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Research Article

Existence of Periodic Solutions of Linear Hamiltonian Systems with Sublinear Perturbation

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We investigate the existence of periodic solutions of linear Hamiltonian systems with a nonlinear perturbation. Under generalized Ahmad-Lazer-Paul type coercive conditions for the nonlinearity on the kernel of the linear part, existence of periodic solutions is obtained by saddle point theorems. A note on a result of Rabinowitz is also given.

1. Introduction

For the second-order Hamiltonian system

$$\ddot{u}(t) + \nabla F(t, u(t)) = 0, \qquad u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \tag{1.1}$$

the existence of periodic solutions is related to the coercive conditions of $F(t, u)$ on u. This fact is first noticed by Berger and Schechter [1] who use the coercive condition $F(t, u) \to -\infty$ as $|u| \to \infty$, uniformly for a.e. $t \in [0, T]$. Subsequently, Mawhin and Willem [2] consider it by using more general coercive conditions of an integral form. More precisely, they assume that $F(t, u) : [0, T] \times \mathbf{R}^N \to \mathbf{R}^N$ is bounded ($|\nabla F(t, u)| \leq g(t)$ for some $g(t) \in L^1(0, T)$) with some additional technical conditions and satisfies one of the following Ahmad-Lazer-Paul type [3] coercive conditions:

$$\lim_{\|u\| \to \infty\ u \in \mathbf{R}^N} \int_0^T F(t, u) dt = \pm\infty, \tag{1.2}$$

then they obtain the existence of at least one solution. How to relax the boundedness of F is a problem which attracted several authors' attention, for example, see [4, 5] and the references therein.

In [6, 7], the nonlinearity is allowed to be unbounded and satisfy

$$|\nabla F(t,u)| \leq g(t)|u|^{\alpha} + h(t), \tag{1.3}$$

where $0 \leq \alpha < 1$ and $g(t), h(t) \in L^2(0,2\pi)$ and satisfy one of the generalized Ahmad-Lazer-Paul type coercive conditions

$$\lim_{\|u\|\to\infty\ u\in \mathbf{R}^N} \|u\|^{-2\alpha} \int_0^{2\pi} F(t,u)dt = \pm\infty, \tag{1.4}$$

the same results are obtained. In fact, a more general system is considered and the above results are just a special case (as $A = 0$) of the results there. The conditions which are useful to deal with problems (1.1) are used in recent years by several authors; see [4, 8] and the references therein for some further information. For some recent developments of the second-order systems (1.1), see [9].

In this paper, we use this kind of condition to consider the existence of periodic solutions of first-order linear Hamiltonian system with a nonlinear perturbation

$$\dot{u} = JA(t)u + J\nabla G(u,t), \tag{1.5}$$

where $A(t)$ is a symmetric 2π-periodic $2N \times 2N$ continuous matrix function, $G(u,t) \in C^1(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ is 2π-periodic for t, and J is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}. \tag{1.6}$$

The 2π-periodic solutions of the problem correspond to the critical points of the functional

$$\Phi(u) = \frac{1}{2}\int_0^{2\pi} (-J\dot{u} - A(t)u) \cdot u\, dt - \int_0^{2\pi} G(u,t)dt \tag{1.7}$$

on the Hilbert space $E := W^{1/2}(S^1, \mathbf{R}^{2N})$. We recall that E is a Sobolev space of 2π-periodic $\mathbf{R}^{2N}$-valued functions

$$u(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt, \quad a_0, a_k, b_k \in \mathbf{R}^{2N} \tag{1.8}$$

with inner product

$$((u,u')) := 2\pi a_0 \cdot a'_0 + \pi \sum_{k=1}^{\infty} k\left(a_k a'_k + b_k b'_k\right), \tag{1.9}$$

and E is compactly (hence continuously) imbedded into $L^s(S^1, \mathbf{R}^{2N})$ for every $s \geq 1$ [10]. That is for every $s \geq 1$

$$E \subset\subset L^s\left(S^1, \mathbf{R}^{2N}\right). \tag{1.10}$$

A compact self-adjoint operator on E can be defined by

$$((Vu, w)) := \int_0^{2\pi} A(t)u \cdot w \, dt. \tag{1.11}$$

Define another self-adjoint operator on E

$$((Uu, w)) := \int_0^{2\pi} -J\dot{u} \cdot w \, dt, \tag{1.12}$$

and denote $U - V$ by L. Hence $\Phi(u)$ has the form

$$\Phi(u) = \frac{1}{2}((Lu, u)) - \varphi(u), \tag{1.13}$$

where $\varphi(u) = \int_0^{2\pi} G(u, t) dt$.

