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Research Article

Transmission Problem in Thermoelasticity

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We show that the energy to the thermoelastic transmission problem decays exponentially as time goes to infinity. We also prove the existence, uniqueness, and regularity of the solution to the system.

1. Introduction

In this paper we deal with the theory of thermoelasticity. We consider the following transmission problem between two thermoelastic materials:

$$\rho_1 u_{tt} - \mu_1 \Delta u - (\mu_1 + \lambda_1) \nabla \operatorname{div} u - m_1 \nabla \widetilde{\theta} = 0 \quad \text{in } \Omega_1 \times [0, \infty), \tag{1.1}$$

$$\rho_2 v_{tt} - \mu_2 \Delta v - (\mu_2 + \lambda_2) \nabla \operatorname{div} v - m_2 \nabla \theta = 0 \quad \text{in } \Omega_2 \times [0, \infty), \tag{1.2}$$

$$\tau_1 \widetilde{\theta}_t - \kappa_1 \Delta \widetilde{\theta} - m_1 \operatorname{div} u_t = 0 \quad \text{in } \Omega_1 \times [0, \infty), \tag{1.3}$$

$$\tau_2 \theta_t - \kappa_2 \Delta \theta - m_2 \operatorname{div} v_t = 0 \quad \text{in } \Omega_2 \times [0, \infty). \tag{1.4}$$

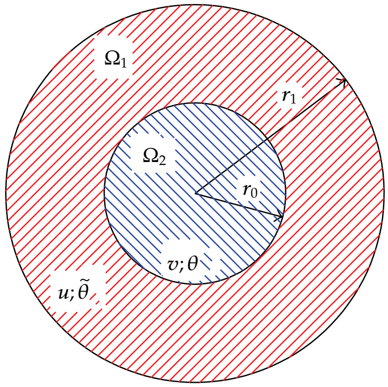

Figure 1: Domains $\Omega_1$and Ω_2 and boundaries of the transmission problem.

We denote by $x = (x_1, \ldots, x_n)$ a point of Ω_i $(i = 1, 2)$ while t stands for the time variable. The displacement in the thermoelasticity parts is denoted by $u : \Omega_1 \times [0, +\infty) \longrightarrow \mathbb{R}^n$, $u = (u_1, \ldots, u_n)$ $(u_i = u_i(x, t)$, $i = 1, \ldots, n)$ and $v =: \Omega_2 \times [0, +\infty) \longrightarrow \mathbb{R}^n$, $v = (v_1, \ldots, v_n)$ $(v_i = v_i(x, t)$, $i = 1, \ldots, n)$, $\widetilde{\theta} : \Omega_1 \times [0, +\infty) \longrightarrow \mathbb{R}$, and $\theta : \Omega_2 \times [0, +\infty) \longrightarrow \mathbb{R}$ is the variation of temperature between the actual state and a reference temperature, respectively. κ_1, κ_2 are the thermal conductivity. All the constants of the system are positive. Let us consider an n-dimensional body which is configured in $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$.

The thermoelastic parts are given by Ω_1 and Ω_2, respectively. The constants $m_1, m_2 > 0$ are the coupling parameters depending on the material properties. The boundary of Ω_1 is denoted by $\partial\Omega_1 = \Gamma_1 \cup \Gamma_2$ and the boundary of Ω_2 by $\partial\Omega_2 = \Gamma_2$. We will consider the boundaries Γ_1 and Γ_2 of class C^2 in the rest of this paper. The thermoelastic parts are given by Ω_1 and Ω_2, respectively, that is (see Figure 1),

$$\Omega_2 = \{x \in \mathbb{R}^n : |x| < r_0\}, \quad \Omega_1 = \{x \in \mathbb{R}^n : r_0 < |x| < r_1\}, \quad 0 < r_0 < r_1. \tag{1.5}$$

We consider for $i = 1, 2$ the operators

$$A_i = \mu_i \Delta + (\mu_i + \lambda_i) \nabla \operatorname{div}, \tag{1.6}$$

$$\frac{\partial u}{\partial \nu_{A_i}} = \mu_i \frac{\partial u}{\partial \nu} + (\mu_i + \lambda_i)(\operatorname{div} u)\nu, \tag{1.7}$$

where μ_i, λ_i $(i = 1, 2)$ are the Lamé moduli satisfying $\mu_i + \lambda_i \geq 0$.

The initial conditions are given by

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \widetilde{\theta}(x, 0) = \widetilde{\theta}_0(x), \quad x \in \Omega_1, \tag{1.8}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega_2. \tag{1.9}$$

The system is subject to the following boundary conditions:

$$u = 0, \quad \widetilde{\theta} = 0 \quad \text{on } \Gamma_1, \tag{1.10}$$

$$u = v, \quad \theta = \widetilde{\theta} \quad \text{on } \Gamma_2 \tag{1.11}$$

and transmission conditions

$$\mu_1 \nabla u + (\mu_1 + \lambda_1) \operatorname{div} u + m_1 \widetilde{\theta} = \mu_2 \nabla v + (\mu_2 + \lambda_2) \operatorname{div} v \ + m_2 \theta \quad \text{on } \Gamma_2, \tag{1.12}$$

$$\frac{\partial \theta}{\partial \nu_{\Gamma_2}} = 0 \quad \text{on } \Gamma_2. \tag{1.13}$$

The transmission conditions are imposed, that express the continuity of the medium and the equilibrium of the forces acting on it. The discontinuity of the coefficients of the equations corresponds to the fact that the medium consists of two physically different materials.

Since the domain $\Omega \subseteq \mathbb{R}^n$ is composed of two different materials, its density is not necessarily a continuous function, and since the stress-strain relation changes from the thermoelastic parts, the corresponding model is not continuous. Taking in consideration this, the mathematical problem that deals with this type of situation is called a transmission problem. From a mathematical point of view, the transmission problem is described by a system of partial differential equations with discontinuous coefficients. The model (1.1)–(1.13) to consider is interesting because we deal with composite materials. From the economical and the strategic point of view, materials are mixed with others in order to get another more convenient material for industry (see [1–3] and references therein). Our purpose in this work is to investigate that the solution of the symmetrical transmission problem decays exponentially as time tends to infinity, no matter how small is the size of the thermoelastic parts. The transmission problem has been of interest to many authors, for instance, in the one-dimensional thermoelastic composite case, we can refer to the papers [4–7]. In the two-, three- or n-dimensional, we refer the reader to the papers [8, 9] and references therein. The method used here is based on energy estimates applied to nonlinear problems, and the differential inequality is obtained by exploiting the symmetry of the solutions and applying techniques for the elastic wave equations, which solve the exponential stability produced by the boundary terms in the interface of the material. This methods allow us to find a Lyapunov functional $\mathcal{L}$ equivalent to the second-order energy for which we have that

$$\frac{d}{dt} \mathcal{L}(t) \le -\gamma \mathcal{L}(t). \tag{1.14}$$

In spite of the obvious importance of the subject in applications, there are relatively few mathematical results about general transmission problem for composite materials. For this reason we study this topic here.

This paper is organized as follows. Before describing the main results, in Section 2, we briefly outline the notation and terminology to be used later on and we present some lemmas. In Section 3 we prove the existence and regularity of radially symmetric solutions to the transmission problem. In Section 4 we show the exponential decay of the solutions and we prove the main theorem.

2. Preliminaries

We will use the following standard notation. Let Ω be a domain in $\mathbb{R}^n$. For $1 \leq p \leq \infty$, $L^p(\Omega)$ are all real valued measurable functions on Ω such that $|u|^p$ is integrable for $1 \leq p < \infty$ and $\sup_{x\in\Omega}\text{ess}|u(x)|$ is finite for $p = \infty$. The norm will be written as

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u(x)|^p dx\right)^{1/p}, \qquad \|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} \text{ess}|u(x)|. \tag{2.1}$$

For a nonnegative integer m and $1 \leq p \leq \infty$, we denote by $W^{m,p}(\Omega)$ the Sobolev space of functions in $L^p(\Omega)$ having all derivatives of order $\leq m$ belonging to $L^p(\Omega)$. The norm in $W^{m,p}(\Omega)$ is given by $\|u\|_{W^{m,p}(\Omega)} = (\sum_{|\alpha|\leq m} \|D^\alpha u(x)\|^p_{L^p(\Omega)} dx)^{1/p}$. $W^{m,2}(\Omega) \equiv H^m(\Omega)$ with norm $\|\cdot\|_{H^m(\Omega)}$, $W^{0,2}(\Omega) \equiv L^2(\Omega)$ with norm $\|\cdot\|_{L^2(\Omega)}$. We write $C^k(I, X)$ for the space of X-valued functions which are k-times continuously differentiable (resp. square integrable) in I, where $I \subseteq \mathbb{R}$ is an interval, X is a Banach space, and k is a nonnegative integer. We denote by $O(n)$ the set of orthogonal $n \times n$ real matrices and by $SO(n)$ the set of matrices in $O(n)$ which have determinant 1.

The following results are going to be used several times from now on. The proof can be found in [10].

Lemma 2.1. *Let $G = (G_{ij})_{n\times n} \in O(2)$ for $n = 2$ or $G = (G_{ij})_{n\times n} \in SO(2)$ for $n \geq 3$ be arbitrary but fixed. Assume that u_0, u_1, $\widetilde{\theta}_0$, v_0, v_1, and θ_0 satisfy*

$$\begin{aligned} &u_0(Gx) = Gu_0(x), \quad u_1(Gx) = Gu_1(x), \quad \widetilde{\theta}_0(Gx) = G\widetilde{\theta}_0(x), \quad \forall x \in \overline{\Omega_1}, \\ &v_0(Gx) = Gv_0(x), \quad v_1(Gx) = Gv_1(x), \quad \theta_0(Gx) = G\theta_0(x), \quad \forall x \in \overline{\Omega_2}. \end{aligned} \tag{2.2}$$

Then the solution u, $\widetilde{\theta}$, v, and θ of (1.1)–(1.13) *has the form*

$$u_i(x,t) = x_i\phi(r,t), \quad \forall x \in \overline{\Omega_1},\ t \geq 0, \tag{2.3}$$

$$\widetilde{\theta}(x,t) = \zeta(r,t), \quad \forall x \in \overline{\Omega_1},\ t \geq 0, \tag{2.4}$$

$$v_i(x,t) = x_i\eta(r,t), \quad v_i(0,t) = 0,\ i = 1, 2, \ldots, n, \quad \forall x \in \overline{\Omega_2},\ t \geq 0, \tag{2.5}$$

$$\widetilde{\theta}(x,t) = \psi(r,t), \quad \forall x \in \overline{\Omega_2},\ t \geq 0, \tag{2.6}$$

where $|x| = r$, for some functions ϕ, ζ, η, and ψ.

Lemma 2.2. *One supposes that $u : \Omega_1 \to \mathbb{R}^n$ is a radially symmetric function satisfying $u|_{\Gamma_1} = 0$. Then there exists a positive constant C such that*

$$\|\nabla u(t)\|_{L^2(\Omega_1)} \le C \|\operatorname{div} u(t)\|_{L^2(\Omega_1)}, \quad t \ge 0. \tag{2.7}$$

Moreover one has the following estimate at the boundary:

$$|\nabla u(t)|^2 \le C\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(n-1)}{r_0}|u|^2 \quad \text{on } \Gamma_2. \tag{2.8}$$

Remark 2.3. From (2.3) we have that

$$\nabla \operatorname{div} u = \Delta u. \tag{2.9}$$

The following straightforward calculations are going to be used several times from now on.

