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Research Article

Multiple Positive Solutions for a Class of Concave-Convex Semilinear Elliptic Equations in Unbounded Domains with Sign-Changing Weights

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We study the existence and multiplicity of positive solutions for the following Dirichlet equations: $-\Delta u + u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where $\lambda > 0$, $1 < q < 2 < p < 2^*(2^* = 2N/(N-2)$ if $N \geq 3; 2^* = \infty$ if $N = 1,2)$, Ω is a smooth unbounded domain in $\mathbb{R}^N$, $a(x)$, $b(x)$ satisfy suitable conditions, and $a(x)$ maybe change sign in Ω.

1. Introduction and Main Results

In this paper, we deal with the existence and multiplicity of positive solutions for the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u + u &= \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{$E_{\lambda a,b}$}$$

where $\lambda > 0$, $1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1,2$), $\Omega \subset \mathbb{R}^N$ is an unbounded domain, and a, b are measurable functions and satisfy the following conditions:

(A1) $a \in C(\Omega) \cap L^{q^*}(\Omega)$ $(q^* = p/(p-q))$ with $a^+ = \max\{a, 0\} \not\equiv 0$ in Ω.

(B1) $b \in C(\Omega) \cap L^{\infty}(\Omega)$ and $b^+ = \max\{b, 0\} \not\equiv 0$ in Ω.

Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are widely studied. For example, Ambrosetti et al. [1] considered the following equation:

$$
\begin{aligned}
-\Delta u &= \lambda u^{q-1} + u^{p-1} \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{E_λ}
$$

where $\lambda > 0$, $1 < q < 2 < p < 2^*$. They proved that there exists $\lambda_0 > 0$ such that (E_λ) admits at least two positive solutions for all $\lambda \in (0, \lambda_0)$ and has one positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$. Actually, Adimurthi et al. [2], Damascelli et al. [3], Ouyang and Shi [4], and Tang [5] proved that there exists $\lambda_0 > 0$ such that (E_λ) in the unit ball $B^N(0;1)$ has exactly two positive solutions for $\lambda \in (0, \lambda_0)$ and has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. For more general results of (E_λ) (involving sign-changing weights) in bounded domains, see Ambrosetti et al. [6], Garcia Azorero et al. [7], Brown and Wu [8], Brown and Zhang [9], Cao and Zhong [10], de Figueiredo et al. [11], and their references. However, little has been done for this type of problem in unbounded domains. For $\Omega = \mathbb{R}^N$, we are only aware of the works [12–15] which studied the existence of solutions for some related concave-convex elliptic problems (not involving sign-changing weights).

Wu in [16] has studied the multiplicity of positive solutions for the following equation involving sign-changing weights:

$$
\begin{aligned}
-\Delta u + u &= f_\lambda(x) u^{q-1} + g_\mu(x) u^{p-1} \quad \text{in } \mathbb{R}^N, \\
u &> 0 \quad \text{in } \mathbb{R}^N, \\
u &\in H^1\left(\mathbb{R}^N\right),
\end{aligned}
\tag{E_{f_λ, g_μ}}
$$

where $1 < q < 2 < p < 2^*$, the parameters $\lambda, \mu \geq 0$. He also assumed that $f_\lambda(x) = \lambda f_+(x) + f_-(x)$ is sign-changing and $g_\mu(x) = a(x) + \mu b(x)$, where a and b satisfy suitable conditions, and proved (E_{f_λ, g_μ}) has at least four positive solutions.

When $\Omega = \Omega' \times \mathbb{R}$ ($\Omega' \subset \mathbb{R}^{N-1}$, $N \geq 2$) is an infinite strip domain, Wu in [17] considered $(E_{\lambda a,b})$ (not involving sign-changing weights) assuming that $0 \not\leq a \in L^{2/(2-q)}(\Omega)$, $0 \leq b \in C(\Omega)$ satisfies $\lim_{|x_N| \to \infty} b(x', x_N) = 1$ in Ω and there exist $\delta > 0$ and $0 < C_0 < 1$ such that $b(x', x_N) \geq 1 - C_0 e^{-2\sqrt{1+\theta_1+\delta}|x_N|}$ for all $x = (x', x_N) \in \Omega$, where θ_1 is the first eigenvalue of the Dirichlet problem $-\Delta$ in Ω'. The author proved that there exists a positive constant Λ_0 such that for $\lambda \in (0, \Lambda_0)$, $(E_{\lambda a,b})$ possesses at least two positive solutions.

Miotto and Miyagaki in [18] have studied $(E_{\lambda a,b})$ in $\Omega = \Omega' \times \mathbb{R}$, under the assumption that $a \in L^{\gamma/(\gamma-q)}(\Omega)$ ($q < \gamma \leq 2^*$) with $a^+ \not\equiv 0$ and a^- is bounded and has a compact support in Ω and $0 \leq b \in L^\infty(\Omega)$ satisfies $\lim_{|x_N| \to \infty} b(x', x_N) = 1$ and there exists $C_0 > 0$ such that $b(x', x_N) \geq 1 - C_0 e^{-2\sqrt{1+\theta_1}|x_N|}$ for all $x = (x', x_N) \in \Omega$, where θ_1 is the first eigenvalue of the Dirichlet problem $-\Delta$ in Ω'. It was obtained there existence of $\Lambda_0 > 0$ such that for $\lambda \in (0, \Lambda_0)$, $(E_{\lambda a,b})$ possesses at least two positive solutions.

In a recent work [19], Hsu and Lin have studied $(E_{\lambda a,b})$ in $\mathbb{R}^N$ under the assumptions (A1)-(A2), (B1), and (Ω_b). They proved that there exists a constant $\Lambda_0 > 0$ such that for

$\lambda \in (0, (q/2)\Lambda_0)$, $(E_{\lambda a,b})$ possesses at least two positive solutions. The main aim of this paper is to study $(E_{\lambda a,b})$ on the general unbounded domains (see the condition (Ω_b)) and extend the results of [19] to more general unbounded domains. We will apply arguments similar to those used in [20] and prove the existence and multiplicity of positive solutions by using Ekeland's variational principle [21].