We make the following assumptions.

(G_1) There exists $0 \leq \alpha < 1$ such that $\nabla G(u, t) = O(|u|^\alpha) + O(1)$ uniformly for $t \in [0, 2\pi]$, $u \in \mathbf{R}$.

(G_2) $\nabla G(u, t) = o(|u|)$ uniformly for $t \in [0, 2\pi]$ as $|u| \to \infty$.

($G_\pm$)G_+G_-

$$\lim_{\|u\| \to \infty,\, u \in N(L)} \frac{\int_0^{2\pi} G(u(t), t) dt}{\|u\|^{2\alpha}} = +\infty, \tag{G_+}$$

$$\lim_{\|u\| \to \infty,\, u \in N(L)} \frac{\int_0^{2\pi} G(u(t), t) dt}{\|u\|^{2\alpha}} = -\infty, \tag{G_-}$$

where $N(L) = \{u \in E \mid Lu = 0\}$. It is easily seen that $u \in N(L)$ if and only if $u \in E$ is a 2π-periodic solution of the following linear problem:

$$\dot{u} = JA(t)u. \tag{1.14}$$

It is a standard result that the self-adjoint operator L on E has discrete eigenvalues: $\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0(\lambda_0) < \lambda_1 \leq \lambda_2 \leq \cdots$. Let $e_{\pm j}$ denote the eigenvectors of L corresponding to

$\lambda_{\pm j}$, respectively. Define $\widetilde{E} = \overline{\operatorname{span}}_{j\geq 1}\{e_{+j}\}$, $\overline{E} = \overline{\operatorname{span}}_{j\geq 1}\{e_{-j}\}$, and $E^0 = \ker L$. Hence there exists a decomposition $E = \overline{E} \oplus E^0 \oplus \widetilde{E}$, where $\dim E^0 < \infty$ and $\widetilde{E}$, $\overline{E}$ are all infinite dimensional. Denote correspondingly for every $u \in E$, $u = \overline{u} + u^0 + \widetilde{u}$. It is more convenient to introduce the following equivalent inner product on E. For $u, v \in E$, $u = \overline{u} + u^0 + \widetilde{u}$, $v = \overline{v} + v^0 + \widetilde{v}$, we define

$$(u, v) = ((L\widetilde{u}, \widetilde{u})) - ((L\overline{u}, \overline{u})) + \left(\left(u^0, v^0\right)\right). \tag{1.15}$$

The induced norm is still denoted by $\|\cdot\|$. Then $\Phi(u)$ has the form

$$\Phi(u) = \frac{1}{2}\|\widetilde{u}\|^2 - \frac{1}{2}\|\overline{u}\|^2 - \varphi(u). \tag{1.16}$$

Now we can state the main results of the paper.

Theorem 1.1. *Suppose that the condition* (G_1) *holds. Furthermore, we assume that one of the conditions* $(G_\pm)$ *holds. Then the Hamiltonian system* (1.5) *has at least one* 2π*-periodic solution.*

Theorem 1.2. *Assume that the linear problem* (1.14) *has only the trivial* 2π*-periodic solution* $u = 0$ *and the condition* (G_2) *holds. Then the Hamiltonian system* (1.5) *has at least one* 2π*-periodic solution.*

Remark 1.3. Theorem 1.2 is essentially known in the literature by various methods, for example, see [11–15]. Here we prove it and Theorem 1.1 by using variational methods in a united framework.

Remark 1.4. When one of the conditions $(G_\pm)$ holds, the critical groups at infinity for the functional (1.16) can be clearly computed, for example, see [8, 16] or [17] for the bounded nonlinearity. Hence at least one critical point of $\Phi(u)$ can be obtained. But for the use of Morse theory, more regularity restrictions than those in the above theorems about $G(t, u)$ have to be used.

