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Research Article

Monotone Positive Solution of Nonlinear Third-Order BVP with Integral Boundary Conditions

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This paper is concerned with the following third-order boundary value problem with integral boundary conditions $u'''(t) + f(t, u(t), u'(t)) = 0, t \in [0, 1]; u(0) = u'(0) = 0, u'(1) = \int_0^1 g(t)u'(t)dt$, where $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$ and $g \in C([0, 1], [0, +\infty))$. By using the Guo-Krasnoselskii fixed-point theorem, some sufficient conditions are obtained for the existence and nonexistence of monotone positive solution to the above problem.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1].

Recently, third-order two-point or multipoint boundary value problems (BVPs for short) have attracted a lot of attention [2–17]. It is known that BVPs with integral boundary conditions cover multipoint BVPs as special cases. Although there are many excellent works on third-order two-point or multipoint BVPs, a little work has been done for third-order BVPs with integral boundary conditions. It is worth mentioning that, in 2007, Anderson and Tisdell [18] developed an interval of λ values whereby a positive solution exists for the following third-order BVP with integral boundary conditions

$$(pu'')'(t) = \lambda f(t, u(t)), \quad t \in [t_1, t_3],$$

$$\alpha u(t_1) - \beta u'(t_1) = \int_{\xi_1}^{\xi_2} g(t)u(t)dt,$$

$$u'(t_2) = 0,$$
$$(pu'')(t_3) = \int_{\eta_1}^{\eta_2} h(t)(pu'')(t)dt \tag{1.1}$$

by using the Guo-Krasnoselskii fixed-point theorem. In 2008, Graef and Yang [19] studied the third-order BVP with integral boundary conditions

$$\begin{aligned} &u'''(t) = g(t)f(u(t)), \quad t \in [0,1], \\ &u(0) = u'(p) = \int_q^1 w(t)u''(t)dt = 0. \end{aligned} \tag{1.2}$$

For second-order or fourth-order BVPs with integral boundary conditions, one can refer to [20–24].

In this paper, we are concerned with the following third-order BVP with integral boundary conditions

$$\begin{aligned} &u'''(t) + f(t,\ u(t),\ u'(t)) = 0, \quad t \in [0,1], \\ &u(0) = u'(0) = 0, \qquad u'(1) = \int_0^1 g(t)u'(t)dt. \end{aligned} \tag{1.3}$$

Throughout this paper, we always assume that $f \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$ and $g \in C([0,1], [0,+\infty))$. Some sufficient conditions are established for the existence and nonexistence of monotone positive solution to the BVP (1.3). Here, a solution u of the BVP (1.3) is said to be monotone and positive if $u'(t) \geq 0$, $u(t) \geq 0$ and $u(t) \not\equiv 0$ for $t \in [0,1]$. Our main tool is the following Guo-Krasnoselskii fixed-point theorem [25].

Theorem 1.1. *Let E be a Banach space and let K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(1) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

(2) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

For convenience, we denote $\mu = \int_0^1 tg(t)dt$.

Lemma 2.1. *Let $\mu \neq 1$. Then for any $h \in C[0,1]$, the BVP*

$$-u'''(t) = h(t), \quad t \in [0,1],$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \int_0^1 g(t)u'(t)dt \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 \left[G_1(t,s) + \frac{t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s)g(\tau)d\tau \right] h(s)ds, \quad t \in [0,1], \tag{2.2}$$

where

$$G_1(t,s) = \frac{1}{2}\begin{cases} (2t - t^2 - s)s, & 0 \le s \le t \le 1, \\ (1-s)t^2, & 0 \le t \le s \le 1, \end{cases}$$
$$G_2(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Proof. Let u be a solution of the BVP (2.1). Then, we may suppose that

$$u(t) = \int_0^1 G_1(t,s)h(s)ds + At^2 + Bt + C, \quad t \in [0,1]. \tag{2.4}$$

By the boundary conditions in (2.1), we have

$$A = \frac{1}{2(1-\mu)} \int_0^1 h(s) \int_0^1 G_2(\tau,s)g(\tau)d\tau ds \quad \text{and } B = C = 0. \tag{2.5}$$

