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Research Article

One-Dimensional Compressible Viscous Micropolar Fluid Model: Stabilization of the Solution for the Cauchy Problem

Nermina Mujaković

Department of Mathematics, University of Rijeka, Omladinska 14, 51000 Rijeka, Croatia

Correspondence should be addressed to Nermina Mujaković, mujakovic@inet.hr

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We consider the Cauchy problem for nonstationary 1D flow of a compressible viscous and heat-conducting micropolar fluid, assuming that it is in the thermodynamical sense perfect and polytropic. This problem has a unique generalized solution on $\mathbf{R}\times]0,T[$ for each $T>0$. Supposing that the initial functions are small perturbations of the constants we derive a priori estimates for the solution independent of T, which we use in proving of the stabilization of the solution.

1. Introduction

In this paper we consider the Cauchy problem for nonstationary 1D flow of a compressible viscous and heat-conducting micropolar fluid. It is assumed that the fluid is thermodynamically perfect and polytropic. The same model has been considered in [1, 2], where the global-in-time existence and uniqueness for the generalized solution of the problem on $\mathbf{R}\times]0,T[$, $T>0$, are proved. Using the results from [1, 3] we can also easily conclude that the mass density and temperature are strictly positive.

Stabilization of the solution of the Cauchy problem for the classical fluid (where microrotation is equal to zero) has been considered in [4, 5]. In [4] was analyzed the Hölder continuous solution. In [5] is considered the special case of our problem. We use here some ideas of Kanel' [4] and the results from [1, 5] as well.

Assuming that the initial functions are small perturbations of the constants, we first derive a priori estimates for the solution independent of T. In the second part of the work we analyze the behavior of the solution as $T\to\infty$. In the last part we prove that the solution of our problem converges uniformly on $\mathbf{R}$ to a stationary one.

The case of nonhomogeneous boundary conditions for velocity and microrotation which is called in gas dynamics "problem on piston" is considered in [6].

2. Statement of the Problem and the Main Result

Let ρ, v, ω, and θ denote, respectively, the mass density, velocity, microrotation velocity, and temperature of the fluid in the Lagrangean description. The problem which we consider has the formulation as follows [1]:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - K \frac{\partial}{\partial x}(\rho\theta), \tag{2.2}$$

$$\rho \frac{\partial \omega}{\partial t} = A\left[\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \omega}{\partial x}\right) - \omega\right], \tag{2.3}$$

$$\rho \frac{\partial \theta}{\partial t} = -K\rho^2\theta \frac{\partial v}{\partial x} + \rho^2\left(\frac{\partial v}{\partial x}\right)^2 + \rho^2\left(\frac{\partial \omega}{\partial x}\right)^2 + \omega^2 + D\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) \tag{2.4}$$

in $R \times R^{+}$, where K, A, and D are positive constants. Equations (2.1)–(2.4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment, and energy. We take the following nonhomogeneous initial conditions:

$$\begin{aligned} \rho(x,0) &= \rho_0(x), \\ v(x,0) &= v_0(x), \\ \omega(x,0) &= \omega_0(x), \\ \theta(x,0) &= \theta_0(x) \end{aligned} \tag{2.5}$$

for $x \in \mathbf{R}$, where ρ_0, v_0, ω_0, and θ_0 are given functions. We assume that there exist $m, M \in \mathbf{R}^{+}$, such that

$$m \le \rho_0(x) \le M, \quad m \le \theta_0(x) \le M, \quad x \in \mathbf{R}. \tag{2.6}$$

In the previous papers [1, 2] we proved that for

$$\rho_0 - 1, v_0, \omega_0, \theta_0 - 1 \in H^1(R) \tag{2.7}$$

the problem (2.1)–(2.5) has, for each $T \in \mathbf{R}^{+}$, a unique generalized solution:

$$(x,t) \longrightarrow (\rho, v, \omega, \theta)(x,t) \quad (x,t) \in \Pi = \mathbf{R} \times \left]0, T\right[, \tag{2.8}$$

with the following properties:

$$\rho - 1 \in L^{\infty}\Big(0, T; H^1(R)\Big) \cap H^1(\Pi),$$
$$v, \omega, \theta - 1 \in L^{\infty}\Big(0, T; H^1(R)\Big) \cap H^1(\Pi) \cap L^2\Big(0, T; H^2(R)\Big). \tag{2.9}$$

Using the results from [1, 3] we can easily conclude that

$$\theta, \rho > 0 \quad \text{in } \Pi. \tag{2.10}$$

We denote by $B^k(R)$, $k \in \mathbf{N_0}$, the Banach space

$$B^k(R) = \left\{ u \in C^k(R) : \lim_{|x| \to \infty} |D^n u(x)| = 0,\, 0 \le n \le k \right\}, \tag{2.11}$$

where D^n is nth derivative; the norm is defined by

$$\|u\|_{B^k(R)} = \sup_{n \le k} \left\{ \sup_{x \in R} |D^n u(x)| \right\}. \tag{2.12}$$

From Sobolev's embedding theorem [7, Chapter IV] and the theory of vector-valued distributions [8, pages 467–480] one can conclude that from (2.9) one has

$$\rho - 1 \in L^{\infty}\Big(0, T; B^0(R)\Big) \cap C\Big([0, T]; L^2(R)\Big), \tag{2.13}$$

$$v, \omega, \theta - 1 \in L^2\Big(0, T; B^1(R)\Big) \cap C\Big([0, T]; H^1(R)\Big) \cap L^{\infty}\Big(0, T; B^0(R)\Big), \tag{2.14}$$

and hence

$$v, \omega, \theta - 1 \in C\Big([0, T]; B^0(R)\Big), \quad \rho \in L^{\infty}(\Pi). \tag{2.15}$$