2. Proofs the Theorems

As to the investigation of (1.1), we need to use the saddle point theorem in the variational methods. But contrary to the functional corresponding to (1.1), which is semidefinite, the functional $\Phi(u)$ is strongly indefinite which means that the positive and negative indees for the linear part are all infinite. Hence we need another version of the saddle point theorem (see Theorem 5.29 and Example 5.22 in [10]) which we state here.

Theorem 2.1. *Let* E *be a real Hilbert space with* $E = E_1 \oplus E_2$ *and* $E_2 = E_1^\perp$. *Suppose* $\Phi \in C^1(E, R)$ *satisfies (PS) condition and*

(1) $\Phi(u) = (1/2)(Lu, u) + b(u)$, *where* $Lu = L_1P_1u + L_2P_2u$ *and* $L_i : E_i \to E_i$ *are bounded and self-adjoint,* $i = 1, 2$,

(2) b' *is compact,*

(3) *there are constants* $\alpha > \omega$ *such that*

$$I|_{(E_2)} \geq \alpha, \quad I|_{\partial Q} \leq \omega, \quad Q = B \cap E_1,\ B \text{ is a ball in } E_1. \tag{2.1}$$

Then Φ *possesses a critical value* $c \geq \alpha$.

Proof of Theorem 1.1. We use Theorem 2.1. and only consider the case where (G_+) holds. The other case can be similarly treated. Set $E = E_1 \oplus E_2 =: (\overline{E} \oplus E^0) \oplus \widetilde{E}$. It is clear that conditions (1) and (2) in Theorem 2.1 hold. Now we prove that the functional Φ satisfies PS condition on E. In the following, C denotes a universal positive constant, and $\langle \cdot, \cdot \rangle$ denotes the paring between E' and E.

Suppose that $\Phi'(u_n) \to 0$ as $n \to \infty$ and $|\Phi(u_n)| \le C$, for all $n \ge 1$.

$$\begin{aligned}
C\|\overline{u}_n\| &\ge \left|\langle \Phi'(u_n), -\overline{u}_n \rangle\right| \\
&= \left|(\overline{u}_n, \overline{u}_n) + \int_0^{2\pi} \nabla G(t, u_n)\overline{u}_n\right| \\
&\ge \|\overline{u}_n\|^2 - \int_0^{2\pi} \left(C + C\left|\overline{u}_n + u_n^0 + \widetilde{u}_n\right|^\alpha\right)|\overline{u}_n| dt \qquad (2.2) \\
&\ge \|\overline{u}_n\|^2 - C\|\overline{u}_n\| - C\int_0^{2\pi} \left(|\overline{u}_n|^{1+\alpha} + \left|u_n^0\right|^\alpha |\overline{u}_n| + |\widetilde{u}_n|^\alpha |\overline{u}_n|\right) dt \\
&\ge \|\overline{u}_n\|^2 - C\|\overline{u}_n\| - C\|\overline{u}_n\|^{1+\alpha} - C\left\|u_n^0\right\|^\alpha \|\overline{u}_n\| - \epsilon\|\overline{u}_n\|^2 - C(\epsilon)\|\widetilde{u}_n\|^{2\alpha} \\
&\ge \frac{1}{2}\|\overline{u}_n\|^2 - C\left\|u_n^0\right\|^\alpha \|\overline{u}_n\| - C\|\widetilde{u}_n\|^{2\alpha}, \qquad (2.3)
\end{aligned}$$

for every $\epsilon > 0$.

In the proof of (2.2), we use the imbedding result (1.10), the finite dimensionality of E^0, and Young inequality. In the proof of

$$\int_0^{2\pi} (|\widetilde{u}_n|^\alpha |\overline{u}_n|) dt \le \epsilon\|\overline{u}_n\|^2 + C(\epsilon)\|\widetilde{u}_n\|^{2\alpha}, \qquad (2.4)$$

we need a little bit of caution. First, as $\alpha = 0$, it is clear. Hence we suppose that $0 < \alpha < 1$. Choosing $p > 1$ sufficiently large such that $p\alpha > 1$, then using Hölder inequality and the imbedding result (1.10), we have

$$\int_0^{2\pi} (|\widetilde{u}_n|^\alpha |\overline{u}_n|) dt \le \left(\int_0^{2\pi} |\widetilde{u}_n|^{p\alpha}\right)^{1/p} \left(\int_0^{2\pi} |\overline{u}_n|^q\right)^{1/q} \le \|\widetilde{u}_n\|^\alpha \|\overline{u}_n\|, \qquad (2.5)$$

where $1/p + 1/q = 1$.