Therefore, the BVP (2.1) has a unique solution

$$u(t) = \int_0^1 \left[G_1(t,s) + \frac{t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s)g(\tau)d\tau \right] h(s)ds, \quad t \in [0,1]. \tag{2.6}$$

□

Lemma 2.2 (see [12]). *For any $(t,s) \in [0,1] \times [0,1]$,*

$$\frac{t^2}{2}(1-s)s \le G_1(t,s) \le \frac{1}{2}(1-s)s. \tag{2.7}$$

Lemma 2.3 (see [26]). *For any* $(t,s) \in [0,1] \times [0,1]$,

$$0 \le G_2(t,s) \le (1-s)s. \tag{2.8}$$

In the remainder of this paper, we always assume that $\mu < 1$, $\alpha \in (0,1)$ and $\beta = \alpha^2/2$.

Lemma 2.4. *If* $h \in C[0,1]$ *and* $h(t) \ge 0$ *for* $t \in [0,1]$, *then the unique solution* u *of the BVP* (2.1) *satisfies*

(1) $u(t) \ge 0$, $t \in [0,1]$,

(2) $u'(t) \ge 0$, $t \in [0,1]$ *and* $\min_{t\in[\alpha,1]} u(t) \ge \beta\|u\|$, *where* $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$.

Proof. Since (1) is obvious, we only need to prove (2). By (2.2), we get

$$u'(t) = \int_0^1 \left[G_2(t,s) + \frac{t}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds, \quad t \in [0,1], \tag{2.9}$$

which indicates that $u'(t) \ge 0$ for $t \in [0,1]$.

On the one hand, by (2.9) and Lemma 2.3, we have

$$\|u'\|_\infty \le \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds. \tag{2.10}$$

On the other hand, in view of (2.2) and Lemma 2.2, we have

$$\|u\|_\infty \le \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\|u\| \le \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds, \tag{2.12}$$

which together with Lemma 2.2 implies that

$$\begin{aligned}
\min_{t\in[\alpha,1]} u(t) &= \min_{t\in[\alpha,1]} \int_0^1 \left[G_1(t,s) + \frac{t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds \\
&\geq \min_{t\in[\alpha,1]} \frac{t^2}{2} \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds \\
&= \frac{\alpha^2}{2} \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] h(s) ds \\
&\geq \beta \|u\|.
\end{aligned} \tag{2.13}$$

□

Let $E = C^1[0,1]$ be equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$. Then E is a Banach space. If we denote

$$K = \left\{ u \in E : u(t) \geq 0,\ u'(t) \geq 0,\ t \in [0,1], \min_{t\in[\alpha,1]} u(t) \geq \beta \|u\| \right\}, \tag{2.14}$$

then it is easy to see that K is a cone in E. Now, we define an operator T on K by

$$(Tu)(t) = \int_0^1 \left[G_1(t,s) + \frac{t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] f(s, u(s), u'(s)) ds, \quad t \in [0,1]. \tag{2.15}$$

Obviously, if u is a fixed point of T, then u is a monotone nonnegative solution of the BVP (1.3).

Lemma 2.5. *$T : K \to K$ is completely continuous.*

Proof. First, by Lemma 2.4, we know that $T(K) \subset K$.