From (2.7) and (2.6) it is easy to see that there exist the constants $E_1, E_2, E_3, M_1 \in \mathbf{R}^+$, $M_1 > 1$, such that

$$\frac{1}{2} \int_R v_0^2 dx + \frac{1}{2A} \int_R \omega_0^2 dx + K \int_R \left(\frac{1}{\rho_0} - \ln \frac{1}{\rho_0} - 1 \right) dx + \int_R (\theta_0 - \ln \theta_0 - 1) dx = E_1, \tag{2.16}$$

$$\frac{1}{2} \int_R \left(v_0'^2 + \frac{1}{A} \omega_0'^2 + \theta_0'^2 \right) dx = E_2, \tag{2.17}$$

$$\frac{1}{2} \int_R \frac{1}{\rho_0^2} \rho_0'^2 dx + \int_R v_0' \ln \frac{1}{\rho_0} dx \le E_3, \tag{2.18}$$

$$\sup_{|x| < \infty} \theta_0(x) < M_1. \tag{2.19}$$

As in [5], we can find out the real numbers $\underline{\eta}$ and $\overline{\eta}$, $\underline{\eta} < 0 < \overline{\eta}$, such that

$$\int_{\underline{\eta}}^{0} \sqrt{e^{\eta} - 1 - \eta}\, d\eta = \int_{0}^{\overline{\eta}} \sqrt{e^{\eta} - 1 - \eta}\, d\eta = E_5, \tag{2.20}$$

where

$$E_5 = 2\left(\frac{E_1 E_4}{K}\right)^{1/2}, \qquad E_4 = 2\mu E_1\left(1 + M_1 + \frac{E_3}{E_1}\right), \quad \mu = \max\left\{\frac{K}{2D}, 1\right\}. \tag{2.21}$$

Using $\underline{\eta}$ and $\overline{\eta}$ we construct the quantities

$$\underline{u} = \exp \underline{\eta}, \qquad \overline{u} = \exp \overline{\eta}. \tag{2.22}$$

The aim of this work is to prove the following theorem.

Theorem 2.1. *Suppose that the initial functions satisfy* (2.6), (2.7), *and the following conditions:*

$$E_1 \int_R v_0'^2 dx < \left(\frac{D}{24}\frac{\underline{u}}{\overline{u}}\right)^2, \tag{2.23}$$

$$E_1 A \int_R \omega_0'^2 dx < \left(\frac{D}{12}\frac{\underline{u}}{\overline{u}}\right)^2, \tag{2.24}$$

$$\begin{aligned} 2E_1 \Bigg\{ & 48 E_4^2 M_1 E_1 \left(\frac{\overline{u}}{\underline{u}}\right)^2 (8 + 9M_1) + 2K M_1 E_4 + \frac{E_1 K^2 M_1^2}{D}\left(\frac{\overline{u}}{\underline{u}} + \frac{3M_1}{2}\right) \\ & + E_1 M_1 \overline{u}^2 \left(1 + \frac{3AE_1}{D}\left(\frac{\overline{u}}{\underline{u}} + \overline{u}^2\right)\right) + E_2 \Bigg\} \\ & < \min\left\{ \left(\frac{D}{12A}\frac{\underline{u}}{\overline{u}}\right)^2, \left(\frac{D}{24}\frac{\underline{u}}{\overline{u}}\right)^2, \left(\int_1^{M_1} \sqrt{s - 1 - \ln s}\, ds\right)^2, \left(\int_0^1 \sqrt{s - 1 - \ln s}\, ds\right)^2 \right\}; \end{aligned} \tag{2.25}$$

then, when $t \longrightarrow \infty$,

$$\rho(x,t) \longrightarrow 1, \qquad v(x,t) \longrightarrow 0, \qquad \omega(x,t) \longrightarrow 0, \qquad \theta(x,t) \longrightarrow 1, \tag{2.26}$$

uniformly with respect to all $x \in \mathbf{R}$.

Remark 2.2. Conditions (2.23)–(2.25) mean that the constants E_1, E_2, E_3, and M_1 are sufficiently small. In other words the initial functions ρ_0, v_0, ω_0, and θ_0 are small perturbations of the constants.

In the proof of Theorem 2.1, we apply some ideas of [4] and obtain the similar results as in [5] where a stabilisation of the generalized solution was proved for the classical model (where $\omega = 0$).

3. A Priori Estimates for ρ, v, ω, and θ

Considering stabilization problem, one has to prove some a priori estimates for the solution independent of the time variable T, which is the main difficulty. When we derive these estimates we use some ideas from [4, 5]. First we construct the energy equation for the solution of problem (2.1)–(2.4) under the conditions indicated above and we estimate the function $1/\rho$.

Lemma 3.1. *For each $t > 0$ it holds that*

$$\frac{1}{2}\int_R v^2 dx + \frac{1}{2A}\int_R \omega^2 dx + \int_R (\theta - \ln\theta - 1)dx + K\int_R \left(\frac{1}{\rho} - \ln\frac{1}{\rho} - 1\right)dx \\ + \int_0^t\int_R \left(\frac{\rho}{\theta}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\rho}{\theta}\left(\frac{\partial \omega}{\partial x}\right)^2 + \frac{\omega^2}{\rho\theta} + D\frac{\rho}{\theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx\,d\tau = E_1, \tag{3.1}$$

where E_1 is defined by (2.16).

Proof. Multiplying (2.1), (2.2), (2.3), and (2.4), respectively, by $K\rho^{-1}(1-\rho^{-1}), v, A^{-1}\rho^{-1}\omega$, and $\rho^{-1}(1-\theta^{-1})$, integrating by parts over $\mathbf{R}$ and over $]0,t[$, and taking into account (2.13) and (2.14), after addition of the obtained equations we easily get equality (3.1) independently of t. □

If we multiply (2.2) by $(\partial/\partial x)\ln(1/\rho)$, integrate it over $\mathbf{R}$ and $]0,t[$, and use some equalities and inequalities which hold by (2.1) and (2.13)–(2.15) together with Young's inequality we get, as in [5], the following formula:

$$\frac{1}{4}\int_R \rho^2\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx + \frac{K}{2}\int_0^t\int_R \theta\rho^3\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx\,d\tau \\ \leq \int_R v^2 dx + \frac{K}{2}\int_0^t\int_R \frac{\rho}{\theta}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau + \int_0^t\int_R \rho\left(\frac{\partial v}{\partial x}\right)^2 dx\,d\tau + \int_R v_0' \ln\left(\frac{1}{\rho_0}\right)dx \\ + \frac{1}{2}\int_R \frac{1}{\rho_0^2}\rho_0'^2 dx \tag{3.2}$$

for each $t > 0$. Using (3.1), (2.16), (2.18), and (2.21) we get easily

$$\frac{1}{4}\int_R \rho^2\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx + \frac{K}{2}\int_0^t\int_R \theta\rho^3\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx\,d\tau \leq K_1\left(\overline{\theta}(t)\right), \tag{3.3}$$

where

$$\overline{\theta}(t) = \sup_{(x,\tau)\in R\times]0,t[} \theta(x,\tau), \qquad K_1\left(\overline{\theta}(t)\right) = 2\mu E_1\left(1+\overline{\theta}(t)+\frac{E_3}{E_1}\right). \tag{3.4}$$

