Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 690342, 21 pages
doi:10.1155/2010/690342

Research Article

Extremal Values of Half-Eigenvalues for p-Laplacian with Weights in L^1 Balls

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Received 24 May 2010; Accepted 21 October 2010

Academic Editor: V. Shakhmurov

For one-dimensional p-Laplacian with weights in $\mathcal{L}^\gamma := L^\gamma([0,1],\mathbb{R})$ $(1 \le \gamma \le \infty)$ balls, we are interested in the extremal values of the mth positive half-eigenvalues associated with Dirichlet, Neumann, and generalized periodic boundary conditions, respectively. It will be shown that the extremal value problems for half-eigenvalues are equivalent to those for eigenvalues, and all these extremal values are given by some best Sobolev constants.

1. Introduction

Occasionally, we need to solve extremal value problems for eigenvalues. A classical example studied by Krein [1] is the infimum and the supremum of the mth Dirichlet eigenvalues of Hill's operator with positive weight

$$\inf\left\{\mu_m^D(w) : w \in E_{r,h}\right\}, \qquad \sup\left\{\mu_m^D(w) : w \in E_{r,h}\right\}, \tag{1.1}$$

where $0 < r \le h < \infty$ and

$$E_{r,h} := \left\{ w \in \mathcal{L}^\gamma : 0 \le w \le h,\ \int_0^1 w(t)dt = r \right\}. \tag{1.2}$$

In this paper, we always use superscripts D, N, P, and G to indicate Dirichlet, Neumann, periodic and generalized periodic boundary value conditions, respectively. Similar extremal value problems for p-Laplacian were studied by Yan and Zhang [2]. For Hill's operator with weight, Lou and Yanagida [3] studied the minimization problem of the positive principal

Neumann eigenvalues, which plays a crucial role in population dynamics. Given constants $\kappa \in (0, \infty)$ and $\alpha \in (0, 1)$, denote

$$S_{\kappa,\alpha} := \left\{ \omega \in \mathcal{L}^\infty : -1 \le \omega \le \kappa, \quad \omega_+ \succ 0, \ \int_0^1 \omega(t)dt \le -\alpha \right\}. \tag{1.3}$$

The positive principal eigenvalue $\mu_0^N(\omega)$ is well-defined for any $\omega \in S_{\kappa,\alpha}$, and the minimization problem in [3] is to find

$$\inf\left\{ \mu_0^N(w) : \omega \in S_{\kappa,\alpha} \right\}. \tag{1.4}$$

In solving the previous three problems, two crucial steps have been employed. The first step is to prove that the extremal values can be attained by some weights. For regular self-adjoint linear Sturm-Liouville problems the continuous dependence of eigenvalues on weights/potentials in the usual L^γ topology is well understood, and so is the Fréchet differentiable dependence. Many of these results are summarized in [4]. It is remarkable that this step cannot be answered immediately by such a continuity results, because the space of weights is infinite-dimensional. The second step is to find the minimizers/maximizers. This step is tricky and it depends on the problem studied. For L^1 weights the solution is suggested by the Pontrjagin's Maximum Principle [5, Sections 48.6–48.8].

For Sturm-Liouville operators and Hill's operators Zhang [6] proved that the eigenvalues are continuous in potentials in the sense of weak topology w_γ. Such a stronger continuity result has been generalized to eigenvalues and half-eigenvalues on potentials/weights for scalar p-Laplacian associated with different types of boundary conditions (see [7–10]).

As an elementary application of such a stronger continuity, the proof of the first step, that is, the existence of minimizers or maximizers, of the extremal value problems as in [1–3] was quite simplified in [9, 10].

Based on the continuity of eigenvalues in weak topology and the Fréchet differentiability, some deeper results have also been obtained by Zhang and his coauthors in [10–12] by using variational method, singular integrals and limiting approach.

The extremal values of eigenvalues for Sturm-Liouville operators with potentials in L^1 balls were studied in [11, 12]. For $\gamma \in [1, \infty]$, $r \ge 0$ and $m \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, denote

$$\begin{aligned} L_{m,\gamma}^F(r) &:= \inf\left\{ \lambda_m^F(q) : q \in \mathcal{L}^\gamma, \ \|q\|_\gamma \le r \right\}, \\ M_{m,\gamma}^F(r) &:= \sup\left\{ \lambda_m^F(q) : q \in \mathcal{L}^\gamma, \ \|q\|_\gamma \le r \right\}, \end{aligned} \tag{1.5}$$

where the superscript F denotes N or P if $m = 0$ and D or N if $m > 0$. By the limiting approach $\gamma \downarrow 1$, the most important extremal values in L^1 balls are proved to be finite real numbers, and they can be evaluated explicitly by using some elementary functions $Z_0(r)$, $Z_1(r)$, $R_m(r)$, and $Y_1(r)$. None of the extremal values $L_{m,1}^F$ can be attained by any potential if $r > 0$, while all extremal values $L_{m,\gamma}^F$, $\gamma \in (1, \infty]$, and $M_{m,\gamma}^F$, $\gamma \in [1, \infty]$, can be attained by some potentials. For details, see [11, 12].

The extremal value of the mth Dirichlet eigenvalue for p-Laplacian with positive weight was studied by Yan and Zhang [10]. It was proved for $\gamma \in [1,\infty]$, $r > 0$, and $m \in \mathbb{N} := \{1,2,3,\ldots\}$ that

$$\begin{aligned}&\inf\left\{\mu_m^D(w) : w \in \mathcal{L}^\gamma,\ w \geq 0,\ \|w\|_\gamma \leq r\right\}\\&\quad = \inf\left\{\mu_m^D(w) : w \in \mathcal{L}^\gamma,\ w \geq 0,\ \|w\|_\gamma = r\right\}\\&\quad = m^p \cdot \frac{K(p\gamma^*, p)}{r},\end{aligned} \tag{1.6}$$

where $\gamma^* := \gamma/(\gamma-1)$ is the conjugate exponent of γ, and $K(\cdot,\cdot)$ is the best Sobolev constant

$$K(\alpha,p) := \inf_{u \in W_0^{1,p}(0,1)} \frac{\|u'\|_p^p}{\|u\|_\alpha^p}, \quad \forall \alpha \in [1,\infty]. \tag{1.7}$$

Moreover, the infimum can be attained by some weight if only $\gamma \in (1,\infty]$. By letting the radius $r \downarrow 0^+$ one sees that the supremum

$$\sup\left\{\mu_m^D(w) : w \in \mathcal{L}^\gamma,\ w \geq 0,\ \|w\|_\gamma \leq r\right\} = \infty, \tag{1.8}$$

so only infimum of weighted eigenvalues is considered.

Our concerns in this paper are the infimum of the mth positive half-eigenvalues $H_{m,\gamma}^F(r)$ and the infimum of the mth positive eigenvalues $E_{m,\gamma}^F(r)$ for p-Laplacian with weights in L^γ ($\gamma \in [1,\infty]$) balls, where F denotes D, N or G, while m is related to the nodal property of the corresponding half-eigenfunctions or eigenfunction. The detailed definitions of $H_{m,\gamma}^F(r)$ and $E_{m,\gamma}^F(r)$ are given by (2.35)–(2.39) and (2.44)–(2.48) in Section 2.

Some results on eigenvalues and half-eigenvalues are collected in Section 2. Compared with the results in [8], the characterizations on antiperiodic half-eigenvalues have been improved, see Theorems 2.2 and 2.4. These characterizations make the definition of $H_{m,\gamma}^G(r)$ clearer and also easier to evaluate; see Remark 2.5.

In Section 3, by using (1.6) and the relationship between Dirichlet, Neumann and generalized periodic eigenvalues (see Lemma 3.2), we will show that

$$E_{m,\gamma}^D(r) = E_{m,\gamma}^N(r) = E_{m,\gamma}^G(r) = m^p \cdot \frac{K(p\gamma^*, p)}{r} \tag{1.9}$$

for any $\gamma \in [1,\infty]$, $m \in \mathbb{N}$ and $r > 0$. It will also be proved that

$$E_{0,\gamma}^N(r) = E_{0,\gamma}^G(r) = 0, \quad \forall \gamma \in [1,\infty],\ \forall r > 0. \tag{1.10}$$

A natural idea to characterize $H_{m,\gamma}^F(r)$ is to employ analogous method as done for $E_{m,\gamma}^F(r)$. However, this idea does not work any more, because the antiperiodic half-eigenvalues cannot be characterized by Dirichlet or Neumann half-eigenvalues by virtue of the jumping terms involved, which is quite different from the eigenvalue case; see Remark 3.3.