(a) From (1.8) we obtain

$$\begin{aligned} \frac{\partial u}{\partial \nu_{A_1}} &= \mu_1 \nabla u \cdot \nu_{\Omega_1} + (\mu_1 + \lambda_1)(\operatorname{div} u)\nu_{\Omega_1}, \\ \frac{\partial v}{\partial \nu_{A_2}} &= \mu_2 \nabla v \cdot \nu_{\Gamma_2} + (\mu_2 + \lambda_2)(\operatorname{div} v)\nu_{\Gamma_2}. \end{aligned} \tag{2.10}$$

(b) Using (1.10) and (1.11) we have that

$$\kappa_1 \int_{\Omega_1} \widetilde{\theta} \Delta \widetilde{\theta}\, d\Omega_1 = -\kappa_1 \int_{\Gamma_2} \theta \nabla \theta \cdot \nu_{\Gamma_2} d\Gamma_2 - \kappa_1 \int_{\Omega_1} \left|\nabla \widetilde{\theta}\right|^2 d\Omega_1, \tag{2.11}$$

$$\kappa_2 \int_{\Omega_2} \theta \Delta \theta\, d\Omega_2 = \kappa_2 \int_{\Gamma_2} \theta \nabla \theta \cdot \nu_{\Gamma_2} d\Gamma_2 - \kappa_2 \int_{\Omega_2} |\nabla \theta|^2 d\Omega_2, \tag{2.12}$$

$$m_1 \int_{\Omega_1} \nabla \widetilde{\theta} \cdot u_t d\Omega_1 + m_1 \int_{\Omega_1} (\operatorname{div} u_t) \widetilde{\theta}\, d\Omega_1 = -m_1 \int_{\Gamma_2} \theta v_t \cdot \nu_{\Gamma_2} d\Gamma_2, \tag{2.13}$$

$$m_2 \int_{\Omega_2} \nabla \theta \cdot v_t d\Omega_2 + m_2 \int_{\Omega_2} (\operatorname{div} v_t) \theta\, d\Omega_2 = m_2 \int_{\Gamma_2} \theta v_t \cdot \nu_{\Gamma_2} d\Gamma_2. \tag{2.14}$$

(c) Using (1.6) we have that

$$
\begin{aligned}
&\int_{\Omega_1} \left[\mu_1 \Delta u + (\mu_1+\lambda_1)\nabla \operatorname{div} u\right]\cdot u_t d\Omega_1 \\
&\quad= \mu_1 \int_{\Omega_1} \Delta u \cdot u_t d\Omega_1 + (\mu_1+\lambda_1)\int_{\Omega_1} \nabla \operatorname{div} u \cdot u_t d\Omega_1 \\
&\quad= \mu_1 \int_{\Omega_1} (\operatorname{div} \nabla u) u_t d\Omega_1 + (\mu_1+\lambda_1)\int_{\Omega_1} \nabla \operatorname{div} u \cdot u_t d\Omega_1 \\
&\quad= \mu_1 \int_{\Omega_1} \nabla\cdot (u_t \nabla u) d\Omega_1 - \mu_1 \int_{\Omega_1} \nabla u \cdot \nabla u_t d\Omega_1 \\
&\qquad+ (\mu_1+\lambda_1)\int_{\Omega_1} \nabla\cdot (u_t \operatorname{div} u) d\Omega_1 - (\mu_1+\lambda_1)\int_{\Omega_1} (\operatorname{div} u)(\operatorname{div} u_t) d\Omega_1 \\
&\quad= \mu_1 \int_{\Gamma_1\cup\Gamma_2} u_t \nabla u \cdot \nu_{\Omega_1} d\Gamma_{\Omega_1} - \frac{1}{2}\mu_1 \frac{d}{dt}\int_{\Omega_1} |\nabla u|^2 d\Omega_1 \\
&\qquad+ (\mu_1+\lambda_1)\int_{\Gamma_1\cup\Gamma_2} (\operatorname{div} u) u_t \cdot \nu_{\Omega_1} d\Gamma_{\Omega_1} - \frac{1}{2}(\mu_1+\lambda_1)\frac{d}{dt}\int_{\Omega_1} |\operatorname{div} u|^2 d\Omega_1.
\end{aligned}
\tag{2.15}
$$

Thus, using (1.10) and (1.11) we have that

$$
\begin{aligned}
\int_{\Omega_1} (A_1 u) u_t d\Omega_1 &= -\frac{1}{2}\mu_1 \frac{d}{dt}\|\nabla u\|^2_{L^2(\Omega_1)} - \frac{1}{2}(\mu_1+\lambda_1)\frac{d}{dt}\|\operatorname{div} u\|^2_{L^2(\Omega_1)} \\
&\quad - (\mu_1+\lambda_1)\int_{\Gamma_2} (\operatorname{div} u) u_t \cdot \nu_{\Gamma_2} d\Gamma_2 - \mu_1 \int_{\Gamma_2} \nabla u \cdot \nu_{\Gamma_2} u_t d\Gamma_2 \\
&= -\frac{1}{2}\mu_1 \frac{d}{dt}\|\nabla u\|^2_{L^2(\Omega_1)} - \frac{1}{2}(\mu_1+\lambda_1)\frac{d}{dt}\|\operatorname{div} u\|^2_{L^2(\Omega_1)} \\
&\quad - (\mu_1+\lambda_1)\int_{\Gamma_2} (\operatorname{div} u) u_t \cdot \nu_{\Gamma_2} d\Gamma_2 - \mu_1 \int_{\Gamma_2} \nabla u \cdot \nu_{\Gamma_2} u_t d\Gamma_2.
\end{aligned}
\tag{2.16}
$$

Similarly, we obtain

$$
\begin{aligned}
\int_{\Omega_2} (A_2 v) v_t d\Omega_2 &= -\frac{1}{2}\mu_2 \frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega_2)} - \frac{1}{2}(\mu_2+\lambda_2)\frac{d}{dt}\|\operatorname{div} v\|^2_{L^2(\Omega_2)} \\
&\quad + (\mu_2+\lambda_2)\int_{\Gamma_2} (\operatorname{div} v) \cdot \nu_{\Gamma_2} d\Gamma_2 + \mu_2 \int_{\Gamma_2} \nabla v \cdot \nu_{\Gamma_2} v_t d\Gamma_2.
\end{aligned}
\tag{2.17}
$$

Throughout this paper C is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

3. Existence and Uniqueness

In this section we establish the existence and uniqueness of solutions to the system (1.1)–(1.13). The proof is based using the standard Galerkin approximation and the elliptic regularity for transmission problem given in [11]. First of all, we define what we will understand for weak solution of the problem (1.1)–(1.13).

We introduce the following spaces:

$$\begin{gathered} H^1_{\Gamma_1} = \left\{ w \in H^1(\Omega_1) : w = 0 \text{ on } \Gamma_1 \right\}, \\ V = \left\{ (f, g) \in H^1_{\Gamma_1} \times H^1(\Omega_2) : f = g \text{ on } \Gamma_2 \right\} \end{gathered} \tag{3.1}$$

for $x \in \Omega \subseteq \mathbb{R}^n$ and $t \geq 0$.

Definition 3.1. One says that $(u, v, \tilde{\theta}, \theta)$ is a weak solution of (1.1)–(1.13) if

$$\begin{gathered} u \in W^{1,\infty}\left([0,T] : L^2(\Omega_1)\right) \cap L^\infty\left([0,T] : H^1_{\Gamma_1}\right), \\ v \in W^{1,\infty}\left([0,T] : L^2(\Omega_2)\right) \cap L^\infty\left([0,T] : H^1(\Omega_2)\right), \\ \tilde{\theta} \in L^\infty\left([0,T] : L^2(\Omega_1)\right) \cap L^2\left([0,T] : H^1_{\Gamma_1}\right), \\ \theta \in L^\infty\left([0,T] : L^2(\Omega_2)\right) \cap L^2\left([0,T] : H^1(\Omega_2)\right), \end{gathered} \tag{3.2}$$

satisfying the identities

$$\begin{aligned} &\int_0^T \int_{\Omega_1} \left[\rho_1 u \cdot \varphi_{tt} + \mu_1 \nabla u \nabla \varphi + (\mu_1 + \lambda_1) \operatorname{div} u \operatorname{div} \varphi + m_1 \tilde{\theta} \operatorname{div} \varphi\right] d\Omega_1 dt \\ &\qquad + \int_0^T \int_{\Omega_2} \left[\rho_2 v \cdot \chi_{tt} + \mu_2 \nabla v \nabla \chi + (\mu_2 + \lambda_2) \operatorname{div} v \operatorname{div} \chi + m_2 \theta \operatorname{div} \chi\right] d\Omega_2 dt \\ &\quad = \int_{\Omega_1} \rho_1 \left[u_1 \varphi(0) - u_0 \varphi_t(0)\right] d\Omega_1 + \int_{\Omega_2} \rho_2 \left[v_1 \chi(0) - v_0 \chi_t(0)\right] d\Omega_2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\int_0^T \int_{\Omega_1} \left[-\tau_1 \tilde{\theta} \tilde{\eta}_t + \kappa_1 \nabla \tilde{\theta} \cdot \nabla \tilde{\eta} - m_1 \tilde{\eta} \operatorname{div} u_t\right] d\Omega_1 dt \\ &\qquad + \int_0^T \int_{\Omega_2} \left[-\tau_2 \theta \eta_t + \kappa_2 \nabla \theta \cdot \nabla \eta - m_2 \eta \operatorname{div} v_t\right] d\Omega_2 dt \\ &\quad = \int_{\Omega_1} \tilde{\theta}_0 \tilde{\eta}(0) d\Omega_1 + \int_{\Omega_2} \theta_0 \eta(0) d\Omega_2 \end{aligned} \tag{3.4}$$

for all $(\varphi, \chi) \in V$, $\tilde{\eta} \in C^1([0,T] : H^1_{\Gamma_1})$, $\eta \in C^1([0,T] : H^1(\Omega_2))$, and almost every $t \in [0,T]$ such that

$$\varphi(T) = \varphi_t(T) = \chi(T) = \chi_t(T) = 0, \qquad \tilde{\eta}(T) = \eta(T) = 0. \tag{3.5}$$

The existence of solutions to the system (1.1)–(1.13) is given in the following theorem.

Theorem 3.2. *One considers the following initial data satisfying*

$$\left(u_0, u_1, \tilde{\theta}\right) \in H^1_{\Gamma_1}(\Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1), \qquad (v_0, v_1, \theta) \in H^1(\Omega_2) \times L^2(\Omega_2) \times L^2(\Omega_2). \tag{3.6}$$

Then there exists only one solution $(u, v, \tilde{\theta}, \theta)$ *of the system* (1.1)–(1.13) *satisfying*

$$\begin{gathered} u \in W^{1,\infty}\left([0,T] : L^2(\Omega_1)\right) \cap L^\infty\left([0,T] : H^1_{\Gamma_1}\right), \\ v \in W^{1,\infty}\left([0,T] : L^2(\Omega_2)\right) \cap L^\infty\left([0,T] : H^1(\Omega_2)\right), \\ \tilde{\theta} \in L^\infty\left([0,T] : L^2(\Omega_1)\right) \cap L^2\left([0,T] : H^1_{\Gamma_1}\right), \\ \theta \in L^\infty\left([0,T] : L^2(\Omega_2)\right) \cap L^2\left([0,T] : H^1(\Omega_2)\right). \end{gathered} \tag{3.7}$$

Moreover, if

$$\begin{gathered} \left(u_0, u_1, \tilde{\theta}_0\right) \in \left(H^2(\Omega_1) \cap H^1_{\Gamma_1}\right) \times H^1_{\Gamma_1} \times \left(H^2(\Omega_1) \cap H^1_{\Gamma_1}\right), \\ (v_0, v_1, \theta_0) \in H^2(\Omega_2) \times H^1(\Omega_2) \times H^2(\Omega_2) \end{gathered} \tag{3.8}$$

verifying the boundary conditions

$$\begin{gathered} u_0 = 0, \quad \tilde{\theta}_0 = 0 \quad \text{on } \Gamma_1, \\ u_0 = v_0, \quad \tilde{\theta}_0 = \theta_0 \quad \text{on } \Gamma_2 \end{gathered} \tag{3.9}$$

and the transmission conditions

$$\begin{gathered} \mu_1 \nabla u + (\mu_1 + \lambda_1) \operatorname{div} u + m_1 \tilde{\theta} = \mu_2 \nabla v + (\mu_2 + \lambda_2) \operatorname{div} v + m_2 \theta \quad \text{on } \Gamma_2, \\ \frac{\partial \theta}{\partial \nu_{\Gamma_2}} = 0 \quad \text{on } \Gamma_2, \end{gathered} \tag{3.10}$$

then the solution satisfies

$$
\begin{aligned}
u &\in L^{\infty}\Big([0,T]:H^2(\Omega_1)\Big)\cap W^{1,\infty}\Big([0,T]:H^1(\Omega_1)\Big)\cap W^{2,\infty}\Big([0,T]:L^2(\Omega_1)\Big),\\
v &\in L^{\infty}\Big([0,T]:H^2(\Omega_2)\Big)\cap W^{1,\infty}\Big([0,T]:H^1(\Omega_2)\Big)\cap W^{2,\infty}\Big([0,T]:L^2(\Omega_2)\Big),\\
\widetilde{\theta} &\in L^{\infty}\Big([0,T]:H^2(\Omega_1)\Big)\cap W^{1,\infty}\Big([0,T]:L^2(\Omega_1)\Big),\\
\theta &\in L^{\infty}\Big([0,T]:H^2(\Omega_2)\Big)\cap W^{1,\infty}\Big([0,T]:L^2(\Omega_2)\Big).
\end{aligned}
\tag{3.11}
$$

Proof. The existence of solutions follows using the standard Galerking approximation.