As in [5], we introduce the increasing function

$$\psi(\eta) = \int_0^\eta \sqrt{e^\xi - 1 - \xi}\, d\xi \tag{3.5}$$

that satisfies the following inequality:

$$\left|\psi\left(\ln\frac{1}{\rho}\right)\right| \le \left(\int_R \left(\frac{1}{\rho} - 1 - \ln\frac{1}{\rho}\right)dx\right)^{1/2}\left(\int_R \rho^2\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx\right)^{1/2} \le K_2\left(\overline{\theta}(t)\right), \tag{3.6}$$

where

$$K_2\left(\overline{\theta}(t)\right) = 2E_1\left(\frac{2\mu}{K}\left(1+\overline{\theta}(t)+\frac{E_3}{E_1}\right)\right)^{1/2}. \tag{3.7}$$

We can easily conclude that there exist the quantities $\underline{\eta}_1(\overline{\theta}(t)) < 0$ and $\overline{\eta}_1(\overline{\theta}(t)) > 0$, such that

$$\int_{\underline{\eta}_1(\overline{\theta}(t))}^0 \sqrt{e^\eta - 1 - \eta}\, d\eta = \int_0^{\overline{\eta}_1(\overline{\theta}(t))} \sqrt{e^\eta - 1 - \eta}\, d\eta = K_2\left(\overline{\theta}(t)\right). \tag{3.8}$$

Comparing (3.8) and (3.6) we obtain, as in [5, Lemma 3.2], the following result.

Lemma 3.2. *For each $t > 0$ there exist the strictly positive quantities $\underline{u}_1 = \exp\underline{\eta}_1(\overline{\theta}(t))$ and $\overline{u}_1 = \exp\overline{\eta}_1(\overline{\theta}(t))$ such that*

$$\underline{u}_1 \le \rho^{-1}(x,\tau) \le \overline{u}_1, \quad (x,\tau) \in \mathbf{R}\times]0,t[. \tag{3.9}$$

Now we find out some estimates for the derivatives of the functions v, ω, and θ.

Lemma 3.3. *For each $t > 0$ it holds that*

$$
\begin{aligned}
&\frac{1}{2}\int_R\left(\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{A}\left(\frac{\partial \omega}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx\\
&\quad+\left(\frac{6\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2dx\right)^2\left(K_3\left(\overline{\theta}(\tau)\right)-\int_R\left(\frac{\partial v}{\partial x}\right)^2dx\right)d\tau\\
&\quad+2\left(\frac{3\overline{u}_1}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial \omega}{\partial x}\right)^2dx\right)^2\left(K_4\left(\overline{\theta}(\tau)\right)-\int_R\left(\frac{\partial \omega}{\partial x}\right)^2dx\right)d\tau\\
&\quad+\frac{D}{12}\int_0^t\int_R\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dx\,d\tau\leq K_5\left(\overline{\theta}(t)\right),
\end{aligned}
\tag{3.10}
$$

where

$$
K_3\left(\overline{\theta}(t)\right)=\left(\frac{D\underline{u}_1}{24\overline{u}_1E_1^{1/2}}\right)^2,
\tag{3.11}
$$

$$
K_4\left(\overline{\theta}(t)\right)=\left(\frac{D\underline{u}_1}{12\overline{u}_1A^{1/2}E_1^{1/2}}\right)^2,
\tag{3.12}
$$

$$
\begin{aligned}
K_5\left(\overline{\theta}(t)\right)&=48K_1^2\left(\overline{\theta}(t)\right)\overline{\theta}(t)E_1\left(\frac{\overline{u}_1}{\underline{u}_1}\right)^2\left(8+9\overline{\theta}(t)\right)+2K\overline{\theta}(t)K_1\left(\overline{\theta}(t)\right)\\
&\quad+\frac{E_1K^2\overline{\theta}^2(t)}{D}\left(\frac{\overline{u}_1}{\underline{u}_1}+\frac{3\overline{\theta}(t)}{2}\right)+E_1\overline{\theta}(t)\overline{u}_1^2\left(1+\frac{3AE_1}{D}\left(\frac{\overline{u}_1}{\underline{u}_1}+\overline{u}_1^2\right)\right)+E_2.
\end{aligned}
\tag{3.13}
$$

Proof. Multiplying (2.2), (2.3), and (2.4), respectively, by $-\partial^2 v/\partial x^2$, $-A^{-1}\rho^{-1}(\partial^2\omega/\partial x^2)$, and $-\rho^{-1}(\partial^2\theta/\partial x^2)$, integrating over $\mathbf{R}\times]0,t[$, and using the following equality:

$$
-\int_0^t\int_R\frac{\partial v}{\partial t}\frac{\partial^2 v}{\partial x^2}dx\,d\tau=\frac{1}{2}\int_R\left(\frac{\partial v}{\partial x}\right)^2dx\Big|_0^t
\tag{3.14}
$$

that is satisfied for the functions θ and ω as well, after addition of the obtained equalities we find that

$$
\begin{aligned}
&\frac{1}{2}\int_R\left(\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{A}\left(\frac{\partial \omega}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx\Big|_0^t+\int_0^t\int_R\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2dx\,d\tau\\
&\quad+\int_0^t\int_R\rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2dx\,d\tau+D\int_0^t\int_R\rho\left(\frac{\partial^2 \theta}{\partial x^2}\right)^2dx\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^t\int_R \frac{\partial\rho}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}dx\,d\tau + K\int_0^t\int_R \rho\frac{\partial\theta}{\partial x}\frac{\partial^2 v}{\partial x^2}dx\,d\tau + K\int_0^t\int_R \theta\frac{\partial\rho}{\partial x}\frac{\partial^2 v}{\partial x^2}dx\,d\tau \\
&\quad -\int_0^t\int_R \frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}dx\,d\tau + \int_0^t\int_R \frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2}dx\,d\tau + K\int_0^t\int_R \rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\,d\tau \\
&\quad -\int_0^t\int_R \rho\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}dx\,d\tau - \int_0^t\int_R \rho\left(\frac{\partial\omega}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}dx\,d\tau - \int_0^t\int_R \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx\,d\tau \\
&\quad - D\int_0^t\int_R \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\,d\tau.
\end{aligned}
\tag{3.15}
$$