Next, we assume that $D \subset K$ is a bounded set. Then there exists a constant $M_1 > 0$ such that $\|u\| \leq M_1$ for any $u \in D$. Now, we will prove that $T(D)$ is relatively compact in K. Suppose that $\{y_k\}_{k=1}^\infty \subset T(D)$. Then there exist $\{x_k\}_{k=1}^\infty \subset D$ such that $Tx_k = y_k$. Let

$$\begin{gathered}
M_2 = \sup\{ f(t,x,y) : (t,x,y) \in [0,1] \times [0,M_1] \times [0,M_1] \}, \\
M_3 = \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau ds.
\end{gathered} \tag{2.16}$$

Then for any k, by Lemma 2.2, we have

$$\begin{aligned}
\left|y_k(t)\right| &= |(Tx_k)(t)| \\
&= \left|\int_0^1 \left[G_1(t,s) + \frac{t^2}{2(1-\mu)}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f\left(s, x_k(s), x_k'(s)\right)ds\right| \\
&\le \frac{M_2}{2}\int_0^1 \left[(1-s)s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right]ds \\
&= \frac{M_2}{2}\left(\frac{1}{6} + M_3\right), \quad t \in [0,1],
\end{aligned} \tag{2.17}$$

which implies that $\{y_k\}_{k=1}^{\infty}$ is uniformly bounded. At the same time, for any k, in view of Lemma 2.3, we have

$$\begin{aligned}
\left|y_k'(t)\right| &= \left|(Tx_k)'(t)\right| \\
&= \left|\int_0^1 \left[G_2(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f\left(s, x_k(s), x_k'(s)\right)ds\right| \\
&\le M_2\left(\int_0^1 \left[(1-s)s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right]ds\right) \\
&= M_2\left(\frac{1}{6} + M_3\right), \quad t \in [0,1],
\end{aligned} \tag{2.18}$$

which shows that $\{y_k'\}_{k=1}^{\infty}$ is also uniformly bounded. This indicates that $\{y_k\}_{k=1}^{\infty}$ is equicontinuous. It follows from Arzela-Ascoli theorem that $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Without loss of generality, we may assume that $\{y_k\}_{k=1}^{\infty}$ converges in $C[0,1]$. On the other hand, by the uniform continuity of $G_2(t,s)$, we know that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1$, we have

$$|G_2(t_1,s) - G_2(t_2,s)| < \frac{\varepsilon}{2(M_2+1)}, \quad s \in [0,1]. \tag{2.19}$$

Let $\delta = \min\{\delta_1, \varepsilon/2(M_2M_3+1)\}$. Then for any k, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
\left|y_k'(t_1) - y_k'(t_2)\right| &= \left|(Tx_k)'(t_1) - (Tx_k)'(t_2)\right| \\
&\le \int_0^1 \left[|G_2(t_1,s) - G_2(t_2,s)| + \frac{|t_1-t_2|}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f\left(s, x_k(s), x_k'(s)\right)ds \\
&\le M_2\int_0^1 |G_2(t_1,s) - G_2(t_2,s)|ds + M_2M_3|t_1-t_2| \\
&\le \frac{M_2\varepsilon}{2(M_2+1)} + M_2M_3|t_1-t_2| \\
&< \varepsilon,
\end{aligned} \tag{2.20}$$

which implies that $\{y_k'\}_{k=1}^{\infty}$ is equicontinuous. Again, by Arzela-Ascoli theorem, we know that $\{y_k'\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Therefore, $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C^1[0,1]$. Thus, we have shown that T is a compact operator.

Finally, we prove that T is continuous. Suppose that $u_m, u \in K$ and $\|u_m - u\| \to 0$ $(m \to \infty)$. Then there exists $M_4 > 0$ such that for any m, $\|u_m\| \le M_4$. Let

$$M_5 = \sup\{f(t,x,y) : (t,x,y) \in [0,1] \times [0,M_4] \times [0,M_4]\}. \tag{2.21}$$

Then for any m and $t \in [0,1]$, in view of Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\left[G_1(t,s) + \frac{t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u_m(s),u_m'(s)) \\
&\qquad \le \frac{M_5}{2}\left[1 + \frac{1}{1-\mu}\int_0^1 g(\tau)d\tau\right](1-s)s, \quad s \in [0,1], \\
&\left[G_2(t,s) + \frac{t}{1-\mu} \int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u_m(s),u_m'(s)) \\
&\qquad \le M_5\left[1 + \frac{1}{1-\mu}\int_0^1 g(\tau)d\tau\right](1-s)s, \quad s \in [0,1].
\end{aligned} \tag{2.22}$$