Faedo-Galerkin Scheme

Given $n \in \mathbb{N}$, denote by P_n and Q_n the projections on the subspaces

$$
\operatorname{span}\{(\varphi_i,\chi_i)\},\quad \operatorname{span}\{(\widetilde{\eta}_i,\eta_i)\},\quad i=1,2,\ldots,n \tag{3.12}
$$

of V and $H^1_{\Gamma_1}$, respectively. Let us write

$$
(u^n,v^n)=\sum_{i=1}^{n}a_i(t)(\varphi_i,\chi_i),\qquad \Big(\widetilde{\theta}^n,\theta^n\Big)=\sum_{i=1}^{n}b_i(t)(\widetilde{\eta}_i,\eta_i), \tag{3.13}
$$

where u^n and v^n satisfy

$$
\begin{aligned}
&\int_{\Omega_1}\Big[\rho_1 u^n_{tt}\cdot\varphi_i+\mu_1\nabla u^n\nabla\varphi_i+(\mu_1+\lambda_1)\operatorname{div}u^n\operatorname{div}\varphi_i+m_1\widetilde{\theta}^n\operatorname{div}\varphi_i\Big]d\Omega_1\\
&\quad+\int_{\Omega_2}\big[\rho_2 v^n_{tt}\cdot\chi_i+\mu_2\nabla v^n\nabla\chi_i+(\mu_2+\lambda_2)\operatorname{div}v^n\operatorname{div}\chi_i+m_2\theta^n\operatorname{div}\chi_i\big]d\Omega_2=0,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&\int_{\Omega_1}\Big[\tau_1\widetilde{\theta}^n_t\widetilde{\eta}_i+\kappa_1\nabla\widetilde{\theta}^n\cdot\nabla\widetilde{\eta}_i-m_1\widetilde{\eta}_i\operatorname{div}u^n_t\Big]d\Omega_1\\
&\quad+\int_{\Omega_2}\big[\tau_2\theta^n_t\eta_i+\kappa_2\nabla\theta^n\cdot\nabla\eta_i-m_2\eta_i\operatorname{div}v^n_t\ \big]d\Omega_2=0,
\end{aligned}
\tag{3.15}
$$

with

$$
u^n(0)=u_0,\qquad v^n(0)=v_0,\qquad u^n_t(0)=u_1,\qquad v^n_t(0)=v_1,\qquad \widetilde{\theta}^n(0)=\widetilde{\theta}_0,\qquad \theta^n(0)=\theta_0 \tag{3.16}
$$

for almost all $t \le T$, where ϕ_0, ψ_0, $\widetilde{\eta}_0$, and η_0 are the zero vectors in the respective spaces. Recasting exactly the classical Faedo-Galerkin scheme, we get a system of ordinary differential equations in the variables $a_i(t)$ and $b_i(t)$. According to the standard existence theory for ordinary differential equations there exists a continuous solution of this system, on some interval $(0, T_n)$. The a priori estimates that follow imply that in fact $t_n = \infty$.

Energy Estimates

Multiplying (3.14) by $a_i'(t)$, summing up over i, and integrating over Ω_1 we obtain

$$\frac{d}{dt}\mathcal{E}_1^n(t) + m_1 \int_{\Omega_1} \widetilde{\theta}^n \operatorname{div} u_t^n d\Omega_1 + m_2 \int_{\Omega_2} \theta^n \operatorname{div} v_t^n d\Omega_2 = 0, \tag{3.17}$$

where

$$\begin{aligned}\mathcal{E}_1^n(t) &= \frac{1}{2}\left[\rho_1 \|u_t^n\|^2_{L^2(\Omega_1)} + \mu_1 \|\nabla u^n\|^2_{L^2(\Omega_1)} + (\mu_1 + \lambda_1)\|\operatorname{div} u^n\|^2_{L^2(\Omega_1)}\right] \\ &\quad + \frac{1}{2}\left[\rho_2 \|v_t^n\|^2_{L^2(\Omega_2)} + \mu_2 \|\nabla v^n\|^2_{L^2(\Omega_2)} + (\mu_2 + \lambda_2)\|\operatorname{div} v^n\|^2_{L^2(\Omega_2)}\right].\end{aligned} \tag{3.18}$$

Multiplying (3.15) by $b_i(t)$, summing up over i, and integrating over Ω_2 we obtain

$$\frac{d}{dt}\mathcal{E}_2^n(t) + \kappa_1 \left\|\nabla \widetilde{\theta}^n\right\|^2_{L^2(\Omega_1)} + \kappa_2 \|\nabla \theta^n\|^2_{L^2(\Omega_2)} - m_1 \int_{\Omega_1} \widetilde{\theta}^n \operatorname{div} u_t^n d\Omega_1 - m_2 \int_{\Omega_2} \theta^n \operatorname{div} v_t^n d\Omega_2 = 0, \tag{3.19}$$

where

$$\mathcal{E}_2^n(t) = \frac{1}{2}\tau_1 \left\|\widetilde{\theta}^n\right\|^2_{L^2(\Omega_1)} + \frac{1}{2}\ \tau_2 \|\theta^n\|^2_{L^2(\Omega_2)}. \tag{3.20}$$

Adding (3.17) with (3.19) we obtain

$$\frac{d}{dt}\mathcal{E}^n(t) + \kappa_1 \left\|\nabla \widetilde{\theta}^n\right\|^2_{L^2(\Omega_1)} + \kappa_2\ \|\nabla \theta^n\|^2_{L^2(\Omega_2)} = 0, \tag{3.21}$$

where

$$\begin{aligned}\mathcal{E}^n(t) &= \mathcal{E}_1^n(u^n, t) + \mathcal{E}_2^n(v^n, t) \\ &= \frac{1}{2}\ \left[\rho_1 \|u_t^n\|^2_{L^2(\Omega_1)} + \rho_2 \|v_t^n\|^2_{L^2(\Omega_2)} + \tau_1 \left\|\widetilde{\theta}^n\right\|^2_{L^2(\Omega_1)} + \tau_2 \|\theta^n\|^2_{L^2(\Omega_2)}\right] \\ &\quad + \frac{1}{2}\left[\mu_1 \|\nabla u^n\|^2_{L^2(\Omega_1)} + \mu_2 \|\nabla v^n\|^2_{L^2(\Omega_2)} + (\mu_1 + \lambda_1)\|\operatorname{div} u^n\|^2_{L^2(\Omega_1)} \right. \\ &\qquad \left. + (\mu_2 + \lambda_2)\|\operatorname{div} v^n\|^2_{L^2(\Omega_2)}\right].\end{aligned} \tag{3.22}$$

Integrating over $(0,t)$, $t \in (0,T)$, we have that

$$\mathcal{E}^n(t) + \kappa_1 \int_0^t \left\| \nabla \tilde{\theta}^n \right\|^2_{L^2(\Omega_1)} dt + \kappa_2 \int_0^t \|\nabla \theta^n\|^2_{L^2(\Omega_2)} dt = \mathcal{E}^n(0). \tag{3.23}$$

Thus,

$$\begin{gathered}\left(u^n, u_t^n, \tilde{\theta}^n\right) \text{ is bounded in } L^\infty\left([0,T] : H^1(\Omega_1)\right) \times L^\infty\left([0,T] : L^2(\Omega_1)\right) \times L^\infty\left([0,T] : L^2(\Omega_1)\right), \\ \left(v^n, v_t^n, \theta^n\right) \text{ is bounded in } L^\infty\left([0,T] : H^1(\Omega_2)\right) \times L^\infty\left([0,T] : L^2(\Omega_2)\right) \times L^\infty\left([0,T] : L^2(\Omega_2)\right).\end{gathered} \tag{3.24}$$

Hence,

$$\begin{aligned} u^n &\rightharpoonup u \text{ weakly}^* \quad \text{in } L^\infty\left([0,T] : H^1(\Omega_1)\right), \\ v^n &\rightharpoonup v \text{ weakly}^* \quad \text{in } L^\infty\left([0,T] : H^1(\Omega_2)\right), \\ u_t^n &\rightharpoonup u_t \text{ weakly}^* \quad \text{in } L^\infty\left([0,T] : L^2(\Omega_1)\right), \\ v_t^n &\rightharpoonup v_t \text{ weakly}^* \quad \text{in } L^\infty\left([0,T] : L^2(\Omega_2)\right), \\ \tilde{\theta}^n &\rightharpoonup \tilde{\theta} \text{ weakly}^* \quad \text{in } L^\infty\left([0,T] : L^2(\Omega_1)\right), \\ \theta^n &\rightharpoonup \theta \text{ weakly}^* \quad \text{in } L^\infty\left([0,T] : L^2(\Omega_2)\right). \end{aligned} \tag{3.25}$$

In particular,

$$\begin{aligned} u^n &\longrightarrow u \text{ strongly} \quad \text{in } L^2\left([0,T] : L^2(\Omega_1)\right), \\ v^n &\longrightarrow v \text{ strongly} \quad \text{in } L^2\left([0,T] : L^2(\Omega_2)\right), \end{aligned} \tag{3.26}$$

and it follows that

$$\begin{aligned} u^n &\longrightarrow u \quad \text{a.e. in } \Omega_1, \\ v^n &\longrightarrow v \quad \text{a.e. in } \Omega_2. \end{aligned} \tag{3.27}$$

The system (1.1)–(1.4) is a linear system, and hence the rest of the proof of the existence of weak solution is a standard matter.

The uniqueness follows using the elliptic regularity for the elliptic transmission problem (see [11]).We suppose that there exist two solutions $(u^1, v^1, \tilde{\theta}^1, \theta^1)$, $(u^2, v^2, \tilde{\theta}^2, \theta^2)$, and we denote

$$U = u^1 - u^2, \qquad V = v^1 - v^2, \qquad \tilde{\Theta} = \tilde{\theta}^1 - \tilde{\theta}^2, \qquad \Theta = \theta^1 - \theta^2. \tag{3.28}$$

Taking

$$u=\int_0^t U\,d\tau,\qquad v=\int_0^t V\,d\tau,\qquad \theta=\int_0^t \widetilde{\Theta}\,d\tau,\qquad \theta=\int_0^t \Theta\,d\tau \tag{3.29}$$

we can see that $(u,v,\widetilde{\theta},\theta)$ satisfies (1.1)–(1.4). Since $(u^1,v^1,\widetilde{\theta}^1,\theta^1)$, $(u^2,v^2,\widetilde{\theta}^2,\theta^2)$ are weak solutions of the system we have that $(u,v,\widetilde{\theta},\theta)$ satisfies

$$\begin{gathered}
u\in L^\infty\left(0,T:H^1_{\Gamma_1}\right),\qquad u_t\in L^\infty\left(0,T:H^1_{\Gamma_1}\right),\qquad u_{tt}\in L^2\left(0,T:L^2(\Omega_1)\right),\\
v\in L^\infty\left(0,T:H^1(\Omega_2)\right),\qquad v_t\in L^\infty\left(0,T:H^1(\Omega_2)\right),\qquad v_{tt}\in L^2\left(0,T:L^2(\Omega_2)\right),\\
\widetilde{\theta}\in L^2\left(0,\ T:H^1_{\Gamma_1}\right),\quad \widetilde{\theta}_t\in L^2\left(0,T:H^1_{\Gamma_1}\right)\Longrightarrow\widetilde{\theta}\in L^2\left(0,T:H^1_{\Gamma_1}\right),\\
\theta\in L^2\left(0,T:H^1(\Omega_2)\right),\quad \theta_t\in L^2\left(0,T:H^1(\Omega_2)\right)\Longrightarrow\theta\in L^2\left(0,T:H^1(\Omega_2)\right).
\end{gathered}\tag{3.30}$$