Hence from (2.3),we get

$$\|\overline{u}_n\|^2 \le C\left\|u_n^0\right\|^{2\alpha} + C\|\widetilde{u}_n\|^{2\alpha}. \qquad (2.6)$$

By estimating $\langle \Phi'(u_n), \widetilde{u}_n \rangle$ and a similar argument as above, we can get

$$\|\widetilde{u}_n\|^2 \le C\left\|u_n^0\right\|^{2\alpha} + C\|\overline{u}_n\|^{2\alpha}. \qquad (2.7)$$

Combining (2.6) and (2.7) and noticing the fact that $0 \le \alpha < 1$, we have

$$\begin{aligned} \|\overline{u}_n\|^2 &\le C\left\|u_n^0\right\|^{2\alpha}, \\ \|\widetilde{u}_n\|^2 &\le C\left\|u_n^0\right\|^{2\alpha}. \end{aligned} \tag{2.8}$$

In order to prove that $\{\|u_n^0\|\}$ and hence $\{\|u_n\|\}$ are bounded, we need much work. By (2.8), we have

$$\begin{aligned} -C &\le \Phi(u_n) \\ &= \frac{1}{2}\|\widetilde{u}_n\|^2 - \frac{1}{2}\|\overline{u}_n\|^2 - \int_0^{2\pi} G(t,u_n)dt \\ &\le C\left\|u_n^0\right\|^{2\alpha} - \int_0^{2\pi}\left[G(t,u_n) - G\left(t,u_n^0\right)\right]dt + \int_0^{2\pi} G\left(t,u_n^0\right)dt. \end{aligned} \tag{2.9}$$

We want to prove that

$$\left|\int_0^{2\pi}\left[G(t,u_n) - G\left(t,u_n^0\right)\right]dt\right| \le C\left\|u_n^0\right\|^{2\alpha} + C. \tag{2.10}$$

In fact

$$\begin{aligned} \left|\int_0^{2\pi}\left[G(t,u_n) - G\left(t,u_n^0\right)\right]dt\right| &= \left|\int_0^{2\pi}\left(\int_0^1 \nabla G\left(t,u_n^0 + s(\widetilde{u}_n + \overline{u}_n)(\widetilde{u}_n + \overline{u}_n)\right)ds\right)dt\right| \\ &\le \int_0^{2\pi}(C + C\left(\left|u_n^0\right|^{\alpha} + |\widetilde{u}_n|^{\alpha} + |\overline{u}_n|^{\alpha}\right)|\widetilde{u}_n + \overline{u}_n|\,dt. \end{aligned} \tag{2.11}$$

Now, to get (2.10), we use a similar argument as that in the proof of (2.2) and the inequalities (2.8).

Hence we get the inequality

$$-C \le C\left\|u_n^0\right\|^{2\alpha} - \int_0^{2\pi} G\left(t,u_n^0\right)dt. \tag{2.12}$$

Hence by condition (G_+) and Lemma 3.1 in [6] or by a direct reasoning, we have that $\{\|u_n^0\|\}$ must be bounded. So $\{\|u_n\|\}$ is bounded in E by (2.8). Using a same argument in [10], we prove that Φ satisfies the PS condition on E.

Finally we verify the conditions (3) in Theorem 2.1.

Recall that we set $E = E_1 \oplus E_2 =: (\overline{E} \oplus E^0) \oplus \widetilde{E}$.