Set

$$\Lambda_0 = \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{(2-q)/(p-2)} \left(\frac{p-2}{(p-q)\|a^+\|_{L^{q^*}}}\right) S_p(\Omega)^{(p(2-q)/2(p-2))+q/2} > 0, \tag{1.1}$$

where $\|b^+\|_{L^\infty} = \sup_{x\in\Omega} b^+(x)$, $\|a^+\|_{L^{q^*}} = (\int_\Omega |a^+|^{q^*} dx)^{1/q^*}$, and $S_p(\Omega)$ is the best Sobolev constant for the imbedding of $H_0^1(\Omega)$ into $L^p(\Omega)$. Now, we state the first main result about the existence of positive solution of $(E_{\lambda a,b})$.

Theorem 1.1. *Assume that (A1) and (B1) hold. If $\lambda \in (0, \Lambda_0)$, then $(E_{\lambda a,b})$ admits at least one positive solution.*

Associated with $(E_{\lambda a,b})$, we consider the energy functional $J_{\lambda a,b}$ in $H_0^1(\Omega)$:

$$J_{\lambda a,b}(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx - \frac{1}{p}\int_\Omega b(x)|u|^p dx, \tag{1.2}$$

where $\|u\|_{H^1} = (\int_\Omega (\nabla u|^2 + u^2) dx)^{1/2}$. By Rabinowitz [22, Proposition B.10], $J_{\lambda a,b} \in C^1(H_0^1(\Omega), \mathbb{R})$. It is well known that the solutions of $(E_{\lambda a,b})$ are the critical points of the energy functional $J_{\lambda a,b}$ in $H_0^1(\Omega)$.

Under the assumptions (A1), (B1), and $\lambda > 0$, $(E_{\lambda a,b})$ can be regarded as a perturbation problem of the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u + u &= b(x)u^{p-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{E_b}$$

where $b(x) \in C(\Omega) \cap L^\infty(\Omega)$ and $b(x) > 0$ for all $x \in \Omega$. We denote by $S_p^b(\Omega)$ the best constant which is given by

$$S_p^b(\Omega) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\left(\int_\Omega b(x)|u|^p dx\right)^{2/p}}. \tag{1.3}$$

A typical approach for solving problem of this kind is to use the following Minimax method:

$$\alpha_{\Gamma}^{b}(\Omega) = \inf_{\gamma\in\Gamma(\Omega)} \max_{t\in[0,1]} J_0^b(\gamma(t)), \tag{1.4}$$

where

$$\Gamma(\Omega) = \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) : \gamma(0) = 0,\ \gamma(1) = e\right\}, \tag{1.5}$$

$J_0^b(e) = 0$ and $e \neq 0$. By the Mountain Pass Lemma due to Ambrosetti and Rabinowitz [23], we called the nonzero critical point $u \in H_0^1(\Omega)$ of J_0^b a ground state solution of (E_b) in Ω if $J_0^b(u) = \alpha_{\Gamma}^b(\Omega)$. We remark that the ground state solutions of (E_b) in Ω can also be obtained by the Nehari minimization problem

$$\alpha_0^b(\Omega) = \inf_{v\in\mathcal{M}_0^b(\Omega)} J_0^b(v), \tag{1.6}$$

where $\mathcal{M}_0^b(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} : \|u\|_{H^1}^2 = \int_\Omega b(x)|u|^p dx\}$. Note that $\mathcal{M}_0^b(\Omega)$ contains every nonzero solution of (E_b) in Ω,

$$\alpha_{\Gamma}^b(\Omega) = \alpha_0^b(\Omega) = \frac{p-2}{2p} S_p^b(\Omega)^{p/(p-2)} > 0 \tag{1.7}$$

(see Willem [24]), and if $b(x) \equiv b^\infty > 0$ is a constant, then J_0^b and $\alpha_0^b(\Omega)$ replace J_0 and $\alpha_0^\infty(\Omega)$, respectively.

The existence of ground state solutions of (E_b) is affected by the shape of the domain Ω and $b(x)$ that satisfies some suitable conditions and has been the focus of a great deal of research in recent years. By the Rellich compactness theorem and the Minimax method, it is easy to obtain a ground state solution for (E_b) in bounded domains. When Ω is an unbounded domain and $b(x) \equiv b^\infty$, the existence of ground state solutions has been established by several authors under various conditions. We mention, in particular, results by Berestycki and Lions [25], Lien et al. [26], Chen and Wang [27], and Del Pino and Felmer [28, 29]. In [25], $\Omega = \mathbb{R}^N$. Actually, Kwong [30] proved that the positive solution of (E_b) in $\mathbb{R}^N$ is unique. In [26], for Ω is a periodic domain. In [26, 27], the domain Ω is required to satisfy

(Ω_1) $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are domains in $\mathbb{R}^N$ and $\Omega_1 \cap \Omega_2$ is bounded;

(Ω_2) $\alpha_0^\infty(\Omega) < \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2)\}$.

In [28, 29] for $1 \leq l \leq N-1$, $\mathbb{R}^N = \mathbb{R}^l \times \mathbb{R}^{N-l}$. For a point $x \in \mathbb{R}^N$, we have $x = (y, z)$, where $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$. Let $y \in \mathbb{R}^l$, we denote by $\Omega^y \subset \mathbb{R}^{N-l}$ the projection of Ω onto $\mathbb{R}^{N-l}$, that is,

$$\Omega^y = \left\{z \in \mathbb{R}^{N-l} : (y, z) \in \Omega\right\}. \tag{1.8}$$

The domain Ω satisfies the following conditions:

(Ω_3) Ω is a smooth subset of $\mathbb{R}^N$ and the projections Ω^y are bounded uniformly in $y \in \mathbb{R}^l$;

(Ω_4) there exists a nonempty closed set $D \subset \mathbb{R}^{N-l}$ such that $D \subset \Omega^y$ for all $y \in \mathbb{R}^l$;

(Ω_5) for each $\delta > 0$, there exists $R_0 > 0$ such that

$$\Omega^y \subset \left\{ z \in \mathbb{R}^{N-l} : \operatorname{dist}(z, D) < \delta \right\} \tag{1.9}$$

for all $|y| \geq R_0$.

When $b(x) \not\equiv b^\infty$ and $b(x) \in C(\Omega) \cap L^\infty(\Omega)$, the existence of ground state solutions of (E_b) has been established by the condition $b(x) \geq b^\infty$ and the existence of ground state solutions of limit equation

$$\begin{aligned} -\Delta u + u &= b^\infty u^{p-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{E_{b^∞}}$$

In order to get the second positive solution of $(E_{\lambda a,b})$, we need some additional assumptions for $a(x)$, $b(x)$, and Ω. We assume the following conditions on $a(x)$, $b(x)$, and Ω:

(Ω_b) $b(x) > 0$ for all $x \in \Omega$ and (E_b) in Ω has a ground state solution w_0 such that $J_0^b(w_0) = \alpha_0^b(\Omega)$.