Using the elliptic regularity for the elliptic transmission problem we conclude that

$$\begin{gathered}
u\in L^\infty\left(0,T:H^1_{\Gamma_1}\cap H^2(\Omega_1)\right),\qquad v\in L^\infty\left(0,T:H^1(\Omega_2)\cap H^2(\Omega_2)\right),\\
\widetilde{\theta}\in L^\infty\left(0,T:H^1_{\Gamma_1}\cap H^2(\Omega_1)\right),\qquad \theta\in L^\infty\left(0,T:H^2(\Omega_2)\right).
\end{gathered}\tag{3.31}$$

Thus $(u,v,\widetilde{\theta},\theta)$ satisfies (1.1)–(1.4) in the strong sense. Multiplying (1.1) by u_t, (1.2) by v_t, (1.3) by $\widetilde{\theta}$, and (1.4) by θ and performing similar calculations as above we obtain $\mathcal{E}(t)=0$, where

$$\begin{aligned}
\mathcal{E}(t)={}&\frac{1}{2}\left[\rho_1\|u_t\|^2_{L^2(\Omega_1)}+\mu_1\|\nabla u\|^2_{L^2(\Omega_1)}+(\mu_1+\lambda_1)\|\operatorname{div}u\|^2_{L^2(\Omega_1)}+\tau_1\left\|\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}\right]\\
&+\frac{1}{2}\left[\rho_2\|v_t\|^2_{L^2(\Omega_2)}+\mu_2\|\nabla v\|^2_{L^2(\Omega_2)}+(\mu_2+\lambda_2)\|\operatorname{div}v\|^2_{L^2(\Omega_2)}+\tau_2\|\theta\|^2_{L^2(\Omega_2)}\right],
\end{aligned}\tag{3.32}$$

which implies that $u^1=u^2$, $\widetilde{\theta}^1=\widetilde{\theta}^2$, $v^1=v^2$, and $\theta^1=\theta^2$. The uniqueness follows.

To obtain more regularity, we differentiate the approximate system (1.1)–(1.4); then multiplying the resulting system by $a_i''(t)$ and $b_i'(t)$ and performing similar calculations as in (3.23) we have that

$$\widetilde{\mathcal{E}}^n(t)\le\widetilde{\mathcal{E}}^n(t)+F=\widetilde{\mathcal{E}}^n(0)\Longrightarrow\widetilde{\mathcal{E}}^n(t)\le C\widetilde{\mathcal{E}}^n(0),\tag{3.33}$$

where

$$\begin{aligned}\mathcal{E}^n(t) &= \frac{1}{2}\left[\rho_1\|u^n_{tt}\|^2_{L^2(\Omega_1)} + \rho_2\|v^n_{tt}\|^2_{L^2(\Omega_2)} + \tau_1\left\|\widetilde{\theta}^n_t\right\|^2_{L^2(\Omega_1)} + \tau_2\|\theta^n_t\|^2_{L^2(\Omega_2)}\right] \\ &+\frac{1}{2}\left[\mu_1\|\nabla u^n_t\|^2_{L^2(\Omega_1)} + \mu_2\|\nabla v^n_t\|^2_{L^2(\Omega_2)} + (\mu_1+\lambda_1)\|\operatorname{div} u^n_t\|^2_{L^2(\Omega_1)}\right. \\ &\quad \left. +(\mu_2+\lambda_2)\|\operatorname{div} v^n_t\|^2_{L^2(\Omega_2)}\right], \\ F &= \kappa_1\int_0^t\left\|\nabla\widetilde{\theta}^n_t\right\|^2_{L^2(\Omega_1)}d\tau + \kappa_2\int_0^t\|\nabla\theta^n_\tau\|^2_{L^2(\Omega_2)}d\tau.\end{aligned} \tag{3.34}$$

Therefore, we find that

$$\begin{aligned} &u^n_{tt}, \widetilde{\theta}^n_t \text{ are bounded} \quad \text{in } L^\infty\left(0,T : L^2(\Omega_1)\right), \\ &v^n_{tt}, \theta^n_t \text{ are bounded} \quad \text{in } L^\infty\left(0,T : L^2(\Omega_2)\right), \\ &u^n_t \text{ is bounded} \quad \text{in } L^\infty\left(0,T : H^1(\Omega_1)\right), \\ &v^n_t \text{ is bounded} \quad \text{in } L^\infty\left(0,T : H^1(\Omega_2)\right), \\ &\widetilde{\theta}^n_t \text{ is bounded} \quad \text{in } L^\infty\left(0,T : H^1(\Omega_1)\right), \\ &\theta^n_t \text{ is bounded} \quad \text{in } L^\infty\left(0,T : H^1(\Omega_2)\right). \end{aligned} \tag{3.35}$$

Finally, our conclusion will follow by using the regularity result for the elliptic transmission problem (see [11]). □

Remark 3.3. To obtain higher regularity we introduce the following definition.

Definition 3.4. One will say that the initial data $(u_0, v_0, \widetilde{\theta}_0, \theta_0)$ is *k-regular* $(k \geq 2)$ if

$$\begin{aligned} &u_j \in H^{k-j}(\Omega_1) \cap H^1_{\Gamma_1}, \quad j = 0, \dots, k-1, \quad u_k \in L^2(\Omega_1), \\ &\widetilde{\theta}_j \in H^{k-j}(\Omega_1) \cap H^1_{\Gamma_1}, \quad j = 0, \dots, k-1, \quad \widetilde{\theta}_k \in L^2(\Omega_2), \end{aligned} \tag{3.36}$$

where the values of u_j and $\widetilde{\theta}_j$ are given by

$$\begin{aligned} &\rho_1 u_{j+2} - A_1 u_j - m_1\nabla\widetilde{\theta}_j = 0 \quad \text{in } \Omega_1 \times [0,\infty), \\ &\rho_2 v_{j+2} - A_2 v_j - m_2\nabla\theta_j = 0 \quad \text{in } \Omega_2 \times [0,\infty), \\ &\tau_1\widetilde{\theta}_{j+1} - \kappa_1\Delta\widetilde{\theta}_j - m_1 \operatorname{div} u_{j+1} = 0 \quad \text{in } \Omega_1 \times [0,\infty), \\ &\tau_2\theta_{j+1} - \kappa_2\Delta\theta_j - m_2 \operatorname{div} v_{j+1} = 0 \quad \text{in } \Omega_2 \times [0,\infty), \end{aligned} \tag{3.37}$$

verifying the boundary conditions

$$u_j = 0, \quad \widetilde{\theta}_j = 0 \quad \text{on } \Gamma_1,$$
$$u_j = v_j, \quad \widetilde{\theta}_j = \theta_j \quad \text{on } \Gamma_2 \tag{3.38}$$

and the transmission conditions

$$\mu_1 \nabla u + (\mu_1 + \lambda_1) \operatorname{div} u + m_1 \widetilde{\theta} = \mu_2 \nabla v + (\mu_2 + \lambda_2) \operatorname{div} v + m_2 \theta \quad \text{on } \Gamma_2,$$
$$\frac{\partial \theta}{\partial \nu_{\Gamma_2}} = 0 \quad \text{on } \Gamma_2 \tag{3.39}$$

for $j = 0, \ldots, k-1$. Using the above notation we say that if the initial data is *k-regular,* then we have that the solution satisfies

$$u, \widetilde{\theta} \in \bigcap_{j=0}^{k} W^{j,\infty}\left(0, T : H^{k-j}(\Omega_1) \cap H^1_{\Gamma_1}\right),$$
$$v, \theta \in \bigcap_{j=0}^{k} W^{j,\infty}\left(0, T : H^{k-j}(\Omega_2)\right). \tag{3.40}$$

Using the same arguments as in Theorem 3.2, the result follows.

4. Exponential Stability

In this section we prove the exponential stability. The great difficulty here is to deal with the boundary terms in the interface of the material. This difficulty is solved using an observability result of the elastic wave equations together with the fact that the solution is radially symmetric.

Lemma 4.1. *Let one suppose that the initial data* $(u_0, v_0, \widetilde{\theta}_0, \theta_0)$ *is 3-regular; then the corresponding solution of the system* (1.1)–(1.13) *satisfies*

$$\frac{d\mathcal{E}_1(t)}{dt} = -\kappa_1 \left\| \nabla \widetilde{\theta} \right\|^2_{L^2(\Omega_1)} - \kappa_2 \| \nabla \theta \|^2_{L^2(\Omega_2)}, \tag{4.1}$$

$$\frac{d\mathcal{E}_2(t)}{dt} = -\kappa_1 \left\| \nabla \widetilde{\theta}_t \right\|^2_{L^2(\Omega_1)} - \kappa_2 \| \nabla \theta_t \|^2_{L^2(\Omega_2)}, \tag{4.2}$$

where $\mathcal{E}_1(t) = \mathcal{E}_1(u, v, \widetilde{\theta}, \theta, t) = \mathcal{E}_1^1(t) + \mathcal{E}_1^2(t)$ *with*

$$\mathcal{E}_1^1(t) = \frac{1}{2}\left[\rho_1 \|u_t\|^2_{L^2(\Omega_1)} + \tau_1 \left\|\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} + \mu_1 \|\nabla u\|^2_{L^2(\Omega_1)} + (\mu_1 + \lambda_1) \|\operatorname{div} u\|^2_{L^2(\Omega_1)}\right],$$
$$\mathcal{E}_1^2(t) = \frac{1}{2}\left[\rho_2 \|v_t\|^2_{L^2(\Omega_2)} + \tau_2 \|\theta\|^2_{L^2(\Omega_2)} + \mu_2 \|\nabla v\|^2_{L^2(\Omega_2)} + (\mu_2 + \lambda_2) \|\operatorname{div} v\|^2_{L^2(\Omega_2)}\right], \tag{4.3}$$

and $\mathcal{E}_2(t) = \mathcal{E}_1(u_t, v_t, \widetilde{\theta}_t, \theta_t, t)$.

Proof. Multiplying (1.1) by u_t, integrating in Ω_1, and using (2.16) we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\rho_1\|u_t\|^2_{L^2(\Omega_1)}+\mu_1\|\nabla u\|^2_{L^2(\Omega_1)}+(\mu_1+\lambda_1)\|\operatorname{div} u\|^2_{L^2(\Omega_1)}\right]\\&\quad+\int_{\Gamma_2}\left[\mu_1\nabla u+(\mu_1+\lambda_1)\operatorname{div} u+m_1\widetilde{\theta}\right]u_t\cdot\nu_{\Gamma_2}d\Gamma_2+m_1\int_{\Omega_1}\widetilde{\theta}\operatorname{div} u_t d\Omega_1=0.\end{aligned}\tag{4.4}$$

Multiplying (1.2) by v_t, integrating in Ω_2, and using (2.17) we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\rho_2\|v_t\|^2_{L^2(\Omega_2)}+\mu_1\|\nabla v\|^2_{L^2(\Omega_2)}+(\mu_2+\lambda_2)\|\operatorname{div} v\|^2_{L^2(\Omega_2)}\right]\\&\quad-\int_{\Gamma_2}\left[\mu_2\nabla v+(\mu_2+\lambda_2)\operatorname{div} v+m_1\theta\right]u_t\cdot\nu_{\Gamma_2}d\Gamma_2+m_2\int_{\Omega_1}\theta\operatorname{div} v_t d\Omega_2=0.\end{aligned}\tag{4.5}$$

Multiplying (1.3) by θ, integrating in Ω_1, and using (2.11) we have that

$$\frac{\tau_1}{2}\frac{d}{dt}\left\|\widetilde{\theta}\right\|^2_{L^2(\Omega_2)}-m_2\int_{\Omega_1}\widetilde{\theta}\operatorname{div} v_t d\Omega_2+\kappa_2\int_{\Gamma_2}\widetilde{\theta}\nabla\widetilde{\theta}\cdot\nu_{\Gamma_2}d\Gamma_2+\kappa_2\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_2)}=0,\tag{4.6}$$