Using (3.3), (3.9), the inequality

$$
\left(\frac{\partial v}{\partial x}\right)^2 \le 2\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2\right)^{1/2}\left(\int_R\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right)^{1/2}, \tag{3.16}
$$

that holds for the functions $\partial\theta/\partial x$, $\partial\omega/\partial x$, v, and ω as well, and applying Young's inequality with a sufficiently small parameter on the right-hand side of (3.15), we find, similarly as in [5], the following estimates:

$$
\begin{aligned}
&\left|\int_0^t\int_R \frac{\partial\rho}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \int_0^t\int_R \frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial v}{\partial x}\right)^2 dx\,d\tau + \frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{2\overline{u}_1^{1/2}}{\underline{u}_1}\int_0^t\left(\int_R \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 dx\right)\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^{1/2}\left(\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\right)^{1/2} d\tau \\
&\qquad + \frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{5}{16}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau + \left(16K_1\left(\overline{\theta}(t)\right)\right)^2\frac{\overline{u}_1}{\underline{u}_1^2}\int_0^t\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\,d\tau \\
&\quad\le \frac{5}{16}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau + \left(16K_1\left(\overline{\theta}(t)\right)\frac{\overline{u}_1}{\underline{u}_1}\right)^2\overline{\theta}(t)\int_0^t\int_R \frac{\rho}{\theta}\left(\frac{\partial v}{\partial x}\right)^2 dx\,d\tau,
\end{aligned}
\tag{3.17}
$$

$$\begin{aligned}
&\left|K\int_0^t\int_R \rho\frac{\partial\theta}{\partial x}\frac{\partial^2 v}{\partial x^2}dx\,d\tau\right| \\
&\quad\le K^2\int_0^t\int_R \rho\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{K^2\overline{u}_1\overline{\theta}^{2}(t)}{\underline{u}_1}\int_0^t\int_R \frac{\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned}\tag{3.18}$$

$$\begin{aligned}
&\left|K\int_0^t\int_R \theta\frac{\partial\rho}{\partial x}\frac{\partial^2 v}{\partial x^2}dx\,d\tau\right| \\
&\quad\le K^2\int_0^t\int_R \frac{\theta^2}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2 dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le K^2\overline{\theta}(t)\int_0^t\int_R \theta\rho^3\left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)^2 dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned}\tag{3.19}$$

$$\begin{aligned}
&\left|\int_0^t\int_R \frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2 \omega}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \int_0^t\int_R \frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial\omega}{\partial x}\right)^2 dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{2\overline{u}_1^{1/2}}{\underline{u}_1}\int_0^t\left(\int_R \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 dx\right)\left(\int_R \left(\frac{\partial\omega}{\partial x}\right)^2 dx\right)^{1/2}\left(\int_R \rho\left(\frac{\partial^2\omega}{\partial x^2}\right)^2 dx\right)^{1/2} d\tau \\
&\qquad+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{3}{8}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau+2\left(\frac{8K_1\left(\overline{\theta}(t)\right)\overline{u}_1^{1/2}}{\underline{u}_1}\right)^2\int_0^t\int_R \left(\frac{\partial\omega}{\partial x}\right)^2 dx\,d\tau \\
&\quad\le \frac{3}{8}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau+2\left(8K_1\left(\overline{\theta}(t)\right)\frac{\overline{u}_1}{\underline{u}_1}\right)^2\overline{\theta}(t)\int_0^t\int_R \frac{\rho}{\theta}\left(\frac{\partial\omega}{\partial x}\right)^2 dx\,d\tau,
\end{aligned}\tag{3.20}$$

$$\begin{aligned}
&\left|\int_0^t\int_R \frac{\omega}{\rho}\frac{\partial^2 \omega}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \int_0^t\int_R \frac{\omega^2}{\rho^3}dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \overline{u}_1^2\overline{\theta}(t)\int_0^t\int_R \frac{\omega^2}{\rho\theta}dx\,d\tau+\frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned}\tag{3.21}$$

$$\begin{aligned}
&\left| K\int_0^t\int_R \rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \frac{3K^2}{2D}\int_0^t\int_R \theta^2\rho\left(\frac{\partial v}{\partial x}\right)^2 dx\,d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{3K^2\overline{\theta}^3(t)}{2D}\int_0^t\int_R \frac{\rho}{\theta}\left(\frac{\partial v}{\partial x}\right)^2 dx\,d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\left|\int_0^t\int_R \rho\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \frac{3}{2D}\int_0^t\int_R \rho\left(\frac{\partial v}{\partial x}\right)^4 dx\,d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{3\overline{u}_1^{1/2}}{D\underline{u}_1}\int_0^t\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^{3/2}\left(\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\right)^{1/2} d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le 4\left(\frac{3\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^3 d\tau + \frac{1}{16}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau \\
&\qquad + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&\left|\int_0^t\int_R \rho\left(\frac{\partial \omega}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \frac{3}{2D}\int_0^t\int_R \rho\left(\frac{\partial \omega}{\partial x}\right)^4 dx\,d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{3\overline{u}_1^{1/2}}{D\underline{u}_1}\int_0^t\left(\int_R\left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)^{3/2}\left(\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\right)^{1/2} d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le 2\left(\frac{3\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)^3 d\tau + \frac{1}{8}\int_0^t\int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\,d\tau \\
&\qquad + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned} \tag{3.24}$$

$$
\begin{aligned}
&\left|\int_0^t\int_R \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \frac{3}{2D}\int_0^t\int_R \frac{\omega^4}{\rho^3}dx\,d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{6AE_1\overline{u}_1^3}{D}\int_0^t\left(\int_R \omega^2 dx\right)^{1/2}\left(\int_R \rho\left(\frac{\partial\omega}{\partial x}\right)^2 dx\right)^{1/2} d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{6AE_1\overline{u}_1^3}{D}\left(\frac{\overline{\theta}(t)}{2\underline{u}_1}\int_0^t\int_R \frac{\omega^2}{\rho\theta}x\,d\tau + \frac{\overline{u}_1\overline{\theta}(t)}{2}\int_0^t\int_R \frac{\rho}{\theta}\left(\frac{\partial\omega}{\partial x}\right)^2 dx\,d\tau\right) \\
&\qquad + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\left|D\int_0^t\int_R \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\,d\tau\right| \\
&\quad\le \frac{3D}{2}\int_0^t\int_R \frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \frac{3D\overline{u}_1^{1/2}}{\underline{u}_1}\int_0^t\left(\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\right)^{1/2}\left(\int_R \left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)^{1/2}\int_R \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 dx\,d\tau \\
&\qquad + \frac{D}{6}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad\le \left(\frac{12DK_1\left(\overline{\theta}(t)\right)\overline{\theta}(t)\overline{u}_1}{\underline{u}_1}\right)^2\frac{3}{D}\int_0^t\int_R \frac{\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau + \frac{D}{4}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau.
\end{aligned}
\tag{3.26}
$$