(A2) $\int_\Omega a(x)|w_0|^q dx > 0$ where w_0 is a positive ground state solution of (E_b) in Ω.

Theorem 1.2. *Assume that (A1)-(A2), (B1), and (Ω_b) hold. If $\lambda \in (0, (q/2)\Lambda_0)$, $(E_{\lambda a,b})$ admits at least two positive solutions.*

Throughout this paper, (A1) and (B1) will be assumed. $H_0^1(\Omega)$ denotes the standard Sobolev space, whose norm $\|\cdot\|_{H^1}$ is induced by the standard inner product. The dual space of $H_0^1(\Omega)$ will be denoted by $H^{-1}(\Omega)$. $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $H_0^1(\Omega)$. We denote the norm in $L^s(\Omega)$ by $\|\cdot\|_{L^s}$ for $1 \leq s \leq \infty$. $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. C, C_i will denote various positive constants, the exact values of which are not important. This paper is organized as follows. In Section 2, we give some properties of Nehari manifold. In Sections 3 and 4, we complete proofs of Theorems 1.1 and 1.2.

2. Nehari Manifold

In this section, we will give some properties of Nehari manifold. As the energy functional $J_{\lambda a,b}$ is not bounded below on $H_0^1(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{M}_{\lambda a,b}(\Omega) = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \left\langle (J_{\lambda a,b})'(u), u \right\rangle = 0 \right\}. \tag{2.1}$$

Thus, $u \in \mathcal{M}_{\lambda a,b}(\Omega)$ if and only if

$$\langle (J_{\lambda a,b})'(u), u\rangle = \|u\|_{H^1}^2 - \lambda \int_\Omega a(x)|u|^q dx - \int_\Omega b(x)|u|^p dx = 0. \tag{2.2}$$

Note that $\mathcal{M}_{\lambda a,b}(\Omega)$ contains every nonzero solution of $(E_{\lambda a,b})$. Moreover, we have the following results.

Lemma 2.1. *The energy functional $J_{\lambda a,b}$ is coercive and bounded below on $\mathcal{M}_{\lambda a,b}(\Omega)$.*

Proof. If $u \in \mathcal{M}_{\lambda a,b}(\Omega)$, then by (A1), (2.2), Hölder and Sobolev inequalities

$$J_{\lambda a,b}(u) = \frac{p-2}{2p}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{pq}\right)\int_\Omega a(x)|u|^q dx \tag{2.3}$$

$$\geq \frac{p-2}{2p}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{pq}\right)S_p(\Omega)^{-q/2}\|a^+\|_{L^{q^*}}\|u\|_{H^1}^q. \tag{2.4}$$

Thus, $J_{\lambda a,b}$ is coercive and bounded below on $\mathcal{M}_{\lambda a,b}(\Omega)$. □

Define

$$\psi_{\lambda a,b}(u) = \langle (J_{\lambda a,b})'(u), u\rangle. \tag{2.5}$$

Then for $u \in \mathcal{M}_{\lambda a,b}(\Omega)$,

$$\begin{aligned}\left\langle (\psi_{\lambda a,b})'(u), u\right\rangle &= 2\|u\|_{H^1}^2 - \lambda q\int_\Omega a(x)|u|^q dx - p\int_\Omega b(x)|u|^p dx\\ &= (2-q)\|u\|_{H^1}^2 - (p-q)\int_\Omega b(x)|u|^p dx\end{aligned} \tag{2.6}$$

$$= \lambda(p-q)\int_\Omega a(x)|u|^q dx - (p-2)\|u\|_{H^1}^2. \tag{2.7}$$

Similar to the method used in Tarantello [20], we split $\mathcal{M}_{\lambda a,b}(\Omega)$ into three parts:

$$\begin{aligned}\mathcal{M}_{\lambda a,b}^{+}(\Omega) &= \left\{u \in \mathcal{M}_{\lambda a,b}(\Omega) : \left\langle (\psi_{\lambda a,b})'(u), u\right\rangle > 0\right\},\\ \mathcal{M}_{\lambda a,b}^{0}(\Omega) &= \left\{u \in \mathcal{M}_{\lambda a,b}(\Omega) : \left\langle (\psi_{\lambda a,b})'(u), u\right\rangle = 0\right\},\\ \mathcal{M}_{\lambda a,b}^{-}(\Omega) &= \left\{u \in \mathcal{M}_{\lambda a,b}(\Omega) : \left\langle (\psi_{\lambda a,b})'(u), u\right\rangle < 0\right\}.\end{aligned} \tag{2.8}$$

Then, we have the following results.

Lemma 2.2. *Assume that u_λ is a local minimizer for $J_{\lambda a,b}$ on $\mathcal{M}_{\lambda a,b}(\Omega)$ and $u_\lambda \notin \mathcal{M}^0_{\lambda a,b}(\Omega)$. Then $(J_{\lambda a,b})'(u_\lambda) = 0$ in $H^{-1}(\Omega)$.*

Proof. Our proof is almost the same as that in Brown and Zhang [9, Theorem 2.3] (or see Binding et al. [31]). □

Lemma 2.3. *We have the following.*

(i) *If $u \in \mathcal{M}^+_{\lambda a,b}(\Omega) \cup \mathcal{M}^0_{\lambda a,b}(\Omega)$, then $\int_\Omega a(x)|u|^q dx > 0$;*

(ii) *If $u \in \mathcal{M}^-_{\lambda a,b}(\Omega)$, then $\int_\Omega b(x)|u|^p dx > 0$.*

Proof. The proof is immediate from (2.6) and (2.7). □

Moreover, we have the following result.

Lemma 2.4. *If $\lambda \in (0, \Lambda_0)$, then $\mathcal{M}^0_{\lambda a,b}(\Omega) = \emptyset$ where Λ_0 is the same as in* (1.1).