Section 4 is devoted to $H^F_{m,\gamma}$. It is possible that for some weights in L^γ balls the mth positive half-eigenvalue does not exist; see Remark 2.3. So it is impossible to utilize directly the continuous dependence of half-eigenvalues in weights in weak topology or the Fréchet differential dependence, as done in [10–12]. Some more fundamental continuous results in weak topology and differentiable results (in Lemma 2.1) will be used instead. We will first show two facts. One is the monotonicity of the half-eigenvalues on the weights (a,b). The other is the infimum $H^F_{m,\gamma}(r)$ can be attained by some weights for any $\gamma \in (1,\infty]$. As consequence of these two facts, for each minimizer (a_γ, b_γ), one sees that a_γ and b_γ do not overlap if $\gamma \in (1,\infty)$. Moreover the extremal problem for half-eigenvalues is reduced to that for eigenvalues. Roughly speaking, for any $\gamma \in (1,\infty]$ and $r > 0$ we have

$$H^F_{m,\gamma}(r) = E^F_{m,\gamma}(r) = m^p \cdot \frac{K(p\gamma^*, p)}{r}, \quad \forall m \in \mathbb{N},\ \forall F \in \{D,N,G\}, \tag{1.11}$$

$$H^F_{0,\gamma}(r) = E^F_{0,\gamma}(r) = 0, \quad \forall F \in \{N,G\}. \tag{1.12}$$

Based on some topological fact on L^γ balls, the extremal values in L^1 balls can be obtained by the limiting approach $\gamma \downarrow 1$. Consequently (1.11) and (1.12) also hold for $\gamma = 1$.

2. Preliminary Results and Extremal Value Problems

Denote by $\phi_p(\cdot)$ the scalar p-Laplacian and let $x_\pm(\cdot) = \max\{\pm x(\cdot), 0\}$. Let us consider the positive half-eigenvalues of

$$(\phi_p(x'))' + \lambda a(t)\phi_p(x_+) - \lambda b(t)\phi_p(x_-) = 0 \quad \text{a.e. } t \in [0,1] \tag{2.1}$$

with respect to the boundary value conditions

$$x(0) = x(1) = 0, \tag{D}$$

$$x'(0) = x'(1) = 0, \tag{N}$$

$$x(0) \pm x(1) = x'(0) \pm x'(1) = 0, \tag{G}$$

respectively.

Denote by $(\cos_p(\theta), \sin_p(\theta))$ the unique solution of the initial value problem

$$\frac{dx}{d\theta} = -\phi_{p^*}(y), \qquad \frac{dy}{d\theta} = \phi_p(x), \qquad (x(0), y(0)) = (1,0). \tag{2.2}$$

The functions $\cos_p(\theta)$ and $\sin_p(\theta)$ are the so-called p-cosine and p-sine. They share several remarkable relations as ordinary trigonometric functions, for instance

(i) both $\cos_p(\theta)$ and $\sin_p(\theta)$ are $2\pi_p$-periodic, where

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}; \tag{2.3}$$

(ii) $\cos_p(\theta) = 0$ if and only if $\theta = \pi_p/2 + m\pi_p$, $m \in \mathbb{Z}$, and $\sin_p(\theta) = 0$ if and only if $\theta = m\pi_p$, $m \in \mathbb{Z}$;

(iii) $|\cos_p(\theta)|^p + (p-1)|\sin_p(\theta)|^{p^*} \equiv 1$.

By setting $\phi_p(x') = -y$ and introducing the Prüfer transformation $x = r^{2/p}\cos_p\theta$, $y = r^{2/p^*}\sin_p\theta$, the scalar equation

$$(\phi_p(x'))' + a(t)\phi_p(x_+) - b(t)\phi_p(x_-) = 0 \quad \text{a.e. } t \in [0,1] \tag{2.4}$$

is transformed into the following equations for r and θ:

$$\begin{aligned}\theta' &= A(t,\theta;a,b)\\ &:= \begin{cases} a(t)\left|\cos_p\theta\right|^p + (p-1)\left|\sin_p\theta\right|^{p^*} & \text{if } \cos_p\theta \geq 0,\\ b(t)\left|\cos_p\theta\right|^p + (p-1)\left|\sin_p\theta\right|^{p^*} & \text{if } \cos_p\theta < 0,\end{cases}\end{aligned} \tag{2.5}$$

$$\begin{aligned}(\log r)' &= G(t,\theta;a,b)\\ &:= \begin{cases} \dfrac{p}{2}(a(t)-1)\phi_p(\cos_p\theta)\phi_{p^*}(\sin_p\theta) & \text{if } \cos_p\theta \geq 0,\\ \dfrac{p}{2}(b(t)-1)\phi_p(\cos_p\theta)\phi_{p^*}(\sin_p\theta) & \text{if } \cos_p\theta < 0.\end{cases}\end{aligned} \tag{2.6}$$

For any $\vartheta_0 \in \mathbb{R}$, denote by $(\theta(t;\vartheta_0,a,b), r(t;\vartheta_0,a,b))$, $t \in [0,1]$, the unique solution of (2.5) + (2.6) satisfying $\theta(0;\vartheta_0,a,b) = \vartheta_0$ and $r(0;\vartheta_0,a,b) = 1$. Let

$$\begin{aligned}\Theta(\vartheta_0,a,b) &:= \theta(1;\vartheta_0,a,b),\\ R(\vartheta_0,a,b) &:= r(1;\vartheta_0,a,b).\end{aligned} \tag{2.7}$$

For any $m \in \mathbb{Z}^+$, denote by $\Sigma_m^+(a,b)$ the set of nonnegative half-eigenvalues of (2.1) + (2.2) for which the corresponding half-eigenfunctions have precisely m zeroes in the interval $[0,1)$. Define

$$\underline{\Theta}(a,b) := \max_{\vartheta_0\in[0,2\pi_p]}\{\Theta(\vartheta_0,a,b) - \vartheta_0\} = \max_{\vartheta_0\in\mathbb{R}}\{\Theta(\vartheta_0,a,b) - \vartheta_0\}, \tag{2.8}$$

$$\overline{\Theta}(a,b) := \min_{\vartheta_0\in[0,2\pi_p]}\{\Theta(\vartheta_0,a,b) - \vartheta_0\} = \min_{\vartheta_0\in\mathbb{R}}\{\Theta(\vartheta_0,a,b) - \vartheta_0\}, \tag{2.9}$$

$$\underline{\lambda}_m^L = \underline{\lambda}_m^L(a,b) := \min\{\lambda > 0 \mid \underline{\Theta}(\lambda a,\lambda b) = m\pi_p\}, \quad m \in \mathbb{N}, \tag{2.10}$$

$$\overline{\lambda}_m^R = \overline{\lambda}_m^R(a,b) := \max\left\{\lambda \geq 0 \mid \overline{\Theta}(\lambda a,\lambda b) = m\pi_p\right\}, \quad m \in \mathbb{Z}^+. \tag{2.11}$$

Similar arguments as in the proof of Lemma 3.2 in [8] show that

$$\underline{\lambda}_m^{L/R}(a,b), \quad \overline{\lambda}_m^{L/R}(a,b) \in \Sigma_m^+(a,b) \tag{2.12}$$

if only these numbers exist.

Lemma 2.1 (see [7, 8]). *Denote by w_γ the weak topology in $\mathcal{L}^\gamma$. Then*

(i) *$\Theta(\vartheta,a,b)$ is jointly continuous in $(\vartheta,a,b) \in \mathbb{R} \times (\mathcal{L}^\gamma, w_\gamma)^2$;*

(ii) *$\underline{\Theta}(\lambda a, \lambda b)$ and $\overline{\Theta}(\lambda a, \lambda b)$ are jointly continuous in $(\lambda, a, b) \in \mathbb{R} \times (\mathcal{L}^\gamma, w_\gamma)^2$, and*

$$\underline{\Theta}(0,0) \in \left(0, \pi_p\right), \quad \overline{\Theta}(0,0) = 0; \tag{2.13}$$

(iii) *$\Theta(\vartheta,a,b)$ is continuously differentiable in $(\vartheta,a,b) \in (\mathcal{L}^\gamma, \|\cdot\|_\gamma)^2$. The derivatives of $\Theta(\vartheta,a,b)$ at ϑ, at $a \in \mathcal{L}^\gamma$ and at $b \in \mathcal{L}^\gamma$ (in the Fréchet sense), denoted, respectively, by $\partial_\vartheta\Theta$, $\partial_a\Theta$, and $\partial_b\Theta$, are*

$$\begin{aligned}
\partial_\vartheta\Theta(\vartheta,a,b) &= \frac{1}{R^2(\vartheta,a,b)}, \\
\partial_a\Theta(\vartheta,a,b) &= X_+^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \\
\partial_b\Theta(\vartheta,a,b) &= X_-^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*,
\end{aligned} \tag{2.14}$$

where $\mathcal{C}^0 := C([0,1],\mathbb{R})$ and

$$X = X(t) = X(t;\vartheta,a,b) := \frac{\{r(t;\vartheta,a,b)\}^{2/p}\cos_p(\theta(t;\vartheta,a,b))}{\{r(1;\vartheta,a,b)\}^{2/p}} \tag{2.15}$$

is a solution of (2.4).