Multiplying (1.4) by θ, integrating in Ω_2, using (2.11), and performing similar calculations as above we have that

$$\frac{\tau_2}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega_1)}-m_1\int_{\Omega_1}\theta\operatorname{div} u_t d\Omega_1+\kappa_1\int_{\Gamma_2}\theta\nabla\theta\cdot\nu_{\Gamma_2}d\Gamma_2+\kappa_1\|\nabla\theta\|^2_{L^2(\Omega_1)}=0.\tag{4.7}$$

Adding up (4.4), (4.5), (4.6), and (4.7) and using (1.12) and (1.13) we obtain

$$\frac{d\mathcal{E}_1(t)}{dt}+\kappa_1\int_{\Omega_1}\left|\nabla\widetilde{\theta}\right|^2 d\Omega_1+\kappa_2\int_{\Omega_2}|\nabla\theta|^2 d\Omega_2=0,\tag{4.8}$$

where

$$\begin{aligned}\mathcal{E}_1(t)=&\frac{1}{2}\left[\rho_1\|u_t\|^2_{L^2(\Omega_1)}+\rho_2\|v_t\|^2_{L^2(\Omega_2)}+\tau_1\left\|\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}+\tau_2\|\theta\|^2_{L^2(\Omega_2)}\right]\\&+\frac{1}{2}\left[\mu_1\|\nabla u\|^2_{L^2(\Omega_1)}+\mu_2\|\nabla v\|^2_{L^2(\Omega_2)}+(\mu_1+\lambda_1)\|\operatorname{div} u\|^2_{L^2(\Omega_1)}+(\mu_2+\lambda_2)\|\operatorname{div} v\|^2_{L^2(\Omega_2)}\right].\end{aligned}\tag{4.9}$$

Thus

$$\frac{d\mathcal{E}_1(t)}{dt}=-\kappa_1\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}-\kappa_2\|\nabla\theta\|^2_{L^2(\Omega_2)}.\tag{4.10}$$

In a similar way we obtain (4.2). □

Lemma 4.2. *Under the same hypotheses as in Lemma 4.1 one has that the corresponding solution of the system* (1.1)–(1.11) *satisfies*

$$\frac{d\mathcal{E}_3(t)}{dt} \leq -\frac{m_1}{3}\|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \frac{\kappa_1^2}{2m_1}\left\|\Delta\tilde{\theta}\right\|^2_{L^2(\Omega_1)} - \frac{\delta(2\mu_1+\lambda_1)}{4\rho_1}\|\Delta u\|^2_{L^2(\Omega_1)}$$
$$+ \left(\frac{\delta\tau_1^2}{\rho_1(2\mu_1+\lambda_1)} + \frac{\tau_1 m_1}{\rho_1} + \frac{m_1^2}{2\delta\rho_1(2\mu_1+\lambda_1)}\right)\left\|\nabla\tilde{\theta}\right\|^2_{L^2(\Omega_1)} \tag{4.11}$$
$$+ \tau_1\int_{\Gamma_2}\tilde{\theta}u_{tt}\cdot\nu_{\Gamma_2}d\Gamma_2 - \delta\int_{\Gamma_2}u_t\frac{\partial u_t}{\partial\nu_{\Gamma_2}}d\Gamma_2,$$

$$\frac{d\tilde{\mathcal{E}}_3(t)}{dt} \leq -\frac{m_2}{3}\|\operatorname{div} v_t\|^2_{L^2(\Omega_2)} + \frac{\kappa_2^2}{2m_2}\|\Delta\theta\|^2_{L^2(\Omega_2)}$$
$$- \frac{\tilde{\delta}(2\mu_2+\lambda_2)}{4\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)} + \left(\frac{\tilde{\delta}\tau_2^2}{\rho_2(2\mu_2+\lambda_2)} + \frac{\tau_2 m_2}{\rho_2} + \frac{m_2^2}{2\tilde{\delta}\rho_2(2\mu_2+\lambda_2)}\right)\|\nabla\theta\|^2_{L^2(\Omega_2)}$$
$$- \tau_2\int_{\Gamma_2}\theta v_{tt}\cdot\nu_{\Gamma_2}d\Gamma_2 - \tilde{\delta}\int_{\Gamma_2}v_t\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2, \tag{4.12}$$

where δ, $\tilde{\delta}$ are positive constants and

$$\mathcal{E}_3(t) = -\int_{\Omega_1}\left(\tau_1\tilde{\theta}\operatorname{div} u_t + \delta u_t\cdot\Delta u\right)d\Omega_1,$$
$$\tilde{\mathcal{E}}_3(t) = -\int_{\Omega_2}(\tau_2\theta\operatorname{div} v_t + \delta v_t\cdot\Delta v)d\Omega_2. \tag{4.13}$$

Proof. Multiplying (1.1) by $-\Delta u$, integrating in Ω_1, and using (1.10) we have that

$$-\rho_1\int_{\Omega_1}u_{tt}\cdot\Delta u\,d\Omega_1 + (2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta u|^2 d\Omega_1 + m_1\int_{\Omega_1}\nabla\tilde{\theta}\cdot\Delta u\,d\Omega_1 = 0. \tag{4.14}$$

Then

$$-\rho_1\frac{d}{dt}\int_{\Omega_1}u_t\cdot\Delta u\,d\Omega_1 + \rho_1\int_{\Omega_1}u_t\cdot\Delta u_t d\Omega_1 + (2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta u|^2 d\Omega_1$$
$$+ m_1\int_{\Omega_1}\nabla\tilde{\theta}\cdot\Delta u\,d\Omega_1 = 0. \tag{4.15}$$

Hence

$$-\rho_1\frac{d}{dt}\int_{\Omega_1} u_t\cdot\Delta u\, d\Omega_1+\rho_1\int_{\Omega_1}\nabla\cdot(u_t\nabla u_t)d\Omega_1-\rho_1\int_{\Omega_1}|\nabla u_t|^2 d\Omega_1 \\ +(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta u|^2 d\Omega_1+m_1\int_{\Omega_1}\nabla\widetilde{\theta}\cdot\Delta u\, d\Omega_1=0. \tag{4.16}$$

Thus

$$-\rho_1\frac{d}{dt}\int_{\Omega_1} u_t\cdot\Delta u\, d\Omega_1+(2\mu_1+\lambda_1)\|\Delta u\|^2_{L^2(\Omega_1)} \\ -\rho_1\|\nabla u_t\|^2_{L^2(\Omega_1)}-\rho_1\int_{\Gamma_2}u_t\nabla u_t\cdot\nu_{\Gamma_2}d\Gamma_2+m_1\int_{\Omega_1}\nabla\widetilde{\theta}\cdot\Delta u\, d\Omega_1=0. \tag{4.17}$$

Hence

$$-\frac{d}{dt}\int_{\Omega_1} u_t\cdot\Delta u\, d\Omega_1=-\frac{(2\mu_1+\lambda_1)}{\rho_1}\|\Delta u\|^2_{L^2(\Omega_1)}+\|\nabla u_t\|^2_{L^2(\Omega_1)} \\ +\int_{\Gamma_2}u_t\,\frac{\partial u_t}{\nu_{\Gamma_2}}d\Gamma_2-\frac{m_1}{\rho_1}\int_{\Omega_1}\nabla\widetilde{\theta}\cdot\Delta u\, d\Omega_1. \tag{4.18}$$

Therefore

$$-\frac{d}{dt}\int_{\Omega_1} u_t\cdot\Delta u\, d\Omega_1\leq-\frac{(2\mu_1+\lambda_1)}{2\rho_1}\|\Delta u\|^2_{L^2(\Omega_1)}+\frac{m_1^2}{2\rho_1(2\mu_1+\lambda_1)}\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} \\ +\|\nabla u_t\|^2_{L^2(\Omega_1)}+\int_{\Gamma_2}u_t\frac{\partial u_t}{\partial\nu_{\Gamma_2}}d\Gamma_2. \tag{4.19}$$

Similarly, multiplying (1.2) by $-\Delta v$, integrating in Ω_2, and performing similar calculations as above we obtain

$$-\frac{d}{dt}\int_{\Omega_2} v_t\cdot\Delta v\, d\Omega_2\leq-\frac{(2\mu_2+\lambda_2)}{2\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)}+\frac{m_2^2}{2\rho_2(2\mu_2+\lambda_2)}\|\nabla\theta\|^2_{L^2(\Omega_2)} \\ +\|\nabla v_t\|^2_{L^2(\Omega_2)}-\int_{\Gamma_2}v_t\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2. \tag{4.20}$$

Multiplying (1.3) by $-\operatorname{div} u_t$ and integrating in Ω_1 we have that

$$-\tau_1\int_{\Omega_1}\widetilde{\theta}_t\operatorname{div} u_t d\Omega_1+\kappa_1\int_{\Omega_1}\Delta\widetilde{\theta}\operatorname{div} u_t d\Omega_1+m_1\int_{\Omega_1}|\operatorname{div} u_t|^2 d\Omega_1=0. \tag{4.21}$$

Hence

$$-\tau_1 \frac{d}{dt}\int_{\Omega_1} \widetilde{\theta}_t \operatorname{div} u_t d\Omega_1 + \tau_1 \int_{\Omega_1} \widetilde{\theta} \operatorname{div} u_{tt} d\Omega_1 + m_1 \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \kappa_1 \int_{\Omega_1} \Delta\widetilde{\theta} \operatorname{div} u_t d\Omega_1 = 0. \tag{4.22}$$

Then

$$\begin{aligned} &-\tau_1 \frac{d}{dt}\int_{\Omega_1} \widetilde{\theta} \operatorname{div} u_t d\Omega_1 + \tau_1 \int_{\Omega_1} \nabla \cdot \left(\widetilde{\theta} u_{tt}\right) d\Omega_1 - \tau_1 \int_{\Omega_1} \nabla\widetilde{\theta} \cdot u_{tt} d\Omega_1 \\ &\quad + m_1 \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \kappa_1 \int_{\Omega_1} \Delta\widetilde{\theta} \operatorname{div} u_t d\Omega_1 = 0. \end{aligned} \tag{4.23}$$

Using (1.10) and (2.9) and performing similar calculations as above we obtain

$$\begin{aligned} -\tau_1 \frac{d}{dt}\int_{\Omega_1} \widetilde{\theta} \operatorname{div} u_t d\Omega_1 = {}& -m_1 \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \tau_1 \int_{\Omega_1} \nabla\widetilde{\theta} \cdot u_{tt} d\Omega_1 \\ & + \tau_1 \int_{\Gamma_2} \widetilde{\theta} u_{tt} \cdot \nu_{\Gamma_2} d\Gamma_2 - \kappa_1 \int_{\Omega_1} \Delta\widetilde{\theta} \operatorname{div} u_t d\Omega_1. \end{aligned} \tag{4.24}$$

Replacing (1.1) in the above equation we obtain

$$\begin{aligned} -\tau_1 \frac{d}{dt}\int_{\Omega_1} \widetilde{\theta} \operatorname{div} u_t d\Omega_1 = {}& -m_1 \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \frac{\tau_1}{\rho_1}(2\mu_1 + \lambda_1) \int_{\Omega_1} \nabla\widetilde{\theta} \Delta u \, d\Omega_1 \\ & + \frac{\tau_1 m_1}{\rho_1} \int_{\Omega_1} \left|\nabla\widetilde{\theta}\right|^2 d\Omega_1 - \kappa_1 \int_{\Omega_1} \Delta\widetilde{\theta} \operatorname{div} u_t d\Omega_1 + \tau_1 \int_{\Gamma_2} \widetilde{\theta} u_{tt} \cdot \nu_{\Gamma_2} d\Gamma_2. \end{aligned} \tag{4.25}$$

On the other hand

$$\kappa_1 \int_{\Omega_1} \Delta\widetilde{\theta} \operatorname{div} u_t d\Omega_1 \leq \frac{\kappa_1^2}{2m_1} \left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} + \frac{m_1}{2} \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)}. \tag{4.26}$$

Therefore

$$\begin{aligned} -\tau_1 \frac{d}{dt}\int_{\Omega_1} \widetilde{\theta} \operatorname{div} u_t d\Omega_1 \leq {}& -\frac{m_1}{2} \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \frac{\tau_1}{\rho_1} m_1 \left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} + \frac{\kappa_1^2}{2m_1} \left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} \\ & + \frac{\tau_1}{\rho_1}(2\mu_1 + \lambda_1) \int_{\Omega_1} \nabla\widetilde{\theta} \Delta u \, d\Omega_1 + \tau_1 \int_{\Gamma_2} \widetilde{\theta} u_{tt} \cdot \nu_{\Gamma_2} d\Gamma_2. \end{aligned} \tag{4.27}$$