By applying Lebesgue Dominated Convergence theorem, we get

$$\begin{aligned}
\lim_{m\to\infty}(Tu_m)(t) &= \lim_{m\to\infty}\int_0^1\left[G_1(t,s) + \frac{t^2}{2(1-\mu)}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u_m(s),u_m'(s))ds \\
&= \int_0^1\left[G_1(t,s) + \frac{t^2}{2(1-\mu)}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u(s),u'(s))ds \\
&= (Tu)(t), \quad t \in [0,1], \\
\lim_{m\to\infty}(Tu_m)'(t) &= \lim_{m\to\infty}\int_0^1\left[G_2(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u_m(s),u_m'(s))ds \\
&= \int_0^1\left[G_2(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u(s),u'(s))ds \\
&= (Tu)'(t), \quad t \in [0,1],
\end{aligned} \tag{2.23}$$

which indicates that T is continuous. Therefore, $T : K \to K$ is completely continuous. □

3. Main Results

For convenience, we define

$$\begin{aligned}
f^0 &= \limsup_{x+y\to 0^+} \max_{t\in[0,1]} \frac{f(t,x,y)}{x+y}, \qquad f_0 = \liminf_{x+y\to 0^+} \min_{t\in[\alpha,1]} \frac{f(t,x,y)}{x+y}, \\
f^\infty &= \limsup_{x+y\to +\infty} \max_{t\in[0,1]} \frac{f(t,x,y)}{x+y}, \qquad f_\infty = \liminf_{x+y\to +\infty} \min_{t\in[\alpha,1]} \frac{f(t,x,y)}{x+y}, \\
H_1 &= 2\int_0^1 \left[(1-s)s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right]ds, \\
H_2 &= \frac{\beta}{2}\int_\alpha^1 \left[(1-s)s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right]ds.
\end{aligned} \tag{3.1}$$

Theorem 3.1. *If $H_1 f^0 < 1 < H_2 f_\infty$, then the BVP* (1.3) *has at least one monotone positive solution.*

Proof. In view of $H_1 f^0 < 1$, there exists $\varepsilon_1 > 0$ such that

$$H_1\left(f^0 + \varepsilon_1\right) \le 1. \tag{3.2}$$

By the definition of f^0, we may choose $\rho_1 > 0$ so that

$$f(t,x,y) \le \left(f^0 + \varepsilon_1\right)(x+y), \ \text{for } t \in [0,1], \ (x+y) \in [0,\rho_1]. \tag{3.3}$$

Let $\Omega_1 = \{u \in E : \|u\| < \rho_1/2\}$. Then for any $u \in K \cap \partial\Omega_1$, in view of (3.2) and (3.3), we have

$$\begin{aligned}
(Tu)'(t) &= \int_0^1 \left[G_2(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u(s),u'(s))ds \\
&\le \int_0^1 \left[(1-s)s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right]\left(f^0 + \varepsilon_1\right)(u(s)+u'(s))ds \\
&\le H_1\left(f^0+\varepsilon_1\right)\|u\| \\
&\le \|u\|, \quad t \in [0,1].
\end{aligned} \tag{3.4}$$

By integrating the above inequality on $[0,t]$, we get

$$(Tu)(t) \le \|u\|, \quad t \in [0,1], \tag{3.5}$$

which together with (3.4) implies that

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.6}$$

On the other hand, since $1 < H_2 f_\infty$, there exists $\varepsilon_2 > 0$ such that

$$H_2(f_\infty - \varepsilon_2) \geq 1. \tag{3.7}$$

By the definition of f_∞, we may choose $\rho_2 > \rho_1$, so that

$$f(t,x,y) \geq (f_\infty - \varepsilon_2)(x+y), \quad \text{for } t \in [\alpha,1],\ (x+y) \in [\rho_2, +\infty). \tag{3.8}$$