As $u \in \widetilde{E}$, $u = \widetilde{u}$, we have

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\|\widetilde{u}\|^2 - \int_0^{2\pi} G(t,\widetilde{u})dt \\
&= \frac{1}{2}\|\widetilde{u}\|^2 - \int_0^{2\pi} G(t,0)dt - \int_0^{2\pi} (G(t,\widetilde{u}) - G(t,0))dt \\
&= \frac{1}{2}\|\widetilde{u}\|^2 - \int_0^{2\pi} G(t,0)dt - \int_0^{2\pi} \left(\int_0^1 \nabla G(t,s\widetilde{u})\widetilde{u}\, ds\right)dt \\
&\geq \frac{1}{2}\|\widetilde{u}\|^2 - C - C\|\widetilde{u}\|^{1+\alpha},
\end{aligned}
\tag{2.13}
$$

where we used condition (G_1). Noticing that $\alpha < 1$, we have that $\Phi(u)$ is bounded below on $\widetilde{E}$.

As $u \in \overline{E} \oplus E^0$, $u = \overline{u} + u^0$, we have

$$
\begin{aligned}
\Phi(u) &= -\frac{1}{2}\|\overline{u}\|^2 - \int_0^{2\pi} G(t,u)dt \\
&= -\frac{1}{2}\|\overline{u}\|^2 - \int_0^{2\pi} G\left(t,u^0\right)dt - \int_0^{2\pi} \left(G(t,u) - G\left(t,u^0\right)\right)dt \\
&= -\frac{1}{2}\|\overline{u}\|^2 - \int_0^{2\pi} G\left(t,u^0\right)dt - \int_0^{2\pi} \left(\int_0^1 \nabla G\left(t,u^0 + s\overline{u}\right)\overline{u}\, ds\right)dt \\
&\leq -\frac{1}{4}\|\overline{u}\|^2 - \int_0^{2\pi} G\left(t,u^0\right)dt + C\left\|u^0\right\|^{2\alpha} + C,
\end{aligned}
\tag{2.14}
$$

where we used Young inequality and condition (G_1) and omitted some simple details. Hence $\Phi(u) \to -\infty$ as $u \in \overline{E} \oplus E^0$ and $\|u\| \to \infty$, by condition (G_+).

This completes the proof. □

Proof of Theorem 1.2. We still use Theorem 2.1. and only consider the case where (G_+) holds. Under the assumption of the theorem, $E^0 = 0$. We set $E = E_1 \oplus E_2 =: \overline{E} \oplus \widetilde{E}$. It is clear that conditions (1) and (2) in Theorem 2.1 hold. Now we prove that the functional Φ satisfies PS condition on E.

By (G_2), for every $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$
|\nabla G(t,u)| \leq \epsilon|u| + C(\epsilon)
\tag{2.15}
$$

for all $t \in \mathbf{R}$, $u \in \mathbf{R}^{2N}$.

Suppose that $\Phi'(u_n) \to 0$ as $n \to \infty$ and $|\Phi(u_n)| \le C$.

$$\begin{aligned}
C\|\overline{u}_n\| &\ge \left|\langle \Phi'(u_n), -\overline{u}_n\rangle\right| \\
&= \left|(\overline{u}_n, \overline{u}_n) + \int_0^{2\pi} \nabla G(t, u_n)\overline{u}_n dt\right| \\
&\ge \|\overline{u}_n\|^2 - \int_0^{2\pi} (\epsilon|u_n| + C(\epsilon))|\overline{u}_n| dt \\
&\ge \|\overline{u}_n\|^2 - C\epsilon\|\overline{u}_n\|^2 - C\epsilon\|\overline{u}_n\|^2 - C\epsilon\|\widetilde{u}_n\|^2 - C(\epsilon)\|\overline{u}_n\|.
\end{aligned} \tag{2.16}$$

Hence we get

$$\|\overline{u}_n\|^2 \le C\epsilon\|\widetilde{u}_n\|^2 + C(\epsilon). \tag{2.17}$$

Similarly, by estimating $\langle \Phi'(u_n), -\widetilde{u}_n\rangle$, we can get

$$\|\widetilde{u}_n\|^2 \le C\epsilon\|\overline{u}_n\|^2 + C(\epsilon). \tag{2.18}$$

By combining the above two inequalities and fixing $\epsilon > 0$ small, we get that $\{\|u_n\|\}$ is bounded in E. Hence an argument in [10] shows that the PS condition hold.