Proof. Suppose the contrary. Then there exists $\lambda \in (0, \Lambda_0)$ such that $\mathcal{M}^0_{\lambda a,b}(\Omega) \neq \emptyset$. Then for $u \in \mathcal{M}^0_{\lambda a,b}(\Omega)$ by (2.6) and Sobolev inequality, we have

$$\frac{2-q}{p-q}\|u\|^2_{H^1} = \int_\Omega b(x)|u|^p dx \le \|b^+\|_{L^\infty} S_p(\Omega)^{-p/2}\|u\|^p_{H^1} \tag{2.9}$$

and so

$$\|u\|_{H^1} \ge \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{1/(p-2)} S_p(\Omega)^{p/2(p-2)}. \tag{2.10}$$

Similarly, using (2.7) and Hölder and Sobolev inequalities, we have

$$\|u\|^2_{H^1} = \lambda\frac{p-q}{p-2}\int_\Omega a(x)|u|^q dx \le \lambda\frac{p-q}{p-2}\|a^+\|_{L^{q^*}} S_p(\Omega)^{-q/2}\|u\|^q_{H^1}, \tag{2.11}$$

which implies

$$\|u\|_{H^1} \le \left(\lambda\frac{p-q}{p-2}\|a^+\|_{L^{q^*}}\right)^{1/(2-q)} S_p(\Omega)^{-q/2(2-q)}. \tag{2.12}$$

Hence, we must have

$$\lambda \ge \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{(2-q)/(p-2)}\left(\frac{p-2}{(p-q)\|a^+\|_{L^{q^*}}}\right) S_p(\Omega)^{(p(2-q)/2(p-2))+q/2} = \Lambda_0, \tag{2.13}$$

which is a contradiction. This completes the proof. □

For each $u \in H_0^1(\Omega)$ with $\int_\Omega b(x)|u|^p dx > 0$, we write

$$t_{\max}(u) = \left(\frac{(2-q)\|u\|_{H^1}^2}{(p-q)\int_\Omega b(x)|u|^p} dx \right)^{1/(p-2)} > 0. \tag{2.14}$$

Then the following lemma holds.

Lemma 2.5. *Let $\lambda \in (0, \Lambda_0)$. For each $u \in H_0^1(\Omega)$ with $\int_\Omega b(x)|u|^p dx > 0$, we have the following.*

(i) *If $\int_\Omega a(x)|u|^q dx \le 0$, then there is a unique $t^- = t^-(u) > t_{\max}(u)$ such that $t^-u \in \mathcal{M}_{\lambda a,b}^-(\Omega)$ and*

$$J_{\lambda a,b}(t^-u) = \sup_{t \ge 0} J_{\lambda a,b}(tu). \tag{2.15}$$

(ii) *If $\int_\Omega a(x)|u|^q dx > 0$, then there are unique*

$$0 < t^+ = t^+(u) < t_{\max}(u) < t^- = t^-(u) \tag{2.16}$$

such that $t^+u \in \mathcal{M}_{\lambda a,b}^+(\Omega)$, $t^-u \in \mathcal{M}_{\lambda a,b}^-(\Omega)$, and

$$J_{\lambda a,b}(t^+u) = \inf_{0 \le t \le t_{\max}(u)} J_{\lambda a,b}(tu), \qquad J_{\lambda a,b}(t^-u) = \sup_{t \ge 0} J_{\lambda a,b}(tu). \tag{2.17}$$

Proof. The proof is almost the same as that in Wu [32, Lemma 5] and is omitted here. □

3. Proof of Theorem 1.1

First, we remark that it follows Lemma 2.4 that

$$\mathcal{M}_{\lambda a,b}(\Omega) = \mathcal{M}_{\lambda a,b}^+(\Omega) \cup \mathcal{M}_{\lambda a,b}^-(\Omega) \tag{3.1}$$

for all $\lambda \in (0, \Lambda_0)$. Furthermore, by Lemma 2.5 it follows that $\mathcal{M}_{\lambda a,b}^+(\Omega)$ and $\mathcal{M}_{\lambda a,b}^-(\Omega)$ are nonempty, and by Lemma 2.1 we may define

$$\alpha_{\lambda a,b} = \inf_{u \in \mathcal{M}_{\lambda a,b}(\Omega)} J_{\lambda a,b}(u); \alpha_{\lambda a,b}^+ = \inf_{u \in \mathcal{M}_{\lambda a,b}^+(\Omega)} J_{\lambda a,b}(u); \alpha_{\lambda a,b}^- = \inf_{u \in \mathcal{M}_{\lambda a,b}^-(\Omega)} J_{\lambda a,b}(u). \tag{3.2}$$

Then we get the following result.

Theorem 3.1. *We have the following.*

(i) *If $\lambda \in (0, \Lambda_0)$, then we have $\alpha_{\lambda a,b}^+ < 0$.*

(ii) *If $\lambda \in (0, (q/2)\Lambda_0)$, then $\alpha_{\lambda a,b}^- > d_0$ for some $d_0 > 0$.*

In particular, for each $\lambda \in (0, (q/2)\Lambda_0)$, we have $\alpha_{\lambda a,b}^+ = \alpha_{\lambda a,b}$.

Proof. (i) Let $u \in \mathcal{M}^{+}_{\lambda a,b}(\Omega)$. By (2.6),

$$\frac{2-q}{p-q}\|u\|^2_{H^1} > \int_{\Omega} b(x)|u|^p dx \tag{3.3}$$

and so

$$\begin{aligned} J_{\lambda}(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2_{H^1} + \left(\frac{1}{q}-\frac{1}{p}\right)\int_{\Omega} b(x)|u|^p dx < \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{p}\right)\left(\frac{2-q}{p-q}\right)\right]\|u\|^2_{H^1} \\ &= -\frac{(p-2)(2-q)}{2pq}\|u\|^2_{H^1} < 0. \end{aligned} \tag{3.4}$$

Therefore, $\alpha^{+}_{\lambda a,b} < 0$.