Multiplying (1.4) by $-\operatorname{div} v_t$, integrating in Ω_2, and performing similar calculations as above we obtain

$$
\begin{aligned}
-\tau_2 \frac{d}{dt}\int_{\Omega_2} \theta \operatorname{div} v_t d\Omega_2 \leq{}& -\frac{m_2}{2}\|\operatorname{div} v_t\|^2_{L^2(\Omega_2)} + \frac{\tau_2}{\rho_2} m_2 \|\nabla\theta\|^2_{L^2(\Omega_2)} + \frac{\kappa_2^2}{2m_2}\ \|\Delta\theta\|^2_{L^2(\Omega_2)} \\
&+ \frac{\tau_2}{\rho_2}(2\mu_2+\lambda_2)\int_{\Omega_2} \nabla\theta\Delta v\, d\Omega_2 - \tau_2 \int_{\Gamma_2} \theta v_{tt}\cdot \nu_{\Gamma_2} d\Gamma_2.
\end{aligned}
\tag{4.28}
$$

Adding (4.19) with (4.27) we have that

$$
\begin{aligned}
\frac{d\mathcal{E}_3(t)}{dt} \leq{}& -\frac{m_1}{2}\|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \tau_1 \int_{\Gamma_2} \tilde{\theta} u_{tt}\cdot \nu_{\Gamma_2} d\Gamma_2 + \int_{\Gamma_2} u_t \frac{\partial u_t}{\partial \nu_{\Gamma_2}} d\Gamma_2 \\
&- \frac{(2\mu_1+\lambda_1)}{2\ \rho_1}\|\Delta u\|^2_{L^2(\Omega_1)} + \|\nabla u_t\|^2_{L^2(\Omega_1)} + \frac{\kappa_1^2}{2m_1}\left\|\Delta\tilde{\theta}\right\|^2_{L^2(\Omega_1)} \\
&+ \frac{\tau_1}{\rho_1}(2\mu_1+\lambda_1)\int_{\Omega_1} \nabla\tilde{\theta}\Delta u\, d\Omega_1 + \left(\frac{\tau_1 m_1}{\rho_1} + \frac{m_1^2}{2\rho_1(2\mu_1+\lambda_1)}\right)\left\|\nabla\tilde{\theta}\right\|^2_{L^2(\Omega_1)}.
\end{aligned}
\tag{4.29}
$$

Adding (4.20) with (4.28) we have that

$$
\begin{aligned}
\frac{d\tilde{\mathcal{E}}_3(t)}{dt} \leq{}& -\frac{m_2}{2}\|\operatorname{div} v_t\|^2_{L^2(\Omega_2)} - \tau_2 \int_{\Gamma_2} \theta v_{tt}\cdot \nu_{\Gamma_2} d\Gamma_2 - \int_{\Gamma_2} v_t \frac{\partial v_t}{\partial \nu_{\Gamma_2}} d\Gamma_2 \\
&- \frac{(2\mu_2+\lambda_2)}{2\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)} + \|\nabla v_t\|^2_{L^2(\Omega_2)} + \frac{\kappa_2^2}{2\ m_2}\ \|\Delta\theta\|^2_{L^2(\Omega_2)} \\
&+ \frac{\tau_2}{\rho_2}(2\mu_2+\lambda_2)\int_{\Omega_2} \nabla\theta\Delta v\, d\Omega_2 + \left(\frac{\tau_2 m_2}{\rho_2} + \frac{m_2^2}{2\rho_2(2\mu_2+\lambda_2)}\right)\|\nabla\theta\|^2_{L^2(\Omega_2)}.
\end{aligned}
\tag{4.30}
$$

Moreover, by Lemma 2.2, there exist positive constants C_1,C_2 such that

$$
\|\nabla u_t\|_{L^2(\Omega_1)} \leq C_1 \|\operatorname{div} u_t\|^2_{L^2(\Omega_1)}, \qquad \|\nabla v_t\|_{L^2(\Omega_2)} \leq C_2 \|\operatorname{div} v_t\|_{L^2(\Omega_2)}.
\tag{4.31}
$$

Therefore we obtain

$$
\begin{aligned}
\frac{d\mathcal{E}_3(t)}{dt} \leq{}& -\frac{m_1}{3}\|\operatorname{div} u_t\|^2_{L^2(\Omega_1)} + \tau_1 \int_{\Gamma_2} \tilde{\theta} u_{tt}\cdot \nu_{\Gamma_2} d\Gamma_2 + \frac{\kappa_1^2}{2m_1}\ \left\|\Delta\tilde{\theta}\right\|^2_{L^2(\Omega_1)} \\
&- \frac{\delta(2\mu_1+\lambda_1)}{4\rho_1}\|\Delta u\|^2_{L^2(\Omega_1)} + \left(\frac{\delta\tau_1^2}{\rho_1(2\mu_1+\lambda_1)} + \frac{\tau_1 m_1}{\rho_1} + \frac{m_1^2}{2\delta\rho_1(2\mu_1+\lambda_1)}\right)\left\|\nabla\tilde{\theta}\right\|^2_{L^2(\Omega_1)}.
\end{aligned}
\tag{4.32}
$$

Similarly

$$\frac{d\widetilde{\mathcal{E}}_3(t)}{dt} \le -\frac{m_2}{3}\|\operatorname{div} v_t\|^2_{L^2(\Omega_2)} - \tau_2\int_{\Gamma_2}\theta v_{tt}\cdot \nu_{\Gamma_2} d\Gamma_2 + \frac{\kappa_2^2}{2m_2}\|\Delta\theta\|^2_{L^2(\Omega_2)}$$
$$-\frac{\delta(2\mu_2+\lambda_2)}{4\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)} + \left(\frac{\delta\tau_2^2}{\rho_2(2\mu_2+\lambda_2)} + \frac{\tau_2 m_2}{\rho_2} + \frac{m_2^2}{2\delta\rho_2(2\mu_2+\lambda_2)}\right)\|\nabla\theta\|^2_{L^2(\Omega_2)}. \tag{4.33}$$

The result follows. □

Lemma 4.3. *Under the same hypotheses of Lemma 4.1 one has that the corresponding solution of the system* (1.1)–(1.13) *satisfies*

$$\begin{aligned}\frac{d\mathcal{E}_4(t)}{dt} \le& -\frac{\kappa_1}{2}(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} - \frac{\kappa_2}{2}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}\\ &-\rho_1(2\mu_1+\lambda_1)\int_{\Gamma_2} u_{tt}\frac{\partial u_t}{\partial\nu_{\Gamma_2}}d\Gamma_2 + \rho_1(2\mu_2+\lambda_2)\int_{\Gamma_2} v_{tt}\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2\\ &+\frac{C}{\varepsilon^3}\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} + \frac{\widetilde{C}}{\varepsilon^3}\|\nabla\theta\|^2_{L^2(\Omega_2)} + 2\varepsilon\|\operatorname{div} v_t\|^2_{L^2(\Gamma_2)},\end{aligned} \tag{4.34}$$

with

$$\begin{aligned}\mathcal{E}_4(t) &= (2\mu_1+\lambda_1)\mathcal{E}_4^1(t) + \frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\mathcal{E}_4^2(t),\\ \mathcal{E}_4^1(t) &= \frac{1}{2}\left[\rho_1\|\nabla u_t\|^2_{L^2(\Omega_1)} + (2\mu_1+\lambda_1)\|\Delta u\|^2_{L^2(\Omega_1)} + \tau_1\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}\right],\\ \mathcal{E}_4^2(t) &= \frac{1}{2}\left[\rho_2\|\nabla v_t\|^2_{L^2(\Omega_2)} + (2\mu_2+\lambda_2)\|\Delta v\|^2_{L^2(\Omega_2)} + \tau_2\|\nabla\theta\|^2_{L^2(\Omega_2)}\right],\end{aligned} \tag{4.35}$$

where ε, $C = C(m_1,\mu_1,\lambda_1)$ and $\widetilde{C} = \widetilde{C}(m_2,\mu_2,\lambda_2)$ are positive constants.

Proof. Multiplying (1.1) by $-(2\mu_1+\lambda_1)\Delta u_t$, integrating in Ω_1, using (2.9), and performing straightforward calculations we have that

$$\begin{aligned}&\frac{1}{2}\rho_1(2\mu_1+\lambda_1)\frac{d}{dt}\|\nabla u_t\|^2_{L^2(\Omega_1)} + \rho_1(2\mu_1+\lambda_1)\int_{\Gamma_2} u_{tt}\nabla u_t\cdot\nu_{\Gamma_2}d\Gamma_2\\ &\quad+\frac{1}{2}(2\mu_1+\lambda_1)^2\frac{d}{dt}\|\Delta u\|^2_{L^2(\Omega_1)} + m_1(2\mu_1+\lambda_1)\int_{\Omega_1}\nabla\widetilde{\theta}\cdot\Delta u_t d\Omega_1 = 0.\end{aligned} \tag{4.36}$$

Using (1.10) we obtain

$$
\begin{aligned}
&\frac{1}{2}(2\mu_1+\lambda_1)\frac{d}{dt}\left[\rho_1\|\nabla u_t\|^2_{L^2(\Omega_1)}+(2\mu_1+\lambda_1)\|\Delta u\|^2_{L^2(\Omega_1)}\right]\\
&\quad=-\rho_1(2\mu_1+\lambda_1)\int_{\Gamma_2}u_{tt}\frac{\partial u_t}{\partial\nu_{\Gamma_2}}d\Gamma_2-m_1(2\mu_1+\lambda_1)\int_{\Omega_1}\nabla\widetilde{\theta}\cdot\Delta u_t d\Omega_1.
\end{aligned}
\tag{4.37}
$$

Multiplying (1.2) by $-(\rho_1/\rho_2)(2\mu_2+\lambda_2)\Delta v_t$, integrating in Ω_2, and performing similar calculations as above we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\frac{d}{dt}\left[\rho_2\|\nabla v_t\|^2_{L^2(\Omega_2)}+(2\mu_2+\lambda_2)\|\Delta v\|^2_{L^2(\Omega_2)}\right]\\
&\quad=\rho_1(2\mu_2+\lambda_2)\int_{\Gamma_2}v_{tt}\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2-m_2\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\int_{\Omega_2}\nabla\theta\cdot\Delta v_t d\Omega_2.
\end{aligned}
\tag{4.38}
$$

Multiplying (1.3) by $-(2\mu_1+\lambda_1)\Delta\widetilde{\theta}$ and integrating in Ω_1, we have that

$$
\begin{aligned}
&-\tau_1(2\mu_1+\lambda_1)\int_{\Omega_1}\widetilde{\theta}_t\Delta\widetilde{\theta}\,d\Omega_1+\kappa_1(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}\\
&\quad+m_1(2\mu_1+\lambda_1)\int_{\Omega_1}\operatorname{div}u_t\Delta\widetilde{\theta}\,d\Omega_1=0.
\end{aligned}
\tag{4.39}
$$

Performing similar calculations as above we obtain

$$
\begin{aligned}
\frac{1}{2}\tau_1(2\mu_1+\lambda_1)\frac{d}{dt}\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}&=-\kappa_1(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}-m_1(2\mu_1+\lambda_1)\int_{\Omega_1}\operatorname{div}u_t\Delta\widetilde{\theta}\,d\Omega_1\\
&\quad-\tau_1(2\mu_1+\lambda_1)\int_{\Gamma_2}\widetilde{\theta}\frac{\partial\widetilde{\theta}}{\partial\nu_{\Gamma_2}}d\Gamma_2.
\end{aligned}
\tag{4.40}
$$

Multiplying (1.4) by $-(\rho_1/\rho_2)(2\mu_2+\lambda_2)\Delta\theta$, integrating in Ω_1, and performing similar calculation as above we obtain

$$
\begin{aligned}
\frac{1}{2}\tau_2\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\frac{d}{dt}\|\nabla\theta\|^2_{L^2(\Omega_2)}&=-\kappa_2\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}\\
&\quad-m_2\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\int_{\Omega_2}\operatorname{div}v_t\Delta\theta\,d\Omega_2\\
&\quad+\tau_2(2\mu_2+\lambda_2)\int_{\Gamma_2}\theta\frac{\partial\theta}{\partial\nu_{\Gamma_2}}d\Gamma_2.
\end{aligned}
\tag{4.41}
$$