Inserting (3.17)–(3.26) into (3.15) and using estimates (3.1), (3.3), and (2.17) we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_R\left(\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{A}\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial x}\right)^2\right)dx + \frac{1}{8}\int_0^t\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,d\tau \\
&\quad + \frac{1}{4}\int_0^t\int_R \rho\left(\frac{\partial^2\omega}{\partial x^2}\right)^2 dx\,d\tau + \frac{D}{12}\int_0^t\int_R \rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx\,d\tau \\
&\le K_5\left(\overline{\theta}(t)\right) + \left(\frac{6\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^3 d\tau + 2\left(\frac{3\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial\omega}{\partial x}\right)^2 dx\right)^3 d\tau,
\end{aligned}
\tag{3.27}
$$

where $K_5(\overline{\theta}(t))$ is defined by (3.10). We also use the following inequality:

$$\begin{aligned}\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx &= -\int_R v\frac{\partial^2 v}{\partial x^2}dx \le \overline{u}_1^{1/2}\left(\int_R v^2 dx\right)^{1/2}\left(\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\right)^{1/2}\\ &\le (2E_1\overline{u}_1)^{1/2}\left(\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\right)^{1/2}\end{aligned} \tag{3.28}$$

that is satisfied for the function $\partial\omega/\partial x$ as well. Therefore we have

$$\begin{aligned}\int_R \rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx &\ge (2E_1\overline{u}_1)^{-1}\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^2,\\ \int_R \rho\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx &\ge (2AE_1\overline{u}_1)^{-1}\left(\int_R\left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)^2.\end{aligned} \tag{3.29}$$

Inserting (3.29) into (3.27) we obtain

$$\begin{aligned}&\frac{1}{2}\int_R\left(\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{A}\left(\frac{\partial \omega}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx+\frac{D}{12}\int_0^t\int_R\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 dx d\tau\\ &\quad+2\left(\frac{3\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)^2\left(\left(\frac{D\underline{u}_1}{12\overline{u}_1A^{1/2}E_1^{1/2}}\right)^2-\int_R\left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)d\tau\\ &\quad+\left(\frac{6\overline{u}_1^{1/2}}{D\underline{u}_1}\right)^2\int_0^t\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)^2\left(\left(\frac{D\underline{u}_1}{24\overline{u}_1E_1^{1/2}}\right)^2-\int_R\left(\frac{\partial v}{\partial x}\right)^2 dx\right)d\tau\le K_5\left(\overline{\theta}(t)\right),\end{aligned} \tag{3.30}$$

and (3.10) is satisfied. □

In the continuation we use the above results and the conditions of Theorem 2.1. Similarly as in [4, 5], we derive the estimates for the solution $(\rho, v, \omega, \theta)$ of problem (2.1)–(2.7), defined by (2.8)–(2.10) in the domain $\Pi = \mathbf{R}\times]0,T[$, for arbitrary $T>0$.

Taking into account assumption (2.19) and the fact that $\theta\in C(\overline{\Pi})$ (see (2.15)) we have the following alternatives: either

$$\sup_{(x,t)\in\Pi}\theta(x,t)=\overline{\theta}(T)\le M_1, \tag{3.31}$$

or there exists t_1, $0<t_1<T$, such that

$$\overline{\theta}(t)<M_1 \quad \text{for } 0\le t<t_1, \qquad \overline{\theta}(t_1)=M_1. \tag{3.32}$$

Now we assume that (3.32) is satisfied and we will show later that because of the choice E_1, E_2, E_3, and M_1 (the conditions of Theorem 2.1), the property (3.32) is impossible.

Because $K_2(\overline{\theta}(t))$, defined by (3.7), increases with increasing $\overline{\theta}(t)$, we can easily conclude that

$$K_2\left(\overline{\theta}(t)\right) < K_2(M_1) \quad \text{for } 0 \le t < t_1, \tag{3.33}$$

and $K_2(M_1) = E_5$. Therefore, comparing (2.20) and (3.8), we obtain

$$\underline{u} < \underline{u}_1\left(\overline{\theta}(t)\right), \qquad \overline{u} > \overline{u}_1\left(\overline{\theta}(t)\right), \tag{3.34}$$

where $\underline{u}$, $\overline{u}$, and $\underline{u}_1(\overline{\theta}(t))$, $\overline{u}_1(\overline{\theta}(t))$ are defined by (2.22) and Lemma 3.2. It is important to point out that the quantities $K_3(\overline{\theta}(t))$ and $K_4(\overline{\theta}(t))$, defined by (3.11)-(3.12), decrease with increasing $\overline{\theta}(t)$ and for $\overline{\theta}(t_1) = M_1$ they become

$$K_3(M_1) = \left(\frac{D\underline{u}}{24\overline{u}E_1^{1/2}}\right)^2, \qquad K_4(M_1) = \left(\frac{D\underline{u}}{12\overline{u}A^{1/2}E_1^{1/2}}\right)^2. \tag{3.35}$$

Now, using these facts we will obtain the estimates for $\int_R (\partial\omega/\partial x)^2 dx$ and $\int_R (\partial v/\partial x)^2 dx$ on $[0, t_1]$. Taking into account the assumptions (2.23) and (2.24) of Theorem 2.1 and the following inclusion (see (2.14)):

$$\frac{\partial v}{\partial x}, \frac{\partial \omega}{\partial x} \in C\left([0,T]; L^2(R)\right), \tag{3.36}$$

we have again the following two alternatives: either

$$\int_R \left(\frac{\partial v}{\partial x}\right)^2 (x,t)dx \le K_3(M_1), \quad \int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t)dx \le K_4(M_1) \quad \text{for } t \in [0, t_1], \tag{3.37}$$

or there exists t_2, $0 < t_2 < t_1$, such that $\partial\omega/\partial x$ and $\partial v/\partial x$ (or conversely) have the following properties:

$$\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t)dx < K_4(M_1) \quad \text{for } 0 \le t < t_2, \tag{3.38}$$

$$\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t_2)dx = K_4(M_1) \quad \text{for } t_2 < t_1, \tag{3.39}$$

$$\int_R \left(\frac{\partial v}{\partial x}\right)^2 (x,t)dx \le K_3(M_1) \quad \text{for } 0 \le t \le t_2. \tag{3.40}$$