Given $a, a_1, a_2, b_1, b_2 \in \mathcal{L}^1$, write $a \succ 0$ if $a \geq 0$ and $\int_0^1 a(t)dt > 0$. Write $(a_1,b_1) \geq (a_2,b_2)$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Write $(a_1,b_1) \succ (a_2,b_2)$ if $(a_1,b_1) \geq (a_2,b_2)$ and both $a_1(t) > a_2(t)$ and $b_1(t) > b_2(t)$ hold for t in a *common* subset of $[0,1]$ of positive measure. Denote

$$\mathcal{W}_+^\gamma := \{(a,b) \mid a,b \in \mathcal{L}^\gamma,\ (a_+,b_+) \succ (0,0)\}. \tag{2.16}$$

Theorem 2.2. *Suppose $(a,b) \in \mathcal{W}_+^1$. There hold the following results.*

(i) *All positive Dirichlet half-eigenvalues of* (2.1) *consist of two sequences $\{\lambda_m^D(a,b)\}_{m\in\mathbb{N}}$ and $\{\lambda_m^D(b,a)\}_{m\in\mathbb{N}}$, where $\lambda_m^D(a,b)$ is the unique solution of*

$$\begin{gathered}
\Theta\left(-\frac{\pi_p}{2}, \lambda a, \lambda b\right) = -\frac{\pi_p}{2} + m\pi_p, \quad \forall m \in \mathbb{N}, \\
\lambda_1^D(a,b) < \lambda_2^D(a,b) < \cdots < \lambda_m^D(a,b) < \cdots (\longrightarrow \infty).
\end{gathered} \tag{2.17}$$

(ii) *All nonnegative Neumann half-eigenvalues of* (2.1) *consist of two sequences* $\{\lambda_m^N(a,b)\}_{m\in\mathbb{Z}^+}$ *and* $\{\lambda_m^N(b,a)\}_{m\in\mathbb{Z}^+}$, *where* $\lambda_m^N(a,b)$ *is determined by*

$$\begin{gathered}\Theta(0,\lambda a,\lambda b)=m\pi_p,\quad \forall m\in\mathbb{Z}^+,\\ (0\le)\lambda_0^N(a,b)<\lambda_1^N(a,b)<\lambda_2^N(a,b)<\cdots<\lambda_m^N(a,b)<\cdots(\longrightarrow\infty).\end{gathered}\tag{2.18}$$

Moreover,

$$\lambda_0^N(a,b)>0\Longleftrightarrow a_+\succ 0,\qquad \int_0^1 a(t)<0.\tag{2.19}$$

(iii) *All solutions of*

$$\begin{gathered}\underline{\Theta}(\lambda a,\lambda b)=m\pi_p,\quad \forall m\in\mathbb{N},\\ \overline{\Theta}(\lambda a,\lambda b)=m\pi_p,\quad \forall m\in\mathbb{Z}^+\end{gathered}\tag{2.20}$$

are contained in $\Sigma_m^+(a,b)$. *Denote* $\underline{\lambda}_0^L:=0$; *then*

$$\left\{\underline{\lambda}_m^{L/R}(a,b),\overline{\lambda}_m^{L/R}(a,b)\right\}\subset\Sigma_m^+(a,b)\subset\left[\underline{\lambda}_m^L(a,b),\overline{\lambda}_m^R(a,b)\right],\quad \forall m\in\mathbb{Z}^+.\tag{2.21}$$

There hold the ordering

$$\begin{gathered}(0<)\ \underline{\lambda}_1^L\le\underline{\lambda}_1^R<\underline{\lambda}_2^L\le\underline{\lambda}_2^R<\cdots<\underline{\lambda}_m^L\le\underline{\lambda}_m^R<\cdots(\longrightarrow\infty),\\ (0=)\ \overline{\lambda}_0^L\le\overline{\lambda}_0^R<\overline{\lambda}_1^L\le\overline{\lambda}_1^R<\cdots<\overline{\lambda}_m^L\le\overline{\lambda}_m^R<\cdots\quad(\longrightarrow\infty),\\ (0\le)\ \overline{\lambda}_0^R<\underline{\lambda}_2^L\le\overline{\lambda}_2^R<\cdots<\underline{\lambda}_{2m}^L\le\overline{\lambda}_{2m}^R<\underline{\lambda}_{2m+2}^L\le\overline{\lambda}_{2m+2}^R\cdots(\longrightarrow\infty).\end{gathered}\tag{2.22}$$

Moreover,

$$\overline{\lambda}_0^R(a,b)>0\Longleftrightarrow\int_0^1 a(t)dt<0\quad or\quad\int_0^1 b(t)dt<0.\tag{2.23}$$

Proof. Compared with results in [8], we need only prove

$$\Sigma_{2m+1}^+(a,b)\in\left[\underline{\lambda}_{2m+1}^L(a,b),\overline{\lambda}_{2m+1}^R(a,b)\right]\quad\forall m\in\mathbb{Z}^+.\tag{2.24}$$

The proof of this is similar to the proof of some stronger results given in Theorem 2.4, so we defer the details until then. □

Remark 2.3. The restriction $(a,b) \in \mathcal{W}_+^1$ in Theorem 2.2 guarantees the existence of such half-eigenvalues, to which the corresponding half-eigenfunction have arbitrary many zeros in $[0,1)$. However, it is possible for some weights $a, b \in \mathcal{L}^1$, for example, $a \succ 0$ and $b = 0$, that only finite of these positive half-eigenvalues exists. We refer this to Remark 2.4 in [8]. In other cases, for example if $a < 0$ and $b < 0$, there exist no positive half-eigenvalues. Since we are going to study the infimum of positive half-eigenvalues, if one of these half-eigenvalues, say $\lambda_m^D(a,b)$, does not exist, we define $\lambda_m^D(a,b) = \infty$ for simplicity.

Theorem 2.4. *Suppose $a, b \in \mathcal{L}^1$. There hold the following results.*

(i) *If $\underline{\lambda}_m^L(a,b) < \infty$ for some $m \in \mathbb{N}$, then*

$$\lambda \geq \underline{\lambda}_m^L(a,b), \quad \forall \lambda \in \Sigma_m^+(a,b); \tag{2.25}$$

(ii) *if $\overline{\lambda}_m^R(a,b) < \infty$ for some $m \in \mathbb{Z}^+$, then*

$$\lambda \leq \overline{\lambda}_m^R(a,b), \quad \forall \lambda \in \Sigma_m^+(a,b). \tag{2.26}$$

Proof. One has the following steps.

Step 1. By checking the proof of Lemma 3.3 in [8], results therein still hold for arbitrary a, $b \in \mathcal{L}^1$, that is,

(1) If $\underline{\Theta}(\mu a, \mu b) = m\pi_p$ for some $\mu > 0$ and $m \in \mathbb{N}$, then there exists $\delta > 0$ such that

$$\underline{\Theta}(\lambda a, \lambda b) > m\pi_p, \quad \forall \lambda \in (\mu, \mu + \delta). \tag{2.27}$$

(2) If $\overline{\Theta}(\mu a, \mu b) = m\pi_p$ for some $\mu > 0$ and $m \in \mathbb{Z}^+$, then there exists $\delta \in (0, \mu)$ such that

$$\overline{\Theta}(\lambda a, \lambda b) < m\pi_p, \quad \forall \lambda \in (\mu - \delta, \mu). \tag{2.28}$$

Step 2. It follows from Step 1 that

$$\underline{\Theta}(\lambda a, \lambda b) \begin{cases} < m\pi_p & \text{if } 0 \leq \lambda < \underline{\lambda}_m^L, \\ \geq m\pi_p & \text{if } \lambda \geq \underline{\lambda}_m^L \end{cases} \quad \forall m \in \mathbb{N}, \tag{2.29}$$

if $\underline{\lambda}_m^L(a,b) < \infty$, and

$$\overline{\Theta}(\lambda a, \lambda b) \begin{cases} \leq m\pi_p & \text{if } 0 \leq \lambda < \overline{\lambda}_m^R, \\ > m\pi_p & \text{if } \lambda > \overline{\lambda}_m^R \end{cases} \quad \forall m \in \mathbb{Z}^+ \tag{2.30}$$

if $\overline{\lambda}_m^R(a,b) < \infty$.