Adding (4.37), (4.38), (4.40), and (4.41), using (1.13), and performing straightforward calculations we obtain

$$\begin{aligned}\frac{d\mathcal{E}_4(t)}{dt} &= -\kappa_1(2\mu_1+\lambda_1)\left\|\Delta\tilde{\theta}\right\|^2_{L^2(\Omega_1)} - \kappa_2\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\nabla\theta\|^2_{L^2(\Omega_2)}\\ &\quad -\rho_1(2\mu_1+\lambda_1)\int_{\Gamma_2} u_{tt}\frac{\partial u_t}{\partial\nu_{\Gamma_2}}d\Gamma_2 + \rho_1(2\mu_2+\lambda_2)\int_{\Gamma_2} v_{tt}\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2\\ &\quad + m_1(2\mu_1+\lambda_1)\int_{\Gamma_2}\frac{\partial\tilde{\theta}}{\partial\nu_{\Gamma_2}}\operatorname{div} u_t d\Gamma_2 - m_2(2\mu_2+\lambda_2)\int_{\Gamma_2}\frac{\partial\theta}{\partial\nu_{\Gamma_2}}\operatorname{div} v_t d\Gamma_2,\end{aligned} \tag{4.42}$$

with

$$\begin{aligned}\mathcal{E}_4(t) &= (2\mu_1+\lambda_1)\mathcal{E}_4^1(t) + \frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\mathcal{E}_4^2(t),\\ \mathcal{E}_4^1(t) &= \frac{1}{2}\left[\rho_1\|\nabla u_t\|^2_{L^2(\Omega_1)} + (2\mu_1+\lambda_1)\|\Delta u\|^2_{L^2(\Omega_1)} + \tau_1\left\|\nabla\tilde{\theta}\right\|^2_{L^2(\Omega_1)}\right],\\ \mathcal{E}_4^2(t) &= \frac{1}{2}\left[\rho_2\|\nabla v_t\|^2_{L^2(\Omega_2)} + (2\mu_2+\lambda_2)\|\Delta v\|^2_{L^2(\Omega_2)} + \tau_2\|\nabla\theta\|^2_{L^2(\Omega_2)}\right].\end{aligned} \tag{4.43}$$

Using the Cauchy inequality we have that

$$\int_{\Gamma_2}\frac{\partial\tilde{\theta}}{\partial\nu_{\Gamma_2}}\operatorname{div} u_t d\Gamma_2 \le \frac{1}{4\varepsilon}\int_{\Gamma_2}\left|\frac{\partial\tilde{\theta}}{\partial\nu_{\Gamma_2}}\right|^2 d\Gamma_2 + \varepsilon\int_{\Gamma_2}|\operatorname{div} u_t|^2 d\Gamma_2, \tag{4.44}$$

and, from trace and interpolation inequalities, we obtain

$$\begin{aligned}\int_{\Gamma_2}\frac{\partial\tilde{\theta}}{\partial\nu_{\Gamma_2}}\operatorname{div} u_t d\Gamma_2 &\le \frac{C_1}{4\varepsilon}\left[\int_{\Omega_1}\left|\nabla\tilde{\theta}\right|^2 d\Omega_1\right]^{1/2}\left[\int_{\Omega_1}\left|\Delta\tilde{\theta}\right|^2 d\Omega_1\right]^{1/2} + \varepsilon\int_{\Gamma_2}|\operatorname{div} u_t|^2 d\Gamma_2\\ &\le \frac{C}{\varepsilon^3}\int_{\Omega_1}\left|\nabla\tilde{\theta}\right|^2 d\Omega_1 + \varepsilon\int_{\Omega_1}\left|\Delta\tilde{\theta}\right|^2 d\Omega_1 + \varepsilon\int_{\Gamma_2}|\operatorname{div} u_t|^2 d\Gamma_2.\end{aligned} \tag{4.45}$$

Similarly

$$\int_{\Gamma_2}\frac{\partial\theta}{\partial\nu_{\Gamma_2}}\operatorname{div} v_t d\Gamma_2 \le \frac{C}{\varepsilon^3}\int_{\Omega_1}|\nabla\theta|^2 d\Omega_1 + \varepsilon\int_{\Omega_1}|\Delta\theta|^2 d\Omega_1 + \varepsilon\int_{\Gamma_2}|\operatorname{div} v_t|^2 d\Gamma_2. \tag{4.46}$$

Replacing in the above equation we obtain

$$\begin{aligned}\frac{d\mathcal{E}_4(t)}{dt} \leq{}& -\kappa_1(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} - \kappa_2\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}\\ &-\rho_1(2\mu_1+\lambda_1)\int_{\Gamma_2} u_{tt}\frac{\partial u_t}{\partial\nu_{\Gamma_2}}d\Gamma_2+\rho_1(2\mu_2+\lambda_2)\int_{\Gamma_2} v_{tt}\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2\\ &+\frac{C}{\varepsilon^3}\int_{\Omega_1}\left|\nabla\widetilde{\theta}\right|^2 d\Omega_1+\varepsilon\int_{\Omega_1}\left|\Delta\widetilde{\theta}\right|^2 d\Omega_1+\varepsilon\int_{\Gamma_2}|\operatorname{div} u_t|^2 d\Gamma_2\\ &+\frac{C}{\varepsilon^3}\int_{\Omega_1}|\nabla\theta|^2 d\Omega_1+\varepsilon\int_{\Omega_1}|\Delta\theta|^2 d\Omega_1+\varepsilon\int_{\Gamma_2}|\operatorname{div} v_t|^2 d\Gamma_2.\end{aligned} \tag{4.47}$$

The result follows.

We introduce the following integrals:

$$\begin{aligned}I_1&=\int_{\Omega_1}\rho_1 u_{tt}(q\cdot\nabla)u_t d\Omega_1,\\ I_2&=\int_{\Omega_1}\rho_1 u_{tt}(h\cdot\nabla)u_t d\Omega_1,\end{aligned} \tag{4.48}$$

where

$$\begin{aligned}q&\in\left[C^2\left(\overline{\Omega}_1\cup\overline{\Omega}_2\right)\right]^3, \qquad q(x)=\begin{cases}\nu & \text{if } x\in\Gamma_1,\\ 0 & \text{if } x\in\Omega_1\cup\Omega_2,\end{cases}\\ h&\in\left[C^2\left(\overline{\Omega}_1\cup\overline{\Omega}_2\right)\right]^3, \qquad h(x)=\begin{cases}0 & \text{if } x\in\Omega_1\setminus\Omega_3,\\ x & \text{if } x\in\Omega_2,\end{cases}\end{aligned} \tag{4.49}$$

with $\Omega_3=\Omega_2\cup[\cup_{x\in\Gamma_2}\mathbb{B}_\varepsilon(x)]$, where $\mathbb{B}_\varepsilon(x)$ is a ball with center x and radius ε. □

Lemma 4.4. *Under the same hypotheses as in Lemma 4.1 one has that the corresponding solution of the system* (1.1)–(1.13) *satisfies*

$$\frac{dI_1(t)}{dt}\leq -k_0\left\|\frac{\partial u_t}{\partial\nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)}+C_{k_0}\left(\|\nabla u_t\|^2_{L^2(\Omega_1)}+\|\Delta u\|^2_{L^2(\Omega_1)}+\left\|\nabla\widetilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right), \tag{4.50}$$

$$\begin{aligned}\frac{dI_2(t)}{dt}\leq{}& -\frac{r_0}{2}\left[(2\mu_1+\lambda_1)\left\|\frac{\partial u_t}{\partial\nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)}+\rho_1\|u_{tt}\|^2_{L^2(\Gamma_2)}\right]\\ &+\frac{(n-1)}{2}(2\mu_1+\lambda_1)\|u_t\|^2_{L^2(\Gamma_2)}+C\left(\|\nabla u_t\|^2_{L^2(\Omega_1)}+\|\Delta u\|^2_{L^2(\Omega_1)}+\left\|\nabla\widetilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right),\end{aligned} \tag{4.51}$$

where κ_0, C_{κ_0}, *and* C *are positive constants and* $r_0=|x_0|$, $x_0\in\Gamma_2$.

Proof. Using Lemma A.1, taking h as above, $\varphi = u_t$, $f = m_1 \nabla \widetilde{\theta}_t$, and $\Omega = \Omega_1$, we obtain

$$\begin{aligned}
\frac{dI_2(t)}{dt} \leq{}& (2\mu_1 + \lambda_1) \int_\Gamma \frac{\partial u_t}{\partial \nu_\Gamma} \left(\sum_{i=1}^n h_i \frac{\partial u_t}{\partial x_i} \right) d\Gamma + \frac{1}{2}\rho_1 \int_\Gamma |u_{tt}|^2 \left(\sum_{i=1}^n h_i \nu_{i_\Gamma} \right) d\Gamma \\
&- \frac{1}{2}(2\mu_1 + \lambda_1) \int_\Gamma |\nabla u_t|^2 \left(\sum_{i=1}^n h_i \nu_{i_\Gamma} \right) d\Gamma \\
&- \frac{1}{2} \int_{\Omega_1} \left(\rho_1 |u_{tt}|^2 - (2\mu_1 + \lambda_1) |\nabla u_t|^2 \right) \left(\sum_{1=1}^n \frac{\partial h_i}{\partial x_i} \right) d\Omega_1 \\
&- (2\mu_1 + \lambda_1) \int_{\Omega_1} \nabla u_t \left(\sum_{i=1}^n \nabla h_i \frac{\partial u_t}{\partial x_i} \right) d\Omega_1 + m_1 \int_{\Omega_1} \nabla \widetilde{\theta}_t \left(\sum_{i=1}^n h_i \frac{\partial u_t}{\partial x_i} \right) d\Omega_1.
\end{aligned} \tag{4.52}$$

Applying the hypothesis on h and since

$$\begin{gathered}
h = -r_0 \nu_{\Gamma_2}, \quad r_0 = |x|, \ \forall x \in \Gamma_2, \ r_0 = \operatorname{diam} \Omega_2, \\
h = 0, \quad \forall\ x \in \Gamma_1,
\end{gathered} \tag{4.53}$$

we have that

$$\begin{aligned}
\frac{dI_2(t)}{dt} \leq{}& -r_0 (2\mu_1 + \lambda_1) \int_{\Gamma_2} \left| \frac{\partial u_t}{\partial \nu_{\Gamma_2}} \right|^2 d\Gamma_2 - \frac{r_0}{2} \rho_1 \int_{\Gamma_2} |u_{tt}|^2 d\Gamma_2 \\
&+ \frac{r_0}{2} (2\mu_1 + \lambda_1) \int_{\Gamma_2} |\nabla u_t|^2 d\Gamma_2 \\
&- \frac{1}{2} \int_{\Omega_1} \left(\rho_1 |u_{tt}|^2 - (2\mu_1 + \lambda_1) |\nabla u_t|^2 \right) d\Omega_1 \\
&- (2\mu_1 + \lambda_1) \int_{\Omega_1} |\nabla u_t|^2 d\Omega_1 + m_1 \int_{\Omega_1} \nabla \widetilde{\theta}_t (h \cdot \nabla u_t) d\Omega_1.
\end{aligned} \tag{4.54}$$

Using (2.8) and the Cauchy-Schwartz inequality in the last term and performing straightforward calculations we obtain

$$\begin{aligned}
\frac{dI_2(t)}{dt} \leq{}& -\frac{r_0}{2} \int_{\Gamma_2} \left[(2\mu_1 + \lambda_1) \left| \frac{\partial u_t}{\partial \nu_{\Gamma_2}} \right|^2 + \rho_1 |u_{tt}|^2 \right] d\Gamma_2 \\
&+ \frac{(n-1)}{2} (2\mu_1 + \lambda_1) \int_{\Gamma_2} |u_t|^2 d\Gamma_2 + C \int_{\Omega_1} \left(|u_{tt}|^2 + |\nabla u_t|^2 + \left| \nabla \widetilde{\theta}_t \right|^2 \right) d\Omega_1.
\end{aligned} \tag{4.55}$$