We assume that (3.38)–(3.40) are satisfied. Then we have

$$\overline{\theta}(t) < M_1, \quad K_4(M_1) < K_4\left(\overline{\theta}(t)\right), \quad K_3(M_1) < K_3\left(\overline{\theta}(t)\right) \tag{3.41}$$

for $t \in [0, t_2]$. Using (3.41) from (3.10), for $t = t_2$, we obtain

$$\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t)dx \leq 2AK_5\left(\overline{\theta}(t_2)\right), \quad 0 \leq t \leq t_2. \tag{3.42}$$

Since $K_5(\overline{\theta}(t))$, defined by (3.13), increases with the increase of $\overline{\theta}(t)$, it holds that

$$K_5\left(\overline{\theta}(t)\right) \leq K_5(M_1), \quad t \in [0, t_2]. \tag{3.43}$$

Using condition (2.25) we get

$$2AK_5\left(\overline{\theta}(t)\right) < K_4(M_1), \quad t \in [0, t_2] \tag{3.44}$$

and conclude that

$$\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t_2)dx < K_4(M_1). \tag{3.45}$$

This inequality contradicts (3.39). Consequently, the only case possible is when

$$t_2 = t_1, \tag{3.46}$$

and then for $\partial\omega/\partial x$ (3.37) is satisfied.

If in (3.38)–(3.40) the functions $\partial\omega/\partial x$ and $\partial v/\partial x$ exchange positions, using assumption (2.25), in the same way as above we obtain that the function $\partial v/\partial x$ satisfies the inequality

$$\int_R \left(\frac{\partial v}{\partial x}\right)^2 (x,t)dx \leq K_3(M_1), \quad t \in [0, t_1]. \tag{3.47}$$

With the help of (3.37) from (3.10) we can easily conclude that

$$\int_R \left(\frac{\partial \theta}{\partial x}\right)^2 (x,t)dx < 2K_5(M_1) \quad \text{for } 0 < t \leq t_1. \tag{3.48}$$

Now, as in [5], we introduce the function Ψ by

$$\Psi(\theta(x,t)) = \int_1^{\theta(x,t)} \sqrt{s - 1 - \ln s}\, ds. \tag{3.49}$$

Taking into account that from (2.14) follows that $\theta(x,t) \to 1$ as $|x| \to \infty$, we have

$$\Psi(\theta(x,t)) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \tag{3.50}$$

Consequently,

$$\begin{aligned}
\psi(\theta(x,t)) &\le \left|\psi(\theta(x,t))\right| \\
&= \left|\int_1^{\theta(x,t)} \frac{d}{ds}\psi(s)ds\right| \\
&= \left|\int_{-\infty}^{x} \sqrt{\theta(x,t)-1-\ln\theta(x,t)}\,\frac{\partial\theta(x,t)}{\partial x}dx\right| \\
&\le \left(\int_R (\theta(x,t)-1-\ln\theta(x,t))dx\right)^{1/2}\left(\int_R \left(\frac{\partial\theta}{\partial x}\right)^2(x,t)dx\right)^{1/2}.
\end{aligned} \tag{3.51}$$

Using (3.32), (3.48), and (3.1) from (3.51) we get

$$\max_{0\le\theta(x,t)\le M_1} \psi(\theta(x,t)) = \psi\left(\overline{\theta}(t_1)\right) = \psi(M_1) \le (2K_5(M_1)E_1)^{1/2}, \tag{3.52}$$

or

$$\int_1^{M_1} \sqrt{s-1-\ln s}\,ds - (2K_5(M_1)E_1)^{1/2} \le 0. \tag{3.53}$$

Since this inequality contradicts (2.25), it remains to assume that $t_1 = T$. Hence we have the following lemma.

Lemma 3.4. *For each $T > 0$ it holds that*

$$\theta(x,t) \le M_1, \quad (x,t) \in \Pi, \tag{3.54}$$

$$\int_R \left(\frac{\partial\omega}{\partial x}\right)^2(x,t)dx \le K_4(M_1), \quad 0 \le t \le T, \tag{3.55}$$

$$\int_R \left(\frac{\partial v}{\partial x}\right)^2(x,t)dx \le K_3(M_1), \quad 0 \le t \le T, \tag{3.56}$$

$$\int_R \left(\frac{\partial\theta}{\partial x}\right)^2(x,t)dx \le 2K_5(M_1), \quad 0 \le t \le T. \tag{3.57}$$

Proof. These conclusions follow from (3.32), (3.37), and (3.48) directly. □

Lemma 3.5. *It holds that*

$$0 < \underline{u} \leq \frac{1}{\rho(x,t)} \leq \overline{u}, \quad (x,t) \in \Pi, \tag{3.58}$$

$$\sup_{(x,t)\in\Pi} |\omega(x,t)| \leq (8AE_1K_4(M_1))^{1/4}, \tag{3.59}$$

$$\sup_{(x,t)\in\Pi} |v(x,t)| \leq (8E_1K_3(M_1))^{1/4}, \tag{3.60}$$

$$\theta(x,t) \geq h > 0, \quad (x,t) \in \Pi, \tag{3.61}$$

where $\underline{u}$ *and* $\overline{u}$ *are defined by* (2.20)–(2.22) *and a constant* h *depends only on the data of problem* (2.1)–(2.5).