(ii) Let $u \in \mathcal{M}^{-}_{\lambda a,b}(\Omega)$. By (2.6),

$$\frac{2-q}{p-q}\|u\|^2_{H^1} < \int_{\Omega} b(x)|u|^p dx. \tag{3.5}$$

Moreover, by (B1) and Sobolev inequality theorem,

$$\int_{\Omega} b(x)|u|^p dx \le \|b^{+}\|_{L^{\infty}} S_p(\Omega)^{-p/2}\|u\|^p_{H^1}. \tag{3.6}$$

This implies

$$\|u\|_{H^1} > \left(\frac{2-q}{(p-q)\|b^{+}\|_{L^{\infty}}}\right)^{1/(p-2)} S_p(\Omega)^{p/2(p-2)} \quad \forall\, u \in \mathcal{M}^{-}_{\lambda a,b}(\Omega). \tag{3.7}$$

By (2.4) and (3.7), we have

$$\begin{aligned} &J_{\lambda a,b}(u) \\ &\quad\ge \|u\|^q_{H^1}\left[\frac{p-2}{2p}\|u\|^{2-q}_{H^1} - \lambda\left(\frac{p-q}{pq}\right)S_p(\Omega)^{-q/2}\|a^{+}\|_{L^{q^*}}\right] \\ &\quad> \left(\frac{2-q}{(p-q)\|b^{+}\|_{L^{\infty}}}\right)^{q/(p-2)} S_p(\Omega)^{pq/2(p-2)} \\ &\quad\times\left[\frac{p-2}{2p}S_p(\Omega)^{p(2-q)/2(p-2)}\left(\frac{2-q}{(p-q)\|b^{+}\|_{L^{\infty}}}\right)^{(2-q)/(p-2)} - \lambda\left(\frac{p-q}{pq}\right)S_p(\Omega)^{-q/2}\|a^{+}\|_{L^{q^*}}\right]. \end{aligned} \tag{3.8}$$

Thus, if $\lambda \in (0, (q/2)\Lambda_0)$, then

$$J_{\lambda a,b}(u) > d_0 \quad \forall\, u \in \mathcal{M}^-_{\lambda a,b}(\Omega), \tag{3.9}$$

for some positive constant d_0. This completes the proof. □

We define the Palais-Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditions in $H^1_0(\Omega)$ for $J_{\lambda a,b}$ as follows.

Definition 3.2. *(i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_c$-sequence in $H^1_0(\Omega)$ for $J_{\lambda a,b}$ if $J_{\lambda a,b}(u_n) = c + o_n(1)$ and $(J_{\lambda a,b})'(u_n) = o_n(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.*

(ii) $c \in \mathbb{R}$ is a (PS)-value in $H^1_0(\Omega)$ for $J_{\lambda a,b}$ if there exists a $(PS)_c$-sequence in $H^1_0(\Omega)$ for $J_{\lambda a,b}$.

(iii) $J_{\lambda a,b}$ satisfies the $(PS)_c$-condition in $H^1_0(\Omega)$ if any $(PS)_c$-sequence $\{u_n\}$ in $H^1_0(\Omega)$ for $J_{\lambda a,b}$ contains a convergent subsequence.

Now, we use the Ekeland variational principle [21] to get the following results.

Proposition 3.3. *(i) If $\lambda \in (0, \Lambda_0)$, then there exists a $(PS)_{\alpha_{\lambda a,b}}$-sequence $\{u_n\} \subset \mathcal{M}_{\lambda a,b}(\Omega)$ in $H^1_0(\Omega)$ for $J_{\lambda a,b}$.*

(ii) If $\lambda \in (0, (q/2)\Lambda_0)$, then there exists a $(PS)_{\alpha^-_{\lambda a,b}}$-sequence $\{u_n\} \subset \mathcal{M}^-_{\lambda a,b}(\Omega)$ in $H^1_0(\Omega)$ for $J_{\lambda a,b}$.

Proof. The proof is almost the same as that in Wu [32, Proposition 9]. □

Now, we establish the existence of a local minimum for $J_{\lambda a,b}$ on $\mathcal{M}^+_{\lambda a,b}(\Omega)$.

Theorem 3.4. *Assume (A1) and (B1) hold. If $\lambda \in (0, \Lambda_0)$, then $J_{\lambda a,b}$ has a minimizer u_λ in $\mathcal{M}^+_{\lambda a,b}(\Omega)$ and it satisfies the following.*

(i) $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b} = \alpha^+_{\lambda a,b}$.

(ii) u_λ *is a positive solution of* $(E_{\lambda a,b})$ *in* Ω.

(iii) $\|u_\lambda\|_{H^1} \to 0$ *as* $\lambda \to 0^+$.

Proof. By Proposition 3.3(i), there is a minimizing sequence $\{u_n\}$ for $J_{\lambda a,b}$ on $\mathcal{M}_{\lambda a,b}(\Omega)$ such that

$$J_{\lambda a,b}(u_n) = \alpha_{\lambda a,b} + o_n(1), \quad (J_{\lambda a,b})'(u_n) = o_n(1) \quad \text{in } H^{-1}(\Omega). \tag{3.10}$$

Since J_λ is coercive on $\mathcal{M}_{\lambda a,b}(\Omega)$ (see Lemma 2.1), we get that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Going if necessary to a subsequence, we can assume that there exists $u_\lambda \in H^1_0(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u_\lambda \quad \text{weakly in } H^1_0(\Omega), \\ &u_n \longrightarrow u_\lambda \quad \text{almost every where in } \Omega, \\ &u_n \longrightarrow u_\lambda \quad \text{strongly in } L^s_{\text{loc}}(\Omega)\ \forall\, 1 \le s < 2^*. \end{aligned} \tag{3.11}$$

By (A1), Egorov theorem, and Hölder inequality, we have

$$\lambda \int_\Omega a(x)|u_n|^q dx = \lambda \int_\Omega a(x)|u_\lambda|^q dx + o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{3.12}$$

First, we claim that u_λ is a nonzero solution of $(E_{\lambda a,b})$. By (3.10) and (3.11), it is easy to see that u_λ is a solution of $(E_{\lambda a,b})$. From $u_n \in \mathcal{M}_{\lambda a,b}(\Omega)$ and (2.3), we deduce that

$$\lambda \int_\Omega a(x)|u_n|^q dx = \frac{q(p-2)}{2(p-q)}\|u_n\|_{H^1}^2 - \frac{pq}{p-q} J_{\lambda a,b}(u_n). \tag{3.13}$$

Let $n \to \infty$ in (3.13); by (3.10), (3.12), and $\alpha_{\lambda a,b} < 0$, we get

$$\lambda \int_\Omega a(x)|u_\lambda|^q dx \geq -\frac{pq}{p-q}\alpha_{\lambda a,b} > 0. \tag{3.14}$$