As $u \in \widetilde{E}$, $u = \widetilde{u}$, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|\widetilde{u}\|^2 - \int_0^{2\pi} G(t, \widetilde{u}) dt \\
&= \frac{1}{2}\|\widetilde{u}\|^2 - \int_0^{2\pi} G(t, 0) dt - \int_0^{2\pi} \left(\int_0^1 \nabla G(t, s\widetilde{u})\widetilde{u}\, ds\right) dt \\
&\ge \frac{1}{2}\|\widetilde{u}\|^2 - C(\epsilon) - C\epsilon\|\widetilde{u}\|^2.
\end{aligned} \tag{2.19}$$

As $u \in \overline{E}$, we have

$$\begin{aligned}
\Phi(u) &= -\frac{1}{2}\|\overline{u}\|^2 - \int_0^{2\pi} G(t, u) dt \\
&= -\frac{1}{2}\|\overline{u}\|^2 - \int_0^{2\pi} G(t, 0) dt - \int_0^{2\pi} \left(\int_0^1 \nabla G(t, s\overline{u})\overline{u}\, ds\right) dt \\
&\le -\frac{1}{2}\|\overline{u}\|^2 + C\epsilon\|\overline{u}\|^2 + C(\epsilon).
\end{aligned} \tag{2.20}$$

By fixing $\epsilon > 0$ such that $C\epsilon < 1/2$, we get that the conditions (3) in Theorem 2.1 hold. Hence we complete the proof. □

Remark 2.2. In order to check the conditions $(G_\pm)$ involving the unknown functions in the kernel $N(L)$, we present the following proposition.

Proposition 2.3. *Suppose that $\nabla G(t,u)$ satisfies (G_1) and there exist $\beta(t), \gamma(t) \in L^1(0,2\pi)$ such that the following limits are uniform for a.e. $t \in [0,2\pi]$:*

$$\beta(t) \le \liminf_{|u|\to\infty} \frac{(\nabla G(t,u),u)}{|u|^{1+\alpha}} \le \limsup_{|u|\to\infty} \frac{(\nabla G(t,u),u)}{|u|^{1+\alpha}} \le \gamma(t). \tag{2.21}$$

Then (i) if $\beta(t) \ge 0$, a.e. $t \in [0,2\pi]$ and $\int_0^{2\pi} \beta(t)dt > 0$, (G_+) holds; (ii) if $\gamma(t) \le 0$, a.e. $t \in [0,2\pi]$ and $\int_0^{2\pi} \gamma(t)dt < 0$, (G_-) holds.

Proof. The case (i) is proved in [4] and the case (ii) can be similarly proved. □

3. A Note on a Result of Rabinowitz

In this Section, we give a note about a result in [18]. Following the same method, we will prove the following result.

Theorem 3.1. *Let $G(t,u)$ satisfy the following conditions: (1) $G(t,u) \ge 0$ for all $t \in [0,2\pi]$ and $u \in \mathbf{R}^{2N}$, (2) $G(t,u) = o(|u|^2)$ as $u \to 0$, uniformly for $t \in [0,2\pi]$, (3) there exists $\mu > 2$, $\overline{r}$ and $1 < \mu^* < \mu$ such that*

$$0 < \mu G(t,u) \le u\nabla G(t,u), \tag{3.1}$$

$$|\nabla G(t,u)| \le C|u|^{\mu^*}, \tag{3.2}$$

for all $|u| \ge \overline{r}$ and $t \in [0,2\pi]$. Then (1.5) has at least one nonzero 2π-periodic solution.

Remark 3.2. When the condition (3.2) is replaced by the following one there are constants $\alpha, R_1 > 0$ such that $|\nabla G(t,u)| \le \alpha(u, \nabla G(t,u))$ for all $t \in \mathbf{R}$, $u \in \mathbf{R}^{2N}$, $|u| > R_1$. The above result is proved by Rabinowitz [18]. When the condition (3.2) is replaced by a condition which measures the difference of the system from an autonomous one, the problem is also considered by [19].

Proof of Theorem 3.1. We basically follow the same method as that in [10, 18]. But under the condition (3.2), we do not need the truncation method there and just use a variant of Theorem 2.1 (generalized mountain pass lemma).

As in Section 1, the solutions of (1.5) correspond to the critical points of

$$\Phi(u) = \frac{1}{2}\|\tilde{u}\|^2 - \frac{1}{2}\|\overline{u}\|^2 - \int_0^{2\pi} G(t,u)dt \tag{3.3}$$

on E. We divide the proof to several steps.