Proof. Because the quantity $\underline{u}_1(\overline{\theta}(t))$ in Lemma 3.2 decreases with increasing $\overline{\theta}(t)$ while $\overline{u}_1(\overline{\theta}(t))$ increases, it follows, in the same way as in [5], from (3.9) and (3.54) that (3.58) is satisfied. Using the inequalities

$$\begin{aligned} v^2 &= 2\int_{-\infty}^{x} v\frac{\partial v}{\partial x}dx \leq 2\left(\int_R v^2 dx\right)^{1/2}\left(\int_R \left(\frac{\partial v}{\partial x}\right)^2 dx\right)^{1/2}, \\ \omega^2 &= 2\int_{-\infty}^{x} \omega\frac{\partial \omega}{\partial x}dx \leq 2\left(\int_R \omega^2 dx\right)^{1/2}\left(\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)^{1/2} \end{aligned} \tag{3.62}$$

and estimations (3.1), (3.55), and (3.56) we get immediately (3.59) and (3.60). From (3.50), (3.53), (3.56), and (3.1) we have, as in [5], for $\theta(x,t) \leq 1$ that

$$\int_{\theta(x,t)}^{1} \sqrt{s-1-\ln s}\, ds \leq (2K_5(M_1)E_1)^{1/2} < \int_0^1 \sqrt{s-1-\ln s}\, ds \tag{3.63}$$

holds because of (2.25). Hence we conclude that there exists the constant $h > 0$ such that $\theta(x,t) \geq h$. □

Remark 3.6. Using the properties of the functions $\underline{u}_1 = \exp \underline{\eta}_1(\overline{\theta}(t))$ and $\overline{u}_1 = \exp \overline{\eta}_1(\overline{\theta}(t))$ defined in Lemma 3.2, from (3.55)-(3.56) and (3.59)-(3.60) we get the following estimates:

$$\begin{gathered}
\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 dx \le \left(\frac{D}{12A^{1/2}E_1^{1/2}}\right)^2 \exp\{-2\lambda(t)\}, \\
\int_R \left(\frac{\partial v}{\partial x}\right)^2 dx \le \left(\frac{D}{24E_1^{1/2}}\right)^2 \exp\{-2\lambda(t)\}, \\
\sup_{(x,t)\in\Pi} |\omega(x,t)| \le \left(\frac{D^2}{18}\right)^{1/4} \exp\left\{-\frac{1}{2\lambda(t)}\right\}, \\
\sup_{(x,t)\in\Pi} |v(x,t)| \le \left(\frac{D^2}{72}\right)^{1/4} \exp\left\{-\frac{1}{2\lambda(t)}\right\},
\end{gathered} \tag{3.64}$$

where $\lambda(t) = \overline{\eta}_1(\overline{\theta}(t)) - \underline{\eta}_1(\overline{\theta}(t)) > 0$.

Lemma 3.7. *For each $T > 0$ it holds that*

$$\int_0^T \int_R \left(\frac{\partial v}{\partial x}\right)^2 dx\, d\tau \le K_6, \tag{3.65}$$

$$\int_0^T \int_R \left(\frac{\partial \theta}{\partial x}\right)^2 dx\, d\tau \le K_7, \tag{3.66}$$

$$\int_0^T \int_R \left(\frac{\partial \omega}{\partial x}\right)^2 dx\, d\tau \le K_8, \tag{3.67}$$

$$\int_0^T \int_R \frac{\omega^2}{\rho\theta} dx\, d\tau \le K_9, \tag{3.68}$$

$$\int_0^T \int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx\, d\tau \le K_{10}, \tag{3.69}$$

$$\int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx \le K_{11}, \quad t \in [0,T], \tag{3.70}$$

$$\int_0^T \int_R \left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 dx\, d\tau \le K_{12}, \tag{3.71}$$

$$\int_0^T \int_R \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\, d\tau \le K_{13}, \tag{3.72}$$

$$\int_0^T \int_R \left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 dx\, d\tau \le K_{14}, \tag{3.73}$$

where the constants $K_6, K_7, \ldots, K_{14} \in \mathbf{R}^+$ are independent of T.

Proof. Taking into account (3.58), (3.61), and (3.54) from (3.1),(3.3), (3.10), and (3.27) we get all above estimates. □

4. Proof of Theorem 2.1

In the following we use the results of Section 3. The conclusions of Theorem 2.1 are immediate consequences of the following lemmas.

Lemma 4.1. *It holds that*

$$\int_R \left(\frac{\partial v}{\partial x}\right)^2 (x,t)dx \longrightarrow 0, \qquad \int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t)dx \longrightarrow 0, \qquad \int_R \left(\frac{\partial \theta}{\partial x}\right)^2 (x,t)dx \longrightarrow 0, \tag{4.1}$$

when $t \to \infty$.

Proof. Let $\varepsilon > 0$ be arbitrary. With the help of (3.65)–(3.69) we conclude that there exists $t_0 > 0$ such that

$$\begin{gathered}\int_{t_0}^t \int_R \left(\frac{\partial v}{\partial x}\right)^2 dx\, d\tau < \varepsilon, \qquad \int_{t_0}^t \int_R \left(\frac{\partial \theta}{\partial x}\right)^2 dx\, d\tau < \varepsilon, \qquad \int_{t_0}^t \int_R \left(\frac{\partial \omega}{\partial x}\right)^2 dx\, d\tau < \varepsilon, \\ \int_{t_0}^t \int_R \frac{\omega^2}{\rho\theta} dx\, d\tau < \varepsilon, \qquad \int_{t_0}^t \int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx\, d\tau < \varepsilon,\end{gathered} \tag{4.2}$$

for $t > t_0$, and

$$\int_R \left(\frac{\partial v}{\partial x}\right)^2 (x,t_0)dx < \varepsilon, \qquad \int_R \left(\frac{\partial \theta}{\partial x}\right)^2 (x,t_0)dx < \varepsilon, \qquad \int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t_0)dx < \varepsilon. \tag{4.3}$$

Similarly to (3.10), we have

$$\begin{aligned}&\frac{1}{2}\int_R \left(\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{A}\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial x}\right)^2\right)dx + \frac{D}{12}\int_{t_0}^t \int_R \rho\left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 dx\, d\tau \\ &\quad + \left(\frac{6\overline{u}^{1/2}}{D\underline{u}}\right)^2 \int_{t_0}^t \left(\int_R \left(\frac{\partial v}{\partial x}\right)^2 dx\right)^2 \left(K_3\left(\overline{\theta}(\tau)\right) - \int_R \left(\frac{\partial v}{\partial x}\right)^2 dx\right) d\tau \\ &\quad + 2\left(\frac{3\overline{u}}{D\underline{u}}\right)^2 \int_{t_0}^t \left(\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 dx\right)^2 \left(K_4\left(\overline{\theta}(\tau)\right) - \int_R \left(\frac{\partial \omega}{\partial x}\right)^2 dx\right) d\tau\end{aligned}$$