Step 3. Suppose $\underline{\lambda}_m^L(a,b) < \infty$ for some $m \in \mathbb{N}$. For any $\lambda \in \Sigma_m^+(a,b)$, there exists $\vartheta \in \mathbb{R}$ (depends on λ) such that

$$\Theta(\vartheta, \lambda a, \lambda b) = \vartheta + m\pi_p. \tag{2.31}$$

Consequently,

$$\underline{\Theta}(\lambda a, \lambda b) = \max_{\vartheta_0 \in \mathbb{R}} \{\Theta(\vartheta_0, \lambda a, \lambda b) - \vartheta_0\} \geq m\pi_p. \tag{2.32}$$

It follows from (2.29) that $\lambda \geq \underline{\lambda}_m^L(a,b)$, which completes the proof of (i). Results (ii) can be proved analogously by using (2.30).

□

In the product space $\mathcal{L}^\gamma \times \mathcal{L}^\gamma$, $1 \leq \gamma \leq \infty$, one can define the norm $|\cdot|_\gamma$ as

$$\begin{aligned} |(a,b)|_\gamma &:= \left\{ \int_0^1 \left(|a(t)|^\gamma + |b(t)|^\gamma \right) dt \right\}^{1/\gamma}, \quad \forall (a,b) \in \mathcal{L}^\gamma \times \mathcal{L}^\gamma,\ \gamma \in [1,\infty), \\ |(a,b)|_\infty &:= \lim_{\gamma \to \infty} |(a,b)|_\gamma = \max\{\|a\|_\infty, \|b\|_\infty\}, \quad \forall (a,b) \in \mathcal{L}^\infty \times \mathcal{L}^\infty. \end{aligned} \tag{2.33}$$

Given $\gamma \in [1,\infty]$, and $r > 0$. We take the notations

$$\begin{aligned} \widetilde{B}^\gamma[r] &:= \left\{ (a,b) \in \mathcal{L}^\gamma \times \mathcal{L}^\gamma : |(a,b)|_\gamma \leq r \right\}, \\ \widetilde{B}_\delta^\gamma(a,b) &:= \left\{ (a_1,b_1) \in \mathcal{L}^\gamma \times \mathcal{L}^\gamma : |(a_1 - a, b_1 - b)|_\gamma \leq \delta \right\}, \\ \widetilde{S}^\gamma(r) &:= \left\{ (a,b) \in \mathcal{L}^\gamma \times \mathcal{L}^\gamma : |(a,b)|_\gamma = r \right\}, \\ \widetilde{S}_+^\gamma(r) &:= \left\{ (a,b) \in \widetilde{S}^\gamma(r) : a \geq 0,\ b \geq 0 \right\}, \\ B^\gamma[r] &:= \left\{ a \in \mathcal{L}^\gamma : \|a\|_\gamma \leq r \right\}, \\ S_+^\gamma(r) &:= \left\{ a \in \mathcal{L}^\gamma : a \geq 0,\ \|a\|_\gamma = r \right\}. \end{aligned} \tag{2.34}$$

Now we can define the infimum of positive half-eigenvalues

$$H_{m,\gamma}^D(r) := \inf\left\{ \lambda_m^D(a,b) : (a,b) \in \widetilde{B}^\gamma[r] \right\}, \quad \forall m \in \mathbb{N}, \tag{2.35}$$

$$H_{m,\gamma}^N(r) := \inf\left\{ \lambda_m^N(a,b) : (a,b) \in \widetilde{B}^\gamma[r] \right\}, \quad \forall m \in \mathbb{N}, \tag{2.36}$$

$$H_{m,\gamma}^{G}(r) := \inf\left\{\lambda \in \Sigma_m^+(a,b) : (a,b) \in \widetilde{B}^{\gamma}[r]\right\}, \quad \forall m \in \mathbb{N}, \tag{2.37}$$

$$H_{0,\gamma}^{N}(r) := \inf\left\{\lambda_m^N(a,b) > 0 : (a,b) \in \widetilde{B}^{\gamma}[r]\right\}, \tag{2.38}$$

$$H_{0,\gamma}^{G}(r) := \inf\left\{\overline{\lambda}_0^R(a,b) > 0 : (a,b) \in \widetilde{B}^{\gamma}[r]\right\}. \tag{2.39}$$

Remark 2.5. (i) It follows from Theorem 2.2 that all the extremal values defined by (2.35)–(2.39) are finite.

(ii) Although there may exist nonvariational half-eigenvalues in $\Sigma_m^+(a,b)$ (cf. [13]), Theorem 2.4 shows that

$$\underline{\lambda}_m^L(a,b) = \inf \Sigma_m^+(a,b) \quad \forall a,b \in \mathcal{L}^1,\ \forall m \in \mathbb{N}. \tag{2.40}$$

Therefore (2.37) can be rewritten as

$$H_{m,\gamma}^{G}(r) := \inf\left\{\underline{\lambda}_m^L(a,b) : (a,b) \in \widetilde{B}^{\gamma}[r]\right\}, \quad \forall m \in \mathbb{N}. \tag{2.41}$$

Notice that if $a = b$, then the half-eigenvalue problem of (2.1) is equivalent to the eigenvalue problem of

$$\left(\phi_p(x')\right)' + \lambda a(t)\phi_p(x) = 0, \quad \text{a.e. } t \in [0,1]. \tag{2.42}$$

If $a_+ \succ 0$, then $(a,a) \in \mathcal{W}_+^1$. Theorem 2.2 shows that all positive Dirichlet eigenvalues of (2.42) consist of a sequence $\{\lambda_m^D(a,a)\}_{m\in\mathbb{N}}$, all nonnegative Neumann eigenvalues of (2.42) consist of $\{0\}\cup\{\lambda_m^N(a,a)\}_{m\in\mathbb{Z}^+}$, while both $\underline{\lambda}_m^L(a,a)$ and $\overline{\lambda}_m^R(a,a)$ are periodic or antiperiodic eigenvalues of (2.42) if m is even or odd, respectively. We take the notations

$$\begin{aligned} \lambda_m^D(a) &:= \lambda_m^D(a,a), & \lambda_m^N(a) &:= \lambda_m^N(a,a), \\ \underline{\lambda}_m^L(a) &:= \underline{\lambda}_m^L(a,a), & \overline{\lambda}_m^R(a) &:= \overline{\lambda}_m^R(a,a) \end{aligned} \tag{2.43}$$

and $\Sigma_m^+(a) := \Sigma_m^+(a,a)$.

Given $\gamma \in [1,\infty]$ and $r > 0$, now we can define the infimum of positive half-eigenvalues

$$E_{m,\gamma}^{D}(r) := \inf\left\{\lambda_m^D(a) : a \in B^{\gamma}[r]\right\}, \quad \forall m \in \mathbb{N}, \tag{2.44}$$

$$E_{m,\gamma}^{N}(r) := \inf\left\{\lambda_m^N(a) : a \in B^{\gamma}[r]\right\}, \quad \forall m \in \mathbb{N}, \tag{2.45}$$

$$\begin{aligned} E^G_{m,\gamma}(r) &:= \inf\{\lambda \in \Sigma^+_m(a) : a \in B^\gamma[r]\} \\ &= \inf\left\{\underline{\lambda}^L_m(a) : a \in B^\gamma[r]\right\}, \quad \forall m \in \mathbb{N}, \end{aligned} \tag{2.46}$$

$$E^N_{0,\gamma}(r) := \inf\left\{\lambda^N_m(a) > 0 : a \in B^\gamma[r]\right\}, \tag{2.47}$$

$$E^G_{0,\gamma}(r) := \inf\left\{\overline{\lambda}^R_0(a) > 0 : a \in B^\gamma[r]\right\}. \tag{2.48}$$

3. Infimum of Eigenvalues with Weight in L^γ Balls

Theorem 3.1. *For any $\gamma \in [1,\infty]$, $m \in \mathbb{N}$ and $r > 0$, one has*

$$E^D_{m,\gamma}(r) = m^p \cdot \frac{K(p\gamma^*, p)}{r}. \tag{3.1}$$

If $\gamma \in (1,\infty]$, then $E^D_{m,\gamma}(r)$ can be attained by some weight, called a minimizer, and each minimizer is contained in $S^\gamma_+(r)$. If $\gamma = 1$, then $E^D_{m,\gamma}(r)$ cannot be attained by any weight in $B^\gamma[r]$.