Thus, $u_\lambda \in \mathcal{M}_{\lambda a,b}(\Omega)$ is a nonzero solution of $(E_{\lambda a,b})$. Now we prove that $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$ and $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b}$. By (3.13), if $u \in \mathcal{M}_{\lambda a,b}(\Omega)$, then

$$J_{\lambda a,b}(u) = \frac{p-2}{2p}\|u\|_{H^1}^2 - \frac{p-q}{pq}\lambda \int_\Omega a(x)|u|^q dx. \tag{3.15}$$

In order to prove that $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b}$, it suffices to recall that $u_n,\ u_\lambda \in \mathcal{M}_{\lambda a,b}(\Omega)$, by (3.15) and by applying Fatou's lemma to get

$$\begin{aligned}
\alpha_{\lambda a,b} \leq J_{\lambda a,b}(u_\lambda) &= \frac{p-2}{2p}\|u_\lambda\|_{H^1}^2 - \frac{p-q}{pq}\lambda \int_\Omega a(x)|u_\lambda|^q dx \\
&\leq \liminf_{n\to\infty}\left(\frac{p-2}{2p}\|u_n\|_{H^1}^2 - \frac{p-q}{pq}\lambda \int_\Omega a(x)|u_n|^q dx\right) \\
&\leq \liminf_{n\to\infty} J_{\lambda a,b}(u_n) = \alpha_{\lambda a,b}.
\end{aligned} \tag{3.16}$$

This implies that $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b}$ and $\lim_{n\to\infty}\|u_n\|_{H^1}^2 = \|u_\lambda\|_{H^1}^2$. Let $v_n = u_n - u_\lambda$; then by Brézis and Lieb, lemma [33] implies that

$$\|u_n\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|u_\lambda\|_{H^1}^2 + o_n(1). \tag{3.17}$$

Therefore, $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$. Moreover, we have $u_\lambda \in \mathcal{M}_{\lambda a,b}^+(\Omega)$. On the contrary, if $u_\lambda \in \mathcal{M}_{\lambda a,b}^-(\Omega)$, then by Lemma 2.5, there are unique t_0^+ and t_0^- such that $t_0^+ u_\lambda \in \mathcal{M}_{\lambda a,b}^+(\Omega)$ and $t_0^- u_\lambda \in \mathcal{M}_{\lambda a,b}^-(\Omega)$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_{\lambda a,b}(t_0^+ u_\lambda) = 0, \quad \frac{d^2}{dt^2}J_{\lambda a,b}(t_0^+ u_\lambda) > 0, \tag{3.18}$$

there exists $t_0^+ < \bar{t} \leq t_0^-$ such that $J_{\lambda a,b}(t_0^+ u_\lambda) < J_{\lambda a,b}(\bar{t} u_\lambda)$. By Lemma 2.5,

$$J_{\lambda a,b}(t_0^+ u_\lambda) < J_{\lambda a,b}\left(\bar{t} u_\lambda\right) \leq J_{\lambda a,b}(t_0^- u_\lambda) = J_{\lambda a,b}(u_\lambda), \tag{3.19}$$

which is a contradiction. Since $J_{\lambda a,b}(u_\lambda) = J_{\lambda a,b}(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{M}^+_{\lambda a,b}(\Omega)$, by Lemma 2.2 we may assume that u_λ is a nonzero nonnegative solution of $(E_{\lambda a,b})$. By Harnack inequality [34], we deduce that $u_\lambda > 0$ in Ω. Finally, by (2.3) and Hölder and Sobolev inequalities,

$$\|u_\lambda\|_{H^1}^{2-q} < \lambda \frac{p-q}{p-2} \|a^+\|_{L^{q^*}} S_p(\Omega)^{-q/2} \tag{3.20}$$

and so $\|u_\lambda\|_{H^1} \to 0$ as $\lambda \to 0^+$. □

Now, we begin the proof of Theorem 1.1. By Theorem 3.4, we obtain that $(E_{\lambda a,b})$ has a positive solution u_λ in $H_0^1(\Omega)$.

4. Proof of Theorem 1.2

In this section, we will establish the existence of the second positive solution of $(E_{\lambda a,b})$ by proving that $J_{\lambda a,b}$ satisfies the $(PS)_{\alpha^-_{\lambda a,b}}$-condition.

Lemma 4.1. *Assume that (A1) and (B1) hold. If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$-sequence for $J_{\lambda a,b}$, then $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. We argue by contradiction. Assume that $\|u_n\|_{H^1} \to \infty$. Let $\hat{u}_n = u_n/\|u_n\|_{H^1}$. We may assume that $\hat{u}_n \rightharpoonup \hat{u}$ weakly in $H_0^1(\Omega)$. This implies that $\hat{u}_n \to \hat{u}$ strongly in $L^s_{\mathrm{loc}}(\Omega)$ for all $1 \leq s < 2^*$. By (A1), Egorov theorem, and Hölder inequality, we have

$$\frac{\lambda}{q}\int_\Omega a(x)|\hat{u}_n|^q dx = \frac{\lambda}{q}\int_\Omega a(x)|\hat{u}|^q dx + o_n(1). \tag{4.1}$$

Since $\{u_n\}$ is a $(PS)_c$-sequence for $J_{\lambda a,b}$ and $\|u_n\|_{H^1} \to \infty$, there hold

$$\frac{1}{2}\|\hat{u}_n\|_{H^1}^2 - \frac{\lambda\|u_n\|_{H^1}^{q-2}}{q}\int_\Omega a(x)|\hat{u}_n|^q dx - \frac{\|u_n\|_{H^1}^{p-2}}{p}\int_\Omega b(x)|\hat{u}_n|^p dx = o_n(1), \tag{4.2}$$

$$\|\hat{u}_n\|_{H^1}^2 - \lambda\|u_n\|_{H^1}^{q-2}\int_\Omega a(x)|\hat{u}_n|^q dx - \|u_n\|_{H^1}^{p-2}\int_\Omega b(x)|\hat{u}_n|^p dx = o_n(1). \tag{4.3}$$

From (4.1)–(4.3), we can deduce that

$$\|\hat{u}_n\|_{H^1}^2 = \frac{2(p-q)}{q(p-2)}\|u_n\|^{q-2}\lambda\int_\Omega a(x)|\hat{u}|^q dx + o_n(1). \tag{4.4}$$

Since $1 < q < 2$ and $\|u_n\|_{H^1} \longrightarrow \infty$, (4.4) implies

$$\|\widehat{u}_n\|_{H^1}^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{4.5}$$

which contradicts with the fact that $\|\widehat{u}_n\|_{H^1} = 1$ for all n. □

We assume the condition (Ω_b) holds and recall

$$S_p^b(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\left(\int_\Omega b(x)|u|^p dx\right)^{2/p}}. \tag{4.6}$$