$$\leq \frac{1}{2}\int_R\left(\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{A}\left(\frac{\partial \omega}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)(x,t_0)dx+K^2\int_{t_0}^t\int_R\rho\left(\frac{\partial \theta}{\partial x}\right)^2dx\,d\tau$$
$$+\left(16K_1\left(\overline{\theta}(t)\right)\right)^2\frac{\overline{u}}{\underline{u}^2}\int_{t_0}^t\int_R\left(\frac{\partial v}{\partial x}\right)^2dx\,d\tau+K^2\int_{t_0}^t\int_R\frac{\theta^2}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2dx\,d\tau$$
$$+2\left(\frac{8K_1^2\left(\overline{\theta}(t)\right)\overline{u}^{1/2}}{\underline{u}}\right)^2\int_{t_0}^t\int_R\left(\frac{\partial \omega}{\partial x}\right)^2dx\,d\tau+\overline{u}^2\overline{\theta}(t)\int_{t_0}^t\int_R\frac{\omega^2}{\rho\theta}dx\,d\tau$$
$$+\frac{3K^2}{2D}\int_{t_0}^t\int_R\theta^2\rho\left(\frac{\partial v}{\partial x}\right)^2dx\,d\tau+\frac{3AE_1\overline{\theta}(t)\overline{u}^3}{D\underline{u}}\int_{t_0}^t\int_R\frac{\omega^2}{\rho\theta}dx\,d\tau$$
$$+\frac{3AE_1\overline{u}^3}{D}\int_{t_0}^t\int_R\left(\frac{\partial \omega}{\partial x}\right)^2dx\,d\tau+\left(\frac{12DK_1\left(\overline{\theta}(t)\right)\overline{u}^{1/2}}{\underline{u}}\right)^2\int_{t_0}^t\int_R\left(\frac{\partial \theta}{\partial x}\right)^2dx\,d\tau. \tag{4.4}$$

Taking into account (3.54), (3.58), (3.61), and (4.2)–(4.3) from (4.4) we obtain

$$\int_R\left(\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{A}\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial \theta}{\partial x}\right)^2\right)dx\leq K_{15}\varepsilon\quad \text{for } t>t_0, \tag{4.5}$$

where K_{15} depends only on the data of our problem and does not depend on t_0. Hence relations (4.1) hold. □

Lemma 4.2. *It holds that*

$$v(x,t)\longrightarrow 0,\qquad \omega(x,t)\longrightarrow 0,\qquad \theta(x,t)\longrightarrow 1 \tag{4.6}$$

when $t\to\infty$, uniformly with respect to all $x, x\in\mathbf{R}$.

Proof. We have (see (3.51) and (3.62))

$$v^2(x,t)\leq 2\left(\int_R v^2(x,t)dx\right)^{1/2}\left(\int_R\left(\frac{\partial v}{\partial x}\right)^2(x,t)dx\right)^{1/2},$$
$$\omega^2(x,t)\leq 2\left(\int_R \omega^2(x,t)dx\right)^{1/2}\left(\int_R\left(\frac{\partial \omega}{\partial x}\right)^2(x,t)dx\right)^{1/2}, \tag{4.7}$$
$$|\psi(\theta(x,t))|\leq\left(\int_R(\theta(x,t)-1-\ln\theta(x,t))dx\right)^{1/2}\left(\int_R\left(\frac{\partial \theta}{\partial x}\right)^2dx\right)^{1/2}.$$

Taking into account (3.1) from (4.7) we get

$$v^2(x,t) \le 2(2E_1)^{1/2}\left(\int_R \left(\frac{\partial v}{\partial x}\right)^2 (x,t)dx\right)^{1/2},$$
$$\omega^2(x,t) \le 2(2AE_1)^{1/2}\left(\int_R \left(\frac{\partial \omega}{\partial x}\right)^2 (x,t)dx\right)^{1/2}, \tag{4.8}$$
$$|\psi(\theta(x,t))| \le E_1^{1/2}\left(\int_R \left(\frac{\partial \theta}{\partial x}\right)^2 dx\right)^{1/2}.$$

Using (4.1) and property (3.50) of the function ψ we can easily obtain that (4.6) holds. □

Now, as in [5], we analyze the behavior of the function ρ as $t \to \infty$. From (3.69) and (3.72) we conclude that for $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$\int_{t_0}^t \int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx\, d\tau < \varepsilon, \qquad \int_{t_0}^t \int_R \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\, d\tau < \varepsilon, \qquad \int_R \left(\frac{\partial \rho}{\partial x}\right)^2 (x,t_0)dx < \varepsilon \tag{4.9}$$

for $t > t_0$. Deriving (2.1) with respect to x, multiplying by $(\partial/\partial x)(1/\rho)$, and integrating over $\mathbf{R}$ and $]t_0, t[$, after using (3.58) and Young's inequality, we obtain

$$\begin{aligned}\frac{\underline{u}^4}{2}\int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx \le \overline{u}^2 \int_{t_0}^t \int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx\, d\tau + \overline{u}^2 \int_{t_0}^t \int_R \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\, d\tau \\ + \frac{\underline{u}^4}{2}\int_R \left(\frac{\partial \rho}{\partial x}\right)^2 (x,t_0)dx.\end{aligned} \tag{4.10}$$

With the help of (4.9) we get easily the following result.

Lemma 4.3. *It holds that*

$$\int_R \left(\frac{\partial \rho}{\partial x}\right)^2 (x,t)dx \longrightarrow 0 \tag{4.11}$$

when $t \to \infty$.

Similarly as for the function $\psi(\theta)$, we have

$$\begin{aligned}\psi\left(\frac{1}{\rho}\right) &= \int_1^{1/\rho} \sqrt{s - 1 - \ln s}\, ds \\ &\le \left(\int_R \left(\frac{1}{\rho} - 1 - \ln\frac{1}{\rho}\right)dx\right)^{1/2}\left(\int_R \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\right)^2 dx\right)^{1/2}.\end{aligned} \tag{4.12}$$

Taking into account (3.1) from (4.12) one has

$$\psi\left(\frac{1}{\rho}\right) \leq \left(K^{-1}E_1\right)^{1/2}\overline{u}\left(\int_R \left(\frac{\partial \rho}{\partial x}\right)^2 dx\right)^{1/2}. \tag{4.13}$$

Using (4.11) we obtain the following conclusion.

Lemma 4.4. *It holds that*

$$\rho(x,t) \longrightarrow 1 \tag{4.14}$$

when $t \to \infty$, *uniformly with respect to* $x \in \mathbf{R}$.

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