Proof. If $a \le 0$, then (2.42) has no positive Dirichlet eigenvalues, that is, $\lambda^D_m(a) = \infty$ by our notations. If $a_+ \succ 0$ and $a_- \succ 0$, then $|a| \succ a$ and

$$\lambda^D_m(|a|) < \lambda^D_m(a) < \infty, \tag{3.2}$$

compare, for example, [9, Theorem 3.9], see also Lemma 4.2(i). Consequently one has

$$E^D_{m,\gamma}(r) = \inf\left\{\lambda^D_m(w) : w \in \mathcal{L}^\gamma,\ w \ge 0,\ \|w\|_\gamma \le r\right\}. \tag{3.3}$$

Now the theorem can be completed by the proof of [10, Theorem 5.6]; see also (1.6). □

Lemma 3.2. *Given $a \in \mathcal{L}^\gamma$, define $a_s(t) := a(s+t)$ for any $s, t \in \mathbb{R}$. Then*

$$\begin{gathered} \underline{\lambda}^L_m(a) = \min_{s\in\mathbb{R}}\left\{\lambda^D_m(a_s)\right\} = \min_{s\in\mathbb{R}}\left\{\lambda^N_m(a_s)\right\}, \quad \forall m \in \mathbb{N}, \\ \overline{\lambda}^R_m(a) = \max_{s\in\mathbb{R}}\left\{\lambda^D_m(a_s)\right\} = \max_{s\in\mathbb{R}}\left\{\lambda^N_m(a_s)\right\}, \quad \forall m \in \mathbb{N}, \\ \overline{\lambda}^R_0(a) = \max_{s\in\mathbb{R}}\left\{\lambda^N_0(a_s)\right\}. \end{gathered} \tag{3.4}$$

Proof. This lemma can be proved as done in [14], where eigenvalues for p-Laplacain with *potential* were studied by employing rotation number functions. □

Remark 3.3. Results in Lemma 3.2 can be generalized to half-eigenvalues exclusively for even integers m. The reason is that $A(t;a,b)$ in (2.5) is $2\pi_p$-periodic in t for general a and b, while for the eigenvalue problem $A(t;a,a)$ is π_p-periodic.

Notice that $a \in B^\gamma[r]$ if and only if $a_s \in B^\gamma[r]$ for any $s \in \mathbb{R}$. One can obtain the following theorem immediately from Theorem 3.1 and Lemma 3.2.

Theorem 3.4. *There holds* (1.9) *for any* $\gamma \in [1,\infty]$, $m \in \mathbb{N}$ *and* $r > 0$. *If* $\gamma \in (1,\infty]$, *any extremal value involved in* (1.9) *can be attained by some weight, and each minimizer is contained in* $S_+^\gamma(r)$. *If* $\gamma = 1$, *none of these extremal values can be attained by any weight in* $B^\gamma[r]$.

However, we cannot characterize $E_{0,\gamma}^N$ and $E_{0,\gamma}^G$ by using Theorem 3.1 and Lemma 3.2, because $\lambda_0^D(a)$ does not exist for any weight $a \in \mathcal{L}^\gamma$.

Theorem 3.5. *There holds* (1.10) *for any* $\gamma \in [1,\infty]$ *and* $r > 0$.

Proof. Choose a sequence of weights

$$a_k(t) = \begin{cases} r & t \in \left[0, \dfrac{1}{2} - \dfrac{1}{k}\right], \\ -r & t \in \left[\dfrac{1}{2} - \dfrac{1}{k}, 1\right], \end{cases} \quad k > 2. \tag{3.5}$$

Then $a_k \in B^\gamma[r]$, $(a_k)_+ \succ 0$ and $\int_0^1 a_k(t)dt < 0$. It follows from Theorem 2.2(ii) that $\nu_k := \lambda_0^N(a_k) = \lambda_0^N(a_k, a_k) > 0$, and ν_k is determined by

$$\Theta(0, \nu_k a_k, \nu_k a_k) = 0. \tag{3.6}$$

Since $a_{k+1} \succ a_k$, by Lemma 4.2(iii) one has $\nu_{k+1} < \nu_k$. Let $k \to \infty$. Then

$$a_k \longrightarrow a_0 = a_0(t) = \begin{cases} r & t \in \left[0, \dfrac{1}{2}\right], \\ -r & t \in \left[\dfrac{1}{2}, 1\right], \end{cases} \quad \text{a.e.} \quad t \in [0,1], \tag{3.7}$$

and $\nu_k \to \nu(\geq 0)$. By Lemma 2.1 the limiting equality of (3.6) is

$$\Theta(0, \nu a_0, \nu a_0) = 0. \tag{3.8}$$

Since $\int_0^1 a_0(t)dt = 0$, it follows from Theorem 2.2(ii) again that $\nu = 0$. Hence $E_{0,\gamma}^N(r) = 0$.

Notice that (2.42) has no positive (Neumann or periodic) eigenvalues if the weight $a \leq 0$. On the other hand, Theorem 2.2 shows that if $a_+ \succ 0$ then

$$\overline{\lambda}_0^R(a) > 0 \Longleftrightarrow \lambda_0^N(a) \Longleftrightarrow \int_0^1 a(t)dt < 0. \tag{3.9}$$

Combining Lemma 3.2 and the definitions in (2.47) and (2.48), one has $E_{0,\gamma}^G(r) = E_{0,\gamma}^N(r) = 0$, completing the proof of the theorem. ◻

4. Infimum of Half-Eigenvalues with Weights in L^γ Balls

4.1. Monotonicity Results of Half-Eigenvalues

Applying Fréchet differentiability of $\lambda_m^D(a,b)$ and $\lambda_m^N(a,b)$ in weights a, $b \in \mathcal{L}^1$, some monotonicity results of eigenvalues have been obtained in [8].

Lemma 4.1 (see [8]). *Given $\gamma \in [1,\infty]$ and $(a_i,b_i) \in \mathcal{W}_+^\gamma$, $i=0,1$, if $(a_0,b_0) \succ (\geq)(a_1,b_1)$, then*

(i) $\lambda_m^D(a_0,b_0) < (\leq)\lambda_m^D(a_1,b_1)$ *for any* $m \in \mathbb{N}$,

(ii) $\lambda_m^N(a_0,b_0) < (\leq)\lambda_m^N(a_1,b_1)$ *for any* $m \in \mathbb{N}$,

(iii) *if moreover* $\int_0^1 a_0(t)dt < 0$, *then* $0 < \lambda_0^N(a_0,b_0) < (\leq)\lambda_0^N(a_1,b_1)$.

By checking the proof in [8] one sees that the restriction $(a,b) \in \mathcal{W}_+^\gamma$ can be weakened. In fact this restriction was used to guarantee the existence of $\lambda_m^D(a,b)$ and $\lambda_m^N(a,b)$ for arbitrary large $m \in \mathbb{N}$. Employing the boundary value conditions and Fréchet differentiability of $\Theta(\vartheta,a,b)$ in weights (Lemma 2.1(iii)), one can prove the following lemma.

Lemma 4.2. *Given $a_i,b_i \in \mathcal{L}^\gamma$, $i=0,1$, $\gamma \in [1,\infty]$. Suppose $(a_0,b_0) \succ (\geq)(a_1,b_1)$, then*

(i) *if* $\lambda_m^D(a_1,b_1) < \infty$ *for some* $m \in \mathbb{N}$, *then* $\lambda_m^D(a_0,b_0) < (\leq)\lambda_m^D(a_1,b_1)$;

(ii) *if* $\lambda_m^N(a_1,b_1) < \infty$ *for some* $m \in \mathbb{N}$, *then* $\lambda_m^N(a_0,b_0) < (\leq)\lambda_m^N(a_1,b_1)$;

(iii) *if* $(a_1)_+ \succ 0$ *and* $\int_0^1 a_0(t)dt < 0$, *then* $0 < \lambda_0^N(a_0,b_0) < (\leq)\lambda_0^N(a_1,b_1)$.