Step 1. Conditions (1) and (2) in Theorem 2.1 hold. It is clear.

Step 2. Set $E = E_1 \oplus E_2 =: (\overline{E} \oplus E^0) \oplus \widetilde{E}$. By conditions (2) and (3), for every $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$|\nabla G(t,u)| \le \epsilon |u|^2 + C(\epsilon)|u|^{\mu^*+1}, \tag{3.4}$$

for all $t \in \mathbf{R}$, $u \in \mathbf{R}^{2N}$. Hence, as $u \in \widetilde{E}$, we have

$$\Phi(u) \ge \frac{1}{2}\|u\|^2 - C\epsilon\|u\|^2 - C(\epsilon)\|u\|^{\mu^*+1}. \tag{3.5}$$

Hence by fixing $\epsilon > 0$ small, we can obtain $\rho > 0, \tau > 0$ such that $\Phi(u) \ge \tau > 0$ for all $u \in \partial B_\rho \cap E_1$.

Step 3. Choose $e \in \partial B_\rho \cap E_1$ and set $Q = \{re \mid 0 \le r \le r_1\} \oplus (B_{r_2} \cap E_2)$. Define $E^* = \operatorname{span}\{e\} \oplus E_2$ so $Q \subset E^*$. Using a same method as [10, Lemma 6.20], we have $\Phi(u) \le 0$ on ∂Q after suitable choices of r_1 and r_2, where the boundary is taken in E^*.

Step 4. By condition (3.1), we have

$$G(t,u) \ge C|u|^\mu - C, \tag{3.6}$$

for some $C > 0$ and all $t \in \mathbf{R}$, $u \in \mathbf{R}^{2N}$.

Now we verify the PS condition.

Suppose that $\Phi'(u_n) \to 0$ as $n \to \infty$ and $|\Phi(u_n)| \le C$, for all $n \ge 1$.Then

$$\begin{aligned} C + \|u_n\| &\ge \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n \\ &= \int_0^{2\pi} \left(\frac{1}{2}u_n \cdot \nabla G(t,u_n) - G(t,u_n)\right)dt \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_0^{2\pi} u_n \cdot \nabla G(t,u_n)dt - C \\ &\ge C\|u_n\|_{L^\mu}^\mu - C, \end{aligned} \tag{3.7}$$

for some $C > 0$. Furthermore, by (3.7), we have

$$C + \|u_n\| \ge C\left(\|u_n\|_{L^2}^2\right)^{\mu/2} \ge C\left\|u_n^0\right\|_{L^2}^\mu \ge C\left\|u_n^0\right\|^\mu. \tag{3.8}$$

Now we turn to estimate other terms.

$$\begin{aligned}\|\tilde{u}_n\|^2 &= \left|\langle \Phi'(u_n), \tilde{u}_n\rangle + \int_0^{2\pi} \nabla G(t, u_n)\tilde{u}_n dt\right| \\ &\le \int_0^{2\pi} |u_n|^{\mu^*} |\tilde{u}_n| dt + C\|\tilde{u}_n\| \\ &\le C\|u_n\|_{L^\mu}^{\mu^*}\|\tilde{u}_n\| + C\|\tilde{u}_n\|.\end{aligned} \tag{3.9}$$

Hence

$$\|\tilde{u}_n\| \le C\|u_n\|_{L^\mu}^{\mu^*} + C. \tag{3.10}$$

Therefore, using (3.7), we get

$$\|\tilde{u}_n\| \le C\|u_n\|^{\mu^*/\mu} + C. \tag{3.11}$$

Similarly, we also have

$$\|\overline{u}_n\| \le C\|u_n\|^{\mu^*/\mu} + C. \tag{3.12}$$

Combining (3.8), (3.11), and (3.12), we have

$$\|u_n\| \le C\|u_n\|^{1/\mu} + C\|u_n\|^{\mu^*/\mu} + C. \tag{3.13}$$

Hence $\{\|u_n\|\}$ must be bounded. By a standard argument, the PS condition holds. Now the theorem is proved by Theorem 5.29 in [10]. □

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