 satisfying

$$\liminf_{n \to \infty} J(s_n) \geq c_1. \tag{3.5}$$

From the definition of J and from (3.5) it follows

$$\liminf_{n \to \infty} \int_0^T \frac{R^2(-G(x,s_n) + f s_n)}{|s_n|} dx \geq 0. \tag{3.6}$$

We note that from (3.2) it follows there exist constants s_+, s_- and functions $A_+(x)$, $A_-(x) \in L^1(0,T)$ such that $A_+(x) \le G(x,s)$, $G(x,s) \le A_-(x)$ for a.e. $x \in (0,T)$ and for all $s \ge s_+$, $s \le s_-$, respectively. We suppose that for this moment $s_n \to +\infty$. Using (3.6) and Fatou's lemma we obtain

$$\int_0^T \left[R(x)^2 f(x)\right] dx \ge \int_0^T \left[R(x)^2 G_+(x)\right] dx, \tag{3.7}$$

a contradiction to (3.2). We proceed for the case $s_n \to -\infty$. Then assumption (a) of Theorem 2.2 is verified.

(b) Now we prove that J is bounded from below on $\widetilde{H}$. For $u \in \widetilde{H}$, we have

$$\int_0^T (u')^2 \, dx = \|u\|^2 \tag{3.8}$$

and assumption (3.3) implies

$$|G(x,s)| \le q(x)|s|, \quad x \in [0,T], \ s \in \mathbb{R}. \tag{3.9}$$

Hence and due to compact imbedding $H \subset C(0,T)(\|u\|_{C(0,T)} \le c_2\|u\|)$ we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^T \left[R^2 (u')^2\right] dx - \int_0^T \left[R^2 G(x,u) - R^2 f u\right] dx \\ &\ge \frac{1}{2}\min_{x\in[0,T]} R(x)^2 \int_0^T (u')^2 dx - \max_{x\in[0,T]} R(x)^2 \int_0^T (|q| + |f|)|u| dx \\ &\ge \frac{1}{2}\min_{x\in[0,T]} R(x)^2 \|u\|^2 - \max_{x\in[0,T]} R(x)^2 (\|q\|_1 + \|f\|_1) c_2 \|u\|. \end{aligned} \tag{3.10}$$

Since the function R is strictly positive equality (3.10) implies that the functional J is bounded from below.

Using (3.4), (3.10) we see that there exists a bounded neighborhood D of 0 in $\mathbb{R} = \widehat{H}$, a constant α such that $J/\partial D \le \alpha$, and there is a constant $\beta > \alpha$ such that $J/\widetilde{H} \ge \beta$.

In order to check assumption (c), we show that J satisfies the Palais-Smale condition. First, we suppose that the sequence (u_n) is unbounded and there exists a constant c_3 such that

$$\left| \frac{1}{2}\int_0^T \left[R^2 (u_n')^2\right] dx - \int_0^T \left[R^2 (G(x,u_n) - f u_n)\right] dx \right| \le c_3, \tag{3.11}$$

$$\lim_{n\to\infty} \|J'(u_n)\| = 0. \tag{3.12}$$

Let (w_k) be an arbitrary sequence bounded in H. It follows from (3.12) and the Schwarz inequality that

$$\begin{aligned}&\left|\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^T\left[R^2u_n'\,w_k'\right]dx-\int_0^T\left[R^2\left(g(x,u_n)w_k-fw_k\right)\right]dx\right|\\&\quad=\left|\lim_{\substack{n\to\infty\\k\to\infty}}J'(u_n)w_k\right|\le\lim_{\substack{n\to\infty\\k\to\infty}}\left\|J'(u_n)\right\|\cdot\|w_k\|=0.\end{aligned}\tag{3.13}$$

From (3.3) we obtain

$$\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^T\left[\frac{R^2g(x,u_n)}{\|u_n\|}w_k-\frac{R^2f}{\|u_n\|}w_k\right]dx=0.\tag{3.14}$$

Put $v_n=u_n/\|u_n\|$ and $w_k=v_n$ then (3.13), (3.14) imply

$$\lim_{n\to\infty}\int_0^T\left[R^2\left(v_n'\right)^2\right]dx=0.\tag{3.15}$$

Due to compact imbedding $H\subset C(0,T)$ and (3.15) we have $|v_n|\to d$ in $C(0,T)$, $d>0$. Suppose that $v_n\to d$ and set $w_k=v_n-d$ in (3.13), we get

$$\lim_{n\to\infty}\int_0^T\left[R^2u_n'v_n'\right]dx-\int_0^T\left[R^2\left(g(x,u_n)-f\right)(v_n-d)\right]dx=0.\tag{3.16}$$

Because the nonlinearity g is bounded (assumption (3.3)) and $v_n\to d$ the second integral in previous equality (3.16) converges to zero. Therefore

$$\lim_{n\to\infty}\int_0^T\left[R^2u_n'v_n'\right]dx=0.\tag{3.17}$$

Now we divide (3.11) by $\|u_n\|$. We get

$$\lim_{n\to\infty}\left\{\frac{1}{2}\int_0^T\left[R^2u_n'v_n'\right]dx-\int_0^T\frac{R^2\left(G(x,u_n)-fu_n\right)}{\|u_n\|}dx\right\}=0.\tag{3.18}$$

Equalities (3.17), (3.18) imply

$$\lim_{n\to\infty}\int_0^T R^2\left(-\frac{G(x,u_n)}{u_n}+f\right)v_n dx=0.\tag{3.19}$$

Because $v_n \to d > 0$, $\lim_{n\to\infty} u_n(x) = +\infty$. Using Fatou's lemma and (3.19) we conclude

$$\int_0^T \left[R(x)^2 f(x)\right] dx \geq \int_0^T \left[R(x)^2 G_+(x)\right] dx, \tag{3.20}$$

a contradiction to (3.2). We proceed for the case $v_n \to -d$ similarly. This implies that the sequence (u_n) is bounded. Then there exists $u_0 \in H$ such that $u_n \rightharpoonup u_0$ in H, $u_n \to u_0$ in $L^2(0,T)$, $C(0,T)$ (taking a subsequence if it is necessary). It follows from equality (3.13) that

$$\lim_{\substack{n\to\infty\\ m\to\infty\\ k\to\infty}} \left\{ \int_0^T \left[R^2 (u_n - u_m)' w_k'\right] dx - \int_0^T \left[R^2 \left(g(x,u_n) - g(x,u_m)\right)\right] w_k dx \right\} = 0. \tag{3.21}$$

The strong convergence $u_n \to u_0$ in $C(0,T)$ and the assumption (3.3) imply

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \int_0^T \left[R^2 \left(g(x,u_n) - g(x,u_m)\right)(u_n - u_m)\right] dx = 0. \tag{3.22}$$

If we set $w_k = u_n$, $w_k = u_m$ in (3.21) and subtract these equalities, then using (3.22) we have

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \int_0^T \left[R^2 \left(u_n' - u_m'\right)^2\right] dx = 0. \tag{3.23}$$

Hence we obtain the strong convergence $u_n \to u_0$ in H. This shows that J satisfies the Palais-Smale condition and the proof of Theorem 3.1 is complete. □

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