Due to the so-called parametric resonance [15] or the so-called coexistence of periodic and antiperiodic eigenvalues [16], half-eigenvalues $\underline{\lambda}_m^L(a,b)$ and $\overline{\lambda}_m^R(a,b)$, $m \in \mathbb{N}$, are not continuously differentiable in (a,b) in general. This add difficulty to the study of monotonicity of $\underline{\lambda}_m^L(a,b)$ and $\overline{\lambda}_m^R(a,b)$ in (a,b). Even if we go back to (2.10) and (2.11) by which $\underline{\lambda}_m^L(a,b)$ and $\overline{\lambda}_m^R(a,b)$ are determined, we find that $\underline{\Theta}(a,b)$ and $\overline{\Theta}(a,b)$ are not differentiable. Finally, we have to resort to the comparison result on $\Theta(\vartheta,a,b)$. It can be proved that

$$(a_0,b_0) \succ (\geq)(a_1,b_1) \Longrightarrow \Theta(\vartheta,a_0,b_0) < (\leq)\Theta(\vartheta,a_1,b_1), \quad \forall \vartheta \in \mathbb{R}. \tag{4.1}$$

New difficulty occurs since the weights are sign-changing, that is, we cannot conclude from $(a_0,b_0) \succ (\geq)(a_1,b_1)$ that

$$\Theta(\vartheta,\lambda a_0,\lambda b_0) < (\leq)\ \Theta(\vartheta,\lambda a_1,\lambda b_1), \quad \forall \vartheta \in \mathbb{R},\ \forall \lambda > 0. \tag{4.2}$$

So we can only obtain some weaker monotonicity results for generalized periodic half-eigenvalues.

Lemma 4.3. *Given $a,b,a_i,b_i \in \mathcal{L}^\gamma$, $i=0,1$, $\gamma \in [1,\infty]$. There hold the following results.*

(i) *If* $\underline{\lambda}_m^L(a,b) < \infty$ *for some* $m \in \mathbb{N}$, *then* $\underline{\lambda}_m^L(a_+,b_+) \leq \underline{\lambda}_m^L(a,b)$.

(ii) *If* $\overline{\lambda}_m^R(a,b) < \infty$ *for some* $m \in \mathbb{N}$, *then* $\overline{\lambda}_m^R(a_+,b_+) \leq \overline{\lambda}_m^R(a,b)$.

(iii) *If* $(a_0, b_0) \succ (\geq)\ (a_1, b_1) \geq (0,0)$ *and* $\underline{\lambda}_m^L(a_1, b_1) < \infty$ *for some* $m \in \mathbb{N}$, *then* $\underline{\lambda}_m^L(a_0, b_0) < (\leq) \underline{\lambda}_m^L(a_1, b_1)$.

(iv) *If* $(a_0, b_0) \succ (\geq)\ (a_1, b_1) \geq (0,0)$ *and* $\overline{\lambda}_m^R(a_1, b_1) < \infty$ *for some* $m \in \mathbb{N}$, *then* $\overline{\lambda}_m^R(a_0, b_0) < (\leq)\ \overline{\lambda}_m^R(a_1, b_1)$.

Proof. Given $a, b \in \mathcal{L}^\gamma$. For any $\lambda \geq 0$, one has $(\lambda a_+, \lambda b_+) \geq (\lambda a, \lambda b)$. It follows from (4.1) that

$$\Theta(\vartheta, \lambda a_+, \lambda b_+) \geq \Theta(\vartheta, \lambda a, \lambda b), \quad \forall \vartheta \in \mathbb{R},\ \forall \lambda \geq 0. \tag{4.3}$$

Notice that $\Theta(\vartheta, a, b) - \vartheta$ is $2\pi_p$-periodic in $\vartheta \in \mathbb{R}$. Combining the definition of $\underline{\Theta}(a, b)$ in (2.8), one has

$$\underline{\Theta}(\lambda a_+, \lambda b_+) \geq \underline{\Theta}(\lambda a, \lambda b), \quad \forall \lambda \geq 0. \tag{4.4}$$

By Lemma 2.1(ii), $\underline{\Theta}(0 \cdot a, 0 \cdot a) \in (0, \pi_p)$ and $\underline{\Theta}(\lambda a, \lambda b)$ is continuous in $\lambda \in \mathbb{R}$. As functions of $\lambda \in [0, \infty)$, the smooth curve $\underline{\Theta}(\lambda a_+, \lambda b_+)$ lies above $\underline{\Theta}(\lambda a, \lambda b)$. By the definition of $\underline{\lambda}_m^L(a, b)$ in (2.10), if $\underline{\lambda}_m^L(a, b) < \infty$ for some $m \in \mathbb{N}$ then $\underline{\lambda}_m^L(a_+, b_+) \leq \underline{\lambda}_m^L(a, b)$. Thus the proof of (i) is completed.

Results (ii), (iii), and (iv) can be proved analogously. □

4.2. The Infimum in $L^\gamma (\gamma \in (1, \infty])$ Balls Can Be Attained

Given $a, b \in \mathcal{L}^\gamma$, $\gamma \in [1, \infty]$, $m \in \mathbb{N}$, and $\tau > 0$, one has

$$\lambda_m^D(\tau a, \tau b) = \frac{\lambda_m^D(a, b)}{\tau}, \qquad \lambda_m^N(\tau a, \tau b) = \frac{\lambda_m^N(a, b)}{\tau}, \qquad \underline{\lambda}_m^L(\tau a, \tau b) = \frac{\underline{\lambda}_m^L(a, b)}{\tau}. \tag{4.5}$$

Hence

$$\frac{H_{m,\gamma}^F(r_1)}{H_{m,\gamma}^F(r_2)} = \frac{r_2}{r_1}, \quad \forall r_1, r_2 \in (0, \infty),\ \forall \gamma \in [1, \infty],\ \forall m \in \mathbb{N}, \tag{4.6}$$

where F denotes D, N or G.

Theorem 4.4. *Given* $\gamma \in (1, \infty]$, $r > 0$, $m \in \mathbb{N}$ *and* $F \in \{D, N, G\}$. *Then* $H_{m,\gamma}^F(r) > 0$ *and it can be attained by some weights. Moreover, any minimizer* $(a_F, b_F) \in \widetilde{S}^\gamma(r)$.

Proof. We only prove for the case $F = G$, other cases can be proved analogously. There exists a sequence of weights $(a_n, b_n) \in \widetilde{B}^\gamma[r]$, $n \in \mathbb{N}$, such that

$$\nu_n := \underline{\lambda}_m^L(a_n, b_n) \longrightarrow \nu_0 := H_{m,\gamma}^G(r) \quad \text{as } n \longrightarrow \infty. \tag{4.7}$$

By the definition of $\underline{\lambda}_m^L$ in (2.10), there exist $\vartheta_n \in [0, 2\pi_p]$, $n \in \mathbb{N}$, such that

$$\begin{gathered}\Theta(\vartheta_n, \nu_n a_n, \nu_n b_n) - \vartheta_n = m\pi_p, \\ \Theta(\vartheta, \nu_n a_n, \nu_n b_n) - \vartheta \le m\pi_p, \quad \forall \vartheta \in [0, 2\pi_p].\end{gathered} \tag{4.8}$$

Notice that $\widetilde{B}^\gamma[r] \subset (\mathcal{L}^\gamma \times \mathcal{L}^\gamma, |\cdot|_\gamma)$, $\gamma \in (1, \infty]$, is sequentially compact in $(\mathcal{L}^\gamma, w_\gamma)^2$. Passing to a subsequence, we may assume $\vartheta_n \to \vartheta_0$ and

$$(a_n, b_n) \longrightarrow (a_0, b_0) \in \widetilde{B}^\gamma[r], \quad \text{in } (\mathcal{L}^\gamma, w_\gamma)^2. \tag{4.9}$$

Let $n \to \infty$ in (4.8). By Lemma 2.1(i), one has

$$\begin{gathered}\Theta(\vartheta_0, \nu_0 a_0, \nu_0 b_0) - \vartheta_0 = m\pi_p, \\ \Theta(\vartheta, \nu_0 a_0, \nu_0 b_0) - \vartheta \le m\pi_p, \quad \forall \vartheta \in [0, 2\pi_p].\end{gathered} \tag{4.10}$$

Thus $\underline{\Theta}(\nu_0 a_0, \nu_0 b_0) = m\pi_p$. It follows from (2.10) and (2.13) that $\hat{r} := |(a_0, b_0)|_\gamma > 0$ and

$$\nu_0 \ge \underline{\lambda}_m^L(a_0, b_0) > 0. \tag{4.11}$$

On the other hand, since $(a_0, b_0) \in \widetilde{B}^\gamma[r]$, one has

$$\underline{\lambda}_m^L(a_0, b_0) \ge \inf\left\{\underline{\lambda}_m^L(a, b) : (a, b) \in \widetilde{B}^\gamma[r]\right\} = \nu_0 = H_{m,\gamma}^G(r). \tag{4.12}$$

To complete the proof of the lemma, it suffices to show $\hat{r} = r$. If this is false, then $0 < \hat{r} < r$ and

$$H_{m,\gamma}^G(\hat{r}) \le \underline{\lambda}_m^L(a_0, b_0) = H_{m,\gamma}^G(r), \tag{4.13}$$

which contradicts (4.6). □

4.3. *Minimizers and Infimum in* $L^\gamma (\gamma \in (1, \infty])$ *Balls*

We have proved that for any $m \in \mathbb{N}$ the infimum $H_{m,\gamma}^F(r)$ can be obtained if only $\gamma \in (1, \infty]$. In the following we will study the property of the minimizers.