Lemma 4.2. *Assume that (A1), (B1), and (Ω_b) hold. If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$-sequence for $J_{\lambda a,b}$ with $c \in (0, \alpha_0^b(\Omega))$, then there exists a subsequence of $\{u_n\}$ converging weakly to a nonzero solution of $(E_{\lambda a,b})$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_c$-sequence for $J_{\lambda a,b}$ with $c \in (0, \alpha_0^b(\Omega))$. We know from Lemma 4.1 that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and then there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_0 \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega), \\ &u_n \longrightarrow u_0 \quad \text{almost every where in } \Omega, \\ &u_n \longrightarrow u_0 \quad \text{strongly in } L_{\text{loc}}^s(\Omega)\ \forall\, 1 \le s < 2^*. \end{aligned} \tag{4.7}$$

It is easy to see that $(J_{\lambda a,b})'(u_0) = 0$, and by (A1), Egorov theorem, and Hölder inequality, we have

$$\lambda \int_\Omega a(x)|u_n|^q dx = \lambda \int_\Omega a(x)|u_0|^q dx + o_n(1). \tag{4.8}$$

Next we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. Setting

$$l = \lim_{n \to \infty} \int_\Omega b(x)|u_n|^p dx. \tag{4.9}$$

Since $(J_{\lambda a,b})'(u_n) = o_n(1)$ and $\{u_n\}$ is bounded, then by (4.8), we can deduce that

$$0 = \lim_{n\to\infty} \langle (J_{\lambda a,b})'(u_n), u_n \rangle = \lim_{n\to\infty} \left(\|u_n\|_{H^1}^2 - \int_\Omega b(x)|u_n|^p dx \right) = \lim_{n\to\infty} \|u_n\|_{H^1}^2 - l, \tag{4.10}$$

that is,

$$\lim_{n\to\infty} \|u_n\|_{H^1}^2 = l. \tag{4.11}$$

If $l = 0$, then we get $c = \lim_{n\to\infty} J_{\lambda a,b}(u_n) = 0$, which contradicts with $c > 0$. Thus we conclude that $l > 0$. Furthermore, by the definition of $S_p^b(\Omega)$ we obtain

$$\|u_n\|_{H^1}^2 \geq S_p^b(\Omega)\left(\int_\Omega b(x)|u_n|^p dx\right)^{2/p}. \tag{4.12}$$

Then as $n \to \infty$ we have

$$l = \lim_{n\to\infty} \|u_n\|_{H^1}^2 \geq S_p^b(\Omega) l^{2/p}, \tag{4.13}$$

which implies that

$$l \geq S_p^b(\Omega)^{p/(p-2)}. \tag{4.14}$$

Hence, from (1.7) and (4.8)–(4.14) we get,

$$\begin{aligned} c &= \lim_{n\to\infty} J_{\lambda a,b}(u_n) = \frac{1}{2}\lim_{n\to\infty}\|u_n\|_{H^1}^2 - \frac{\lambda}{q}\lim_{n\to\infty}\int_\Omega a(x)|u_n|^q dx - \frac{1}{p}\lim_{n\to\infty}\int_\Omega b(x)|u_n|^p dx \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) l \geq \frac{p-2}{2p} S_p^b(\Omega)^{p/(p-2)} = \alpha_0^b(\Omega). \end{aligned} \tag{4.15}$$

This is a contradiction to $c < \alpha_0^b(\Omega)$. Therefore u_0 is a nonzero solution of $(E_{\lambda a,b})$. □

Lemma 4.3. *Assume that (A1)-(A2), (B1), and (Ω_b) hold. Let w_0 be a positive ground state solution of (E_b); then*

(i) $\sup_{t\geq 0} J_{\lambda a,b}(tw_0) < \alpha_0^b(\Omega)$ *for all* $\lambda > 0$;

(ii) $\alpha_{\lambda a,b}^- < \alpha_0^b(\Omega)$ *for all* $\lambda \in (0, \Lambda_0)$.

Proof. (i) First, we consider the functional $Q : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$Q(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_\Omega b(x)|u|^p dx \quad \forall\, u \in H_0^1(\Omega). \tag{4.16}$$

Then, from (1.3) and (1.7), we conclude that

$$\sup_{t\geq 0} Q(tw_0) = \frac{p-2}{2p}\left(\frac{\|w_0\|_{H^1}^2}{\left(\int_\Omega b(x)|w_0|^p dx\right)^{2/p}}\right)^{p/(p-2)} = \frac{p-2}{2p} S_p^b(\Omega)^{p/(p-2)} = \alpha_0^b(\Omega), \tag{4.17}$$

where the following fact has been used:

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A - \frac{t^p}{p}B\right) = \frac{p-2}{2p}\left(\frac{A}{B^{2/p}}\right)^{p/(p-2)} \quad \text{where } A, B > 0. \tag{4.18}$$

Using the definitions of $J_{\lambda a,b}$, w_0 and $b(x) > 0$ for all $x \in \Omega$, for any $\lambda > 0$, we have

$$J_{\lambda a,b}(tw_0) \longrightarrow -\infty \quad \text{as } t \longrightarrow \infty. \tag{4.19}$$

From this we know that there exists $t_0 > 0$ such that

$$\sup_{t\geq 0} J_{\lambda a,b}(tw_0) = \sup_{0\leq t\leq t_0} J_{\lambda a,b}(tw_0). \tag{4.20}$$

By the continuity of $J_{\lambda a,b}(tw_0)$ as a function of $t \geq 0$ and $J_{\lambda a,b}(0) = 0$, we can find some $t_1 \in (0, t_0)$ such that

$$\sup_{0\leq t\leq t_1} J_{\lambda a,b}(tw_0) < \alpha_0^b(\Omega). \tag{4.21}$$

Thus, we only need to show that

$$\sup_{t_1\leq t\leq t_0} J_{\lambda a,b}(tw_0) < \alpha_0^b(\Omega). \tag{4.22}$$

To this end, by (A2) and (4.17), we have

$$\sup_{t_1\leq t\leq t_0} J_{\lambda a,b}(tw_0) \leq \sup_{t\geq 0} Q(tw_0) - \frac{t_1^q}{q}\int_\Omega a(x)|w_0|^q dx < \alpha_0^b(\Omega). \tag{4.23}$$

Hence (i) holds.