Let $\Omega_2 = \{u \in E : \|u\| < \rho_2/\beta\}$. Then for any $u \in K \cap \partial\Omega_2$, in view of (3.7) and (3.8), we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 \left[G_1(1,s) + \frac{1}{2(1-\mu)}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right] f(s,u(s),u'(s))ds \\
&\geq \frac{1}{2}\int_\alpha^1 \left[(1-s)s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)g(\tau)d\tau\right](f_\infty - \varepsilon_2)(u(s)+u'(s))ds \\
&\geq H_2(f_\infty - \varepsilon_2)\|u\| \\
&\geq \|u\|,
\end{aligned} \tag{3.9}$$

which implies that

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{3.10}$$

Therefore, it follows from (3.6), (3.10), and Theorem 1.1 that the operator T has one fixed point $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is a monotone positive solution of the BVP (1.3). □

Theorem 3.2. *If $H_1 f^\infty < 1 < H_2 f_0$, then the BVP* (1.3) *has at least one monotone positive solution.*

Proof. The proof is similar to that of Theorem 3.1 and is therefore omitted. □

Theorem 3.3. *If $H_1 f(t,x,y) < (x+y)$ for $t \in [0,1]$ and $(x+y) \in [0,+\infty)$, then the BVP* (1.3) *has no monotone positive solution.*

Proof. Suppose on the contrary that u is a monotone positive solution of the BVP (1.3). Then $u(t) \geq 0$ and $u'(t) \geq 0$ for $t \in [0,1]$, and

$$\begin{aligned} u'(t) &= \int_0^1 \left[G_2(t,s) + \frac{t}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] f(s,u(s),u'(s)) ds \\ &\leq \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] f(s,u(s),u'(s)) ds \\ &< \frac{1}{H_1} \int_0^1 \left[(1-s)s + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) g(\tau) d\tau \right] (u(s)+u'(s)) ds \\ &\leq \|u\|, \quad t \in [0,1]. \end{aligned} \tag{3.11}$$

By integrating the above inequality on $[0,t]$, we get

$$u(t) < \|u\|, \quad t \in [0,1], \tag{3.12}$$

which together with (3.11) implies that

$$\|u\| < \|u\|. \tag{3.13}$$

This is a contradiction. Therefore, the BVP (1.3) has no monotone positive solution. □

Similarly, we can prove the following theorem.

Theorem 3.4. *If $H_2 f(t,x,y) > (x+y)$ for $t \in [\alpha,1]$ and $(x+y) \in [0,+\infty)$, then the BVP* (1.3) *has no monotone positive solution.*

Example 3.5. Consider the following BVP:

$$\begin{gathered} u'''(t) + \frac{1}{1+t} \left[\frac{u(t)+u'(t)}{e^{u(t)+u'(t)}} + \frac{1000(u(t)+u'(t))^2}{1+u(t)+u'(t)} \right] = 0, \quad t \in [0,1], \\ u(0) = u'(0) = 0, \qquad u'(1) = \int_0^1 t u'(t) dt. \end{gathered} \tag{3.14}$$

Since $f(t,x,y) = 1/(1+t)[((x+y)/e^{x+y}) + (1000(x+y)^2/(1+x+y))]$ and $g(t) = t$, if we choose $\alpha = 1/2$, then it is easy to compute that

$$f^0 = 1, \qquad f_\infty = 500, \qquad H_1 = \frac{11}{24}, \qquad H_2 = \frac{91}{12288}, \tag{3.15}$$

which shows that

$$H_1 f^0 < 1 < H_2 f_\infty. \tag{3.16}$$

So, it follows from Theorem 3.1 that the BVP (3.14) has at least one monotone positive solution.

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