Theorem 4.5. *Given $\gamma \in (1, \infty)$, $r > 0$, $m \in \mathbb{N}$, and $F \in \{D, N, G\}$, if (a, b) is the minimizer of $H_m^F(r)$, then $(a, b) \in \widetilde{S}_+^\gamma(r)$. Moreover, a and b do not overlap, that is,*

$$\begin{gathered}a(t) = 0 \quad \text{a.e. } t \in J_b := \{t \mid b(t) > 0\}, \\ b(t) = 0 \quad \text{a.e. } t \in J_a := \{t \mid a(t) > 0\}.\end{gathered} \tag{4.14}$$

Proof. We only prove for the case $F = G$, other cases can be proved analogously.

Step 1 (Nonnegative). Suppose $a(t) < 0$ a.e. $t \in J_0 \subset [0,1]$, where J_0 is of positive measure. Let

$$a_1(t) = \begin{cases} \dfrac{|a(t)|}{2}, & \text{if } t \in J_0, \\ a(t), & \text{otherwise,} \end{cases}$$
$$b_1(t) = \begin{cases} b(t) + \varepsilon, & \text{if } t \in J_0, \\ b(t), & \text{otherwise,} \end{cases} \tag{4.15}$$

where $\varepsilon = \varepsilon(\gamma) > 0$ can be chosen arbitrary small such that $|(a_1, b_1)|_\gamma \leq r$. Then $(a_1, b_1) \succ (a, b)$ and it follows from Lemma 4.3(iii) that

$$\underline{\lambda}_m^L(a_1, b_1) < \underline{\lambda}_m^L(a, b) = H_{m,\gamma}^F(r), \tag{4.16}$$

which is in contradiction to the definition of $H_{m,\gamma}^F(r)$. Thus a is nonnegative. Analogously b is also nonnegative. Then it follows from Theorem 4.4 that $(a, b) \in \widetilde{S}_+^\gamma(r)$.

Step 2 (Nonoverlap). If a and b overlap, then $(a, b) \succ (0, 0)$, that is, there exists $J_0 \subset [0,1]$ with positive measure such that

$$a(t) > 0, \quad b(t) > \quad \text{a.e. } \ t \in J_0 \subset [0,1]. \tag{4.17}$$

Let $X(t)$ be the half-eigenfunction corresponding to $\nu := \underline{\lambda}_m^L(a, b)$. Without loss of generality, we may assume that

$$X(t) > 0 \quad \text{a.e. } \ t \in \widetilde{J}_0 \subset J_0 \tag{4.18}$$

for some $\widetilde{J}_0$ with positive measure. Let

$$a_1(t) = a(t), \qquad b_1(t) = \begin{cases} 0 & \text{if } t \in \widetilde{J}_0, \\ b(t), & \text{otherwise.} \end{cases} \tag{4.19}$$

Then $\widehat{r} := |(a_1, b_1)|_\gamma < |(a, b)|_\gamma = r$ and

$$(\phi_p(X'))' + \nu a_1(t)\phi_p(X_+) + \nu b_1(t)\phi_p(X_-) = 0. \tag{4.20}$$

Therefore $\underline{\lambda}_m^L(a_1, b_1) \le \nu = H_{m,\gamma}^G(r)$. It follows that

$$H_{m,\gamma}^G(\hat{r}) \le \underline{\lambda}_m^L(a_1, b_1) \le H_{m,\gamma}^G(r), \tag{4.21}$$

which contradicts (4.6). Thus a and b do not overlap. □

Corollary 4.6. *Given $\gamma \in (1, \infty)$, $r > 0$, $m \in \mathbb{N}$, and $F \in \{D, N, G\}$, if (a, b) is the minimizer of $H_m^F(r)$ and X is the corresponding half-eigenfunction, then*

$$X(t) > 0 \quad a.e. \quad t \in J_a := \{a(t) > 0\}, \tag{4.22}$$

$$X(t) < 0 \quad a.e. \quad t \in J_b := \{b(t) > 0\}. \tag{4.23}$$

Proof. If (4.23) does not hold. Then there exist $\widetilde{J}_0 \subset J_b$ such that $\widetilde{J}_0$ is of positive measure and

$$X(t) > 0 \quad \text{a.e. } t \in \widetilde{J}_0. \tag{4.24}$$

Define a_1 and b_1 as in (4.19). A contradiction can be obtained by similar arguments as in the proof of Theorem 4.5. Thus (4.23) holds. One can prove (4.22) analogously. □

Theorem 4.7. *Given $r > 0$, then* (1.11) *holds for any $\gamma \in (1, \infty]$ and* (1.12) *holds for any $\gamma \in [1, \infty]$.*

Proof. By the monotonicity results in Lemmas 4.2 and 4.3, $H_{m,\infty}^F(r)$ can be attained by the minimizer $(a, b) = (r, r)$ for any $F \in \{D, N, G\}$ and $m \in \mathbb{N}$. Thus (1.11) holds for $\gamma = \infty$.

Now we prove (1.11) for $\gamma \in (1, \infty)$. Suppose (a_0, b_0) is the minimizer of $\nu := H_m^F(r)$ and X is the corresponding half-eigenfunction. Let $w_0 = a_0 + b_0$. By Theorem 4.5, $(a_0, b_0) \in S_+^\gamma(r)$ and a_0 and b_0 do not overlap, thus

$$\|w_0\|_\gamma = |(a_0, b_0)|_\gamma = r. \tag{4.25}$$

Combining Corollary 4.6, one has

$$(\phi_p(X'))' + \nu w_0(t) \phi_p(X) = 0. \tag{4.26}$$

Hence

$$H_{m,\gamma}^F(r) = \nu \ge E_{m,\gamma}^F(r). \tag{4.27}$$

On the other hand, for any $w \in B^\gamma[r]$ and $\lambda \in \mathbb{R}$, one has $|(w_+, w_-)|_\gamma = \|w\|_\gamma$ and

$$\begin{aligned} &(\phi_p(x'))' + \lambda w(t) \phi_p(x) = 0 \\ &\quad \Longleftrightarrow (\phi_p(x'))' + \lambda w_+(t) \phi_p(x_+) + \lambda w_-(t) \phi_p(x_-) = 0. \end{aligned} \tag{4.28}$$

Take the notations $\lambda_m^G(a,b) := \underline{\lambda}_m^L(a,b)$ and $\lambda_m^G(a) := \underline{\lambda}_m^L(a)$ for any $a, b \in \mathcal{L}^\gamma$. Then $\lambda_m^F(w) = \lambda_m^F(w_+, w_-)$ for any $F \in \{D, N, G\}$ and

$$
\begin{aligned}
E_{m,\gamma}^F(r) &= \inf\left\{\lambda_m^F(w) : w \in B^\gamma[r]\right\} \\
&= \inf\left\{\lambda_m^F(w_+, w_-) : w \in B^\gamma[r]\right\} \\
&\geq \inf\left\{\lambda_m^F(a,b) : (a,b)_\gamma \in \widetilde{B}^\gamma[r]\right\} = H_{m,\gamma}^F(r).
\end{aligned}
\tag{4.29}
$$

Therefore (1.11) is proved for $\gamma \in (1, \infty)$.

One can obtain (1.12) for any $\gamma \in [1, \infty]$ by the fact that the half-eigenfunction corresponding to $\lambda_0^N(a,b)$ or $\overline{\lambda}_0^R(a,b)$ does not change its sign. □

4.4. The Infimum in L^1 Balls

We cannot handle extremal problem in L^1 balls in the same way as done for $L^\gamma (\gamma > 1)$ case, because L^1 balls are not sequentially compact even in the sense of weak topology.