(ii) By (A1), (A2), and the definition of w_0, we have

$$\int_\Omega b(x)|w_0|^p dx > 0, \quad \int_\Omega a(x)|w_0|^q dx > 0. \tag{4.24}$$

Combining this with lemma 2.5(ii), from the definition of $\alpha^-_{\lambda a,b}$ and part (i), for all $\lambda \in (0, \Lambda_0)$, we obtain that there exists $t_0 > 0$ such that $t_0 w_0 \in \mathcal{M}^-_{\lambda a,b}(\Omega)$ and

$$\alpha^-_{\lambda a,b} \leq J_{\lambda a,b}(t_0 w_0) \leq \sup_{t\geq 0} J_{\lambda a,b}(t w_0) < \alpha^b_0(\Omega). \tag{4.25}$$

Therefore, (ii) holds. □

Now, we establish the existence of a local minimum of J_λ on $\mathcal{M}^-_{\lambda a,b}(\Omega)$.

Theorem 4.4. *Assume that (A1)-(A2), (B1), and (Ω_b) hold. If $\lambda \in (0, (q/2)\Lambda_0)$, then $J_{\lambda a,b}$ has a minimizer U_λ in $\mathcal{M}^-_{\lambda a,b}(\Omega)$ and it satisfies the following.*

(i) $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$.

(ii) *U_λ is a positive solution of $(E_{\lambda a,b})$ in Ω.*

Proof. If $\lambda \in (0, (q/2)\Lambda_0)$, then by Theorem 3.1(ii), Proposition 3.3(ii), and Lemma 4.3(ii), there exists a $(PS)_{\alpha^-_{\lambda a,b}}$-sequence $\{u_n\} \subset \mathcal{M}^-_{\lambda a,b}(\Omega)$ in $H^1_0(\Omega)$ for $J_{\lambda a,b}$ with $\alpha^-_{\lambda a,b} \in (0, \alpha^b_0(\Omega))$. From Lemma 4.2, there exist a subsequence still denoted by $\{u_n\}$ and a nonzero solution $U_\lambda \in H^1_0(\Omega)$ of $(E_{\lambda a,b})$ such that $u_n \rightharpoonup U_\lambda$ weakly in $H^1_0(\Omega)$. Now we prove that $u_n \to U_\lambda$ strongly in $H^1_0(\Omega)$ and $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$. By (3.15), if $u \in \mathcal{M}_{\lambda a,b}(\Omega)$, then

$$J_{\lambda a,b}(u) = \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_\Omega a(x)|u|^q dx. \tag{4.26}$$

First, we prove that $U_\lambda \in \mathcal{M}^-_{\lambda a,b}(\Omega)$. On the contrary, if $U_\lambda \in \mathcal{M}^+_{\lambda a,b}(\Omega)$, then by $\mathcal{M}^-_{\lambda a,b}(\Omega)$ being closed in $H^1_0(\Omega)$, we have $\|U_\lambda\|^2_{H^1} < \liminf_{n\to\infty}\|u_n\|^2_{H^1}$. From Lemma 2.3(i) and $b(x) > 0$ for all $x \in \Omega$, we get

$$\int_\Omega a(x)|U_\lambda|^q dx > 0, \qquad \int_\Omega b(x)|U_\lambda|^p dx > 0. \tag{4.27}$$

By Lemma 2.5(ii), there exists a unique t^-_λ such that $t^-_\lambda U_\lambda \in \mathcal{M}^-_{\lambda a,b}(\Omega)$. Since $u_n \in \mathcal{M}^-_{\lambda a,b}(\Omega)$, $J_{\lambda a,b}(u_n) \geq J_{\lambda a,b}(t u_n)$ for all $t \geq 0$ and by (4.26), we have

$$\alpha^-_{\lambda a,b} \leq J_{\lambda a,b}(t^-_\lambda U_\lambda) < \lim_{n\to\infty} J_{\lambda a,b}(t^-_\lambda u_n) \leq \lim_{n\to\infty} J_{\lambda a,b}(u_n) = \alpha^-_{\lambda a,b} \tag{4.28}$$

and this is a contradiction. In order to prove that $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$, it suffices to recall that u_n, $U_\lambda \in \mathcal{M}^-_{\lambda a,b}$ for all n, by (4.26) and applying Fatou's lemma to get

$$\begin{aligned}
\alpha^-_{\lambda a,b} \leq J_{\lambda a,b}(U_\lambda) &= \frac{p-2}{2p}\|U_\lambda\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_\Omega a(x)|U_\lambda|^q dx \\
&\leq \liminf_{n\to\infty}\left(\frac{p-2}{2p}\|u_n\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_\Omega a(x)|u_n|^q dx\right) \\
&\leq \liminf_{n\to\infty} J_{\lambda a,b}(u_n) = \alpha^-_{\lambda a,b}.
\end{aligned} \tag{4.29}$$

This implies that $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$ and $\lim_{n\to\infty}\|u_n\|^2_{H^1} = \|U_\lambda\|^2_{H^1}$. Let $v_n = u_n - U_\lambda$; then by Brézis and Lieb, lemma [33] implies that

$$\|v_n\|^2_{H^1} = \|u_n\|^2_{H^1} - \|U_\lambda\|^2_{H^1} + o_n(1). \tag{4.30}$$

Therefore, $u_n \to U_\lambda$ strongly in $H^1_0(\Omega)$.

Since $J_{\lambda a,b}(U_\lambda) = J_{\lambda a,b}(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{M}^-_{\lambda a,b}(\Omega)$, by Lemma 2.2 we may assume that U_λ is a nonzero nonnegative solution of $(E_{\lambda a,b})$. Finally, By the Harnack inequality [34] we deduce that $U_\lambda > 0$ in Ω. □

Now, we complete the proof of Theorem 1.2: by Theorems 3.4, 4.4, we obtain that $(E_{\lambda a,b})$ has two positive solutions u_λ and U_λ such that $u_\lambda \in \mathcal{M}^+_{\lambda a,b}(\Omega)$, $U_\lambda \in \mathcal{M}^-_{\lambda a,b}(\Omega)$. Since $\mathcal{M}^+_{\lambda a,b}(\Omega) \cap \mathcal{M}^-_{\lambda a,b}(\Omega) = \emptyset$, this implies that u_λ and U_λ are distinct.

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