Finally, considering (1.1) and applying the trace theorem we obtain

$$\|u_t\|_{L^2(\Gamma_2)} \leq \widetilde{C} \|\nabla u_t\|_{L^2(\Omega_1)}, \tag{4.56}$$

with $\widetilde{C} > 0$; there exists a positive constant C which proves (4.51). □

We now introduce the integrals

$$I_3(t) = \rho_2 \int_{\Omega_2} v_{tt}(x \cdot \nabla) v_t \, d\Omega_2, \qquad \Phi(t) = I_3(t) + \frac{(n-1)}{2}\rho_2 \int_{\Omega_2} v_t \cdot v_{tt} \, d\Omega_2. \tag{4.57}$$

Lemma 4.5. *With the same hypotheses as in Lemma 4.1, the following equality holds:*

$$\begin{aligned}
\frac{d\Phi(t)}{dt} &= \frac{r_0}{2}\left[(2\mu_2+\lambda_2)\left\|\frac{\partial v_t}{\partial \nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)} + \rho_2 \|v_{tt}\|^2_{L^2(\Gamma_2)}\right] \\
&\quad + \frac{(n-1)(2\mu_2+\lambda_2)}{2}\int_{\Gamma_2} v_t \frac{\partial v_t}{\partial \nu_{\Gamma_2}} d\Gamma_2 + \frac{(n-1)(2\mu_2+\lambda_2)}{2}\|v_t\|^2_{L^2(\Gamma_2)} \\
&\quad - \frac{1}{2}\left[\rho_2\|v_{tt}\|^2_{L^2(\Omega_2)} + (2\mu_2+\lambda_2)\|\nabla v_t\|^2_{L^2(\Omega_2)}\right] + \frac{(n-1)}{2} m_2 \int_{\Omega_2} v_t \cdot \nabla\theta_t \, d\Omega_2.
\end{aligned} \tag{4.58}$$

Proof. Differentiating (1.2) in the t-variable we have that

$$\rho_2 v_{ttt} = (2\mu_2+\lambda_2)\Delta v_t + m_2 \nabla\theta_t. \tag{4.59}$$

Multiplying the above equation by v_t and integrating in Ω_2 we obtain

$$\rho_2 \int_{\Omega_2} v_t \cdot v_{ttt} \, d\Omega_2 = (2\mu_2+\lambda_2)\int_{\Omega_2} v_t \cdot \Delta v_t \, d\Omega_2 + m_2 \int_{\Omega_2} v_t \cdot \nabla\theta_t \, d\Omega_2. \tag{4.60}$$

Hence

$$\begin{aligned}
\rho_2 \frac{d}{dt}\int_{\Omega_2} v_t \cdot v_{tt} \, d\Omega_2 &= \rho_2 \int_{\Omega_2} |v_{tt}|^2 d\Omega_2 + \rho_2 \int_{\Omega_2} v_t \cdot v_{ttt} \, d\Omega_2 \\
&= \rho_2 \int_{\Omega_2} |v_{tt}|^2 d\Omega_2 + (2\mu_2+\lambda_2)\int_{\Omega_2} v_t \cdot \Delta v_t \, d\Omega_2 \\
&\quad + m_2 \int_{\Omega_2} v_t \cdot \nabla\theta_t \, d\Omega_2 \\
&= \rho_2 \|v_{tt}\|^2_{L^2(\Omega_2)} - (2\mu_2+\lambda_2)\|\nabla v_t\|^2_{L^2(\Omega_2)} \\
&\quad + (2\mu_2+\lambda_2)\int_{\Gamma_2} v_t \frac{\partial v_t}{\partial \nu_{\Gamma_2}} d\Gamma_2 + m_2 \int_{\Omega_2} v_t \cdot \nabla\theta_t \, d\Omega_2.
\end{aligned} \tag{4.61}$$

On the other hand, using Lemma A.1 for $h = x$, $\varphi = v_t$, $f = 0$, and $\Omega = \Omega_2$ we obtain

$$\begin{aligned}\frac{dI_3(t)}{dt} &= \frac{r_0}{2}\left[(2\mu_2+\lambda_2)\left\|\frac{\partial v_t}{\partial \nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)} + \rho_2\|v_{tt}\|^2_{L^2(\Gamma_2)}\right] \\ &\quad - \frac{(n-1)(2\mu_2+\lambda_2)}{2}\|v_t\|^2_{L^2(\Gamma_2)} + \frac{(n-1)}{2}\left[(2\mu_2+\lambda_2)\|\nabla v_t\|^2_{L^2(\Omega_2)} - \rho_2\|v_{tt}\|^2_{L^2(\Omega_2)}\right] \\ &\quad - \frac{1}{2}\left[(2\mu_2+\lambda_2)\|\nabla v_t\|^2_{L^2(\Omega_2)} + \rho_2\|v_{tt}\|^2_{L^2(\Omega_2)}\right].\end{aligned} \tag{4.62}$$

Multiplying (4.61) by $(n-1)/2$ and adding with (4.62) we obtain

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= \frac{r_0}{2}\left[(2\mu_2+\lambda_2)\left\|\frac{\partial v_t}{\partial \nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)} + \rho_2\|v_{tt}\|^2_{L^2(\Gamma_2)}\right] \\ &\quad + \frac{(n-1)(2\mu_2+\lambda_2)}{2}\int_{\Gamma_2} v_t\frac{\partial v_t}{\partial \nu_{\Gamma_2}}d\Gamma_2 + \frac{(n-1)(2\mu_2+\lambda_2)}{2}\|v_t\|^2_{L^2(\Gamma_2)} \\ &\quad - \frac{1}{2}\left[\rho_2\|v_{tt}\|^2_{L^2(\Omega_2)} + (2\mu_2+\lambda_2)\|\nabla v_t\|^2_{L^2(\Omega_2)}\right] + \frac{(n-1)}{2}m_2\int_{\Omega_2} v_t\cdot\nabla\theta_t d\Omega_2.\end{aligned} \tag{4.63}$$

The result follows. □

We introduce the integral

$$\mathcal{M}(t) = \mathcal{E}_4(t) + \frac{m_1(2\mu_1+\lambda_1)}{2\kappa_1}\,\mathcal{E}_3(t) + \frac{m_2(2\mu_2+\lambda_2)}{2\kappa_2}\widetilde{\mathcal{E}}_3(t) + \delta_1 I_1(t) + \delta_2 I_2(t), \tag{4.64}$$

where δ_1 and δ_2 are positive constants.

Lemma 4.6. *Under the same hypotheses as in Lemma 4.1 one has that the corresponding solution of the system* (1.1)–(1.13) *satisfies*

$$\begin{aligned}\frac{d\mathcal{M}(t)}{dt} &\le -\frac{\kappa_1}{4}(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} - \frac{\kappa_2}{4}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)} \\ &\quad - \left[C_1 - \delta_1 C_{\kappa_0} - \delta_2 c - \frac{\delta C_p c}{2}\right]\|\nabla u_t\|^2_{L^2(\Omega_1)} - C_2\|\nabla v_t\|^2_{L^2(\Omega_2)} \\ &\quad - \frac{\delta m_1(2\mu_1+\lambda_1)^2}{32\rho_1\kappa_1}\|\Delta u\|^2_{L^2(\Omega_1)} - \frac{\delta m_2(2\mu_2+\lambda_2)^2}{32\rho_2\kappa_2}\frac{\rho_1}{\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)} \\ &\quad + \widetilde{C}_\varepsilon\left(\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)} + \left\|\nabla\widetilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right) + C_\varepsilon\left(\|\nabla\theta\|^2_{L^2(\Omega_2)} + \|\nabla\theta_t\|^2_{L^2(\Omega_2)}\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{r_0\rho_1}{4}\|v_{tt}\|^2_{L^2(\Gamma_2)}+\frac{\delta_2(2\mu_1+\lambda_1)}{2}\|v_t\|^2_{L^2(\Gamma_2)}\\
&+\left[\frac{\delta_2(2\mu_1+\lambda_1)}{2}-\frac{\delta c}{2}+C+\frac{\delta_1\kappa_0}{4}\right]\left\|\frac{\partial v_t}{\partial \nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)}.
\end{aligned}
\tag{4.65}
$$

Proof. From (4.11), (4.12), and (4.34), using the Cauchy-Schwartz inequality and performing straightforward calculations we have that

$$
\begin{aligned}
&\frac{d}{dt}\left[\mathcal{E}_4(t)+\frac{m_1(2\mu_1+\lambda_1)}{2\kappa_1}\mathcal{E}_3(t)+\frac{m_2(2\mu_2+\lambda_2)}{2\kappa_2}\widetilde{\mathcal{E}}_3(t)\right]\\
&\quad\le-\frac{\kappa_1}{4}(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}-\frac{\kappa_2}{4}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}\\
&\qquad-\frac{m_1^2(2\mu_1+\lambda_1)}{6\kappa_1}\|\operatorname{div} u_t\|^2_{L^2(\Omega_1)}-\frac{m_2^2(2\mu_2+\lambda_2)}{6\kappa_2}\frac{\rho_1}{\rho_2}\|\operatorname{div} v_t\|^2_{L^2(\Omega_2)}\\
&\qquad-\frac{\delta m_1(2\mu_1+\lambda_1)^2}{8\rho_1\kappa_1}\|\Delta u\|^2_{L^2(\Omega_1)}-\frac{\widetilde{\delta} m_2(2\mu_2+\lambda_2)^2}{8\rho_2\kappa_2}\frac{\rho_1}{\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)}\\
&\qquad+\widetilde{C}_\varepsilon\left(\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}+\left\|\nabla\widetilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right)+C_\varepsilon\left(\|\nabla\theta\|^2_{L^2(\Omega_2)}+\|\nabla\theta_t\|^2_{L^2(\Omega_2)}\right)\\
&\qquad+\varepsilon\|\operatorname{div} v_t\|^2_{L^2(\Gamma_2)}+\varepsilon\|v_{tt}\|^2_{L^2(\Gamma_2)}\\
&\qquad-\left[\frac{\delta m_1\tau_1(2\mu_1+\lambda_1)}{2\kappa_1}+\frac{\widetilde{\delta} m_2\tau_2(2\mu_2+\lambda_2)}{2\kappa_2}\frac{\rho_1}{\rho_2}\right]\int_{\Gamma_2}v_t\cdot\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2,
\end{aligned}
\tag{4.66}
$$

where C_ε, $\widetilde{C}_\varepsilon$, and ε are positive constants. By Lemma 2.2, there exist positive constants C_1 and C_2 such that

$$
\begin{aligned}
&-\frac{m_1^2(2\mu_1+\lambda_1)}{6\kappa_1}\|\operatorname{div} u_t\|^2_{L^2(\Omega_1)}\le-C_1\|\nabla u_t\|^2_{L^2(\Omega_1)},\\
&-\frac{m_2^2(2\mu_1+\lambda_1)}{6\kappa_2}\frac{\rho_1}{\rho_2}\|\operatorname{div} v_t\|^2_{L^2(\Omega_2)}\le-C_2\|\nabla v_t\|^2_{L^2(\Omega_2)}.
\end{aligned}
\tag{4.67}
$$

Then

$$
\begin{aligned}
&\frac{d}{dt}\left[\mathcal{E}_4(t)+\frac{m_1(2\mu_1+\lambda_1)}{2\kappa_1}\mathcal{E}_3(t)+\frac{m_2(2\mu_2+\lambda_2)}{2\kappa_2}\widetilde{\mathcal{E}}_3(t)\right]\\
&\quad\le-\frac{\kappa_1}{4}(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}-\frac{\kappa_2}{4}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}
\end{aligned}
$$

$$
\begin{aligned}
&- C_1\|\nabla u_t\|^2_{L^2(\Omega_1)} - C_2\|\nabla v_t\|^2_{L^2(\Omega_2)} \\
&- \frac{\delta m_1(2\mu_1+\lambda_1)^2}{8\rho_1\kappa_1}\|\Delta u\|^2_{L^2(\Omega_1)} - \frac{\tilde{\delta} m_2(2\mu_2+\lambda_2)^2}{8\rho_2\kappa_2}\frac{\rho_1}{\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)} \\
&+ \widetilde{C}_\varepsilon\left(\left\|\nabla\tilde{\theta}\right\|^2_{L^2(\Omega_1)} + \left\|\nabla\tilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right) + C_\varepsilon\left(\|\nabla\theta