Lemma 4.8. *Given $\gamma \in (1, \infty)$, $r > 0$, and $m \in \mathbb{N}$, there hold the following properties.*

(i) *If $\underline{\lambda}_m^L(a_0, b_0) < \infty$, then there exists $\delta > 0$ such that*

$$
\underline{\lambda}_m^L(a,b) < \infty, \quad \forall (a,b) \in \widetilde{B}_\delta^\gamma(a_0, b_0). \tag{4.30}
$$

(ii) *If $\lambda_m^{D/N}(a_0, b_0) < \infty$, then there exists $\delta > 0$ such that*

$$
\lambda_m^{D/N}(a,b) < \infty, \quad \forall (a,b) \in \widetilde{B}_\delta^\gamma(a_0, b_0). \tag{4.31}
$$

Proof. (i) Suppose $\underline{\lambda}_m^L(a_0, b_0) < \infty$. By Theorem 2.4(i) there exist $\varepsilon > 0$ and $\nu > \underline{\lambda}_m^L(a_0, b_0)$ such that

$$
\underline{\Theta}(\nu a_0, \nu b_0) > m\pi_p + 2\varepsilon. \tag{4.32}
$$

By the definition of $\underline{\Theta}$ in (2.8), there is $\vartheta_0 \in \mathbb{R}$ such that

$$
\Theta(\vartheta_0, \nu a_0, \nu b_0) - \vartheta_0 > m\pi_p + 2\varepsilon. \tag{4.33}
$$

By Lemma 2.1(i), that is, the continuous dependence of $\Theta(\vartheta, a, b)$ in the weights (a,b), there exists $\delta > 0$ such that

$$
\Theta(\vartheta_0, \nu a, \nu b) - \vartheta_0 > m\pi_p + \varepsilon, \quad \forall (a,b) \in \widetilde{B}_\delta^\gamma(a_0, b_0). \tag{4.34}
$$

Therefore,

$$\underline{\Theta}(\nu a, \nu b) > m\pi_p + \varepsilon, \quad \forall (a,b) \in \widetilde{B}_\delta^\gamma(a_0, b_0). \tag{4.35}$$

We conclude from (2.29) that

$$\underline{\lambda}_m^L(a,b) < \nu < \infty, \quad \forall (a,b) \in \widetilde{B}_\delta^\gamma(a_0, b_0), \tag{4.36}$$

completing the proof of (i).

(ii) Suppose $\mu := \lambda_m^{D/N}(a_0, b_0) < \infty$. Let X be the half-eigenfunction corresponding to μ. Then X satisfies Dirichlet or Neumann boundary value conditions and

$$(\phi_p(X'))' + \mu a_0(t)\phi_p(X_+) - \mu b_0 \phi_p(X_-) = 0. \tag{4.37}$$

Multiplying (4.37) by X and integrating over $[0,1]$, one has

$$\int_0^1 \left(a_0 X_+^p + b_0 X_-^p\right) dt = \frac{1}{\mu}\int_0^1 |X'|^p dt > 0. \tag{4.38}$$

Let $\vartheta^D = -\pi_p/2$ and $\vartheta^N = 0$. By Lemma 2.1(iii), one has

$$\frac{d}{d\lambda}\Theta\left(\vartheta^{D/N}, \lambda a, \lambda b\right)\Big|_{\lambda=\mu} = \int_0^1 \left(a_0 X_+^p + b_0 X_-^p\right) dt > 0. \tag{4.39}$$

Notice that $\Theta(\vartheta^{D/N}, \mu a_0, \mu b_0) = \vartheta^{D/N} + m\pi_p$. Then there exist $\varepsilon > 0$ and $\nu > \mu = \lambda_m^{D/N}(a_0, b_0)$ such that

$$\Theta\left(\vartheta^{D/N}, \nu a_0, \nu b_0\right) > \vartheta^{D/N} + m\pi_p + 2\varepsilon. \tag{4.40}$$

By Lemma 2.1(i), there exists $\delta > 0$ such that for any $(a,b) \in \widetilde{B}_\delta^\gamma(a_0, b_0)$, one has

$$\Theta\left(\vartheta^{D/N}, \nu a, \nu b\right) > \vartheta^{D/N} + m\pi_p + \varepsilon, \tag{4.41}$$

and hence $\lambda_m^{D/N}(a,b) < \infty$, completing the proof of (ii). □

As a function of α, $\mathrm{K}(\alpha, p)$ is continuous in $\alpha \in [1, \infty]$. Explicit formula of $\mathrm{K}(\alpha, p)$ can be found in [17, Theorem 4.1]. For instance, $\mathrm{K}(p,p) = \pi_p^p$ and $\mathrm{K}(\infty, p) = 2^p$.

Theorem 4.9. *For any $r > 0$, (1.11) holds for $\gamma = 1$, that is,*

$$H_{m,1}^F(r) = E_{m,1}^F(r) = \frac{(2m)^p}{r}, \quad \forall m \in \mathbb{N}, \quad \forall F \in \{D, N, G\}. \tag{4.42}$$

Proof. By Theorem 4.7, (1.11) holds for any $\gamma \in (1,\infty]$. As the Sobolev constant $\mathrm{K}(\alpha,p)$ is continuous in $\alpha \in [1,\infty]$, one has $\lim_{\gamma\downarrow 1} H^F_{m,\gamma}(r) = (2m)^p/r$.

Our first aim is to prove

$$H^F_{m,1}(r) \geq \lim_{\gamma\downarrow 1} H^F_{m,\gamma}(r) = \frac{(2m)^p}{r}. \tag{4.43}$$

Any $(a_0,b_0) \in \widetilde{B}^1[r]$ can be approximated by elements in $\widetilde{B}^\gamma[r]$, $\gamma > 1$, in the sense that there exists $(a_\gamma,b_\gamma) \in \widetilde{B}^\gamma[r]$ such that

$$\lim_{\gamma\downarrow 1}\left|(a_\gamma,b_\gamma) - (a_0,b_0)\right|_1 = 0. \tag{4.44}$$

For instance, one can choose

$$a_\gamma = r^{1/\gamma^*}|a_0(t)|^{1/\gamma}\cdot \operatorname{sign}(a_0(t)), \qquad b_\gamma = r^{1/\gamma^*}|b_0(t)|^{1/\gamma}\cdot \operatorname{sign}(b_0(t)), \tag{4.45}$$

compare, for example, [11, Lemma 2.1]. For simplicity, we take the notation

$$\lambda^G_m(a,b) := \underline{\lambda}^L_m(a,b). \tag{4.46}$$

Given m and F. Suppose $\lambda^F_m(a_0,b_0) < \infty$. By Lemma 4.8, there exists $\delta > 0$ such that $\lambda^F_m(a,b)$ exists for any $(a,b) \in B^\gamma_\delta(a_0,b_0)$. We can assume that $\lambda^F_m(a_\gamma,b_\gamma)$ exists for any $\gamma \in (1,\infty)$ due to (4.44). Furthermore, by Lemma 2.1(iii), one can prove that $\lambda^F_m(a,b)$ is continuously differentiable in $(a,b) \in B^\gamma_\delta(a_0,b_0)$ in $|\cdot|_1$ topology. In particular, $\lambda^F_m(a,b)$ is continuous in $(a,b) \in B^\gamma_\delta(a_0,b_0)$ in $|\cdot|_1$ topology. Thus we obtain

$$\lambda^F_m(a_0,b_0) = \lim_{\gamma\downarrow 1}\lambda^F_m(a_\gamma,b_\gamma) \geq \lim_{\gamma\downarrow 1} H^F_{m,\gamma}(r), \tag{4.47}$$

and therefore,

$$H^F_{m,1}(r) = \inf\left\{\lambda^F_m(a_0,b_0) \mid (a_0,b_0) \in \widetilde{B}^1[r]\right\} \geq \lim_{\gamma\downarrow 1} H_{m,\gamma}(r) = \frac{(2m)^p}{r}. \tag{4.48}$$

On the other hand, we prove

$$H^F_{m,1}(r) \leq \frac{(2m)^p}{r}. \tag{4.49}$$

Notice that $\widetilde{B}^\gamma[2^{1/\gamma-1}r] \subset \widetilde{B}^1[r]$ for all $\gamma > 1$ and all $r > 0$, because

$$|(a,b)|_1 = \|a\|_1 + \|b\|_1 \leq \|a\|_\gamma + \|b\|_\gamma \leq 2^{1-1/\gamma}\left(\|a\|^\gamma_\gamma + \|b\|^\gamma_\gamma\right)^{1/\gamma} \tag{4.50}$$

for any $(a, b) \in \mathcal{L}^\gamma \times \mathcal{L}^\gamma$. Thus we obtain

$$H_{m,1}^F(r) \le H_{m,\gamma}^F\left(2^{1-1/\gamma} r\right) = m^p \cdot \frac{\mathrm{K}(p\gamma^*, p)}{2^{1-1/\gamma} r}. \tag{4.51}$$

Inequality (4.49) follows immediately by letting $\gamma \downarrow 1$. The desired result is proved by combining (4.43) and (4.49). □

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant no. 10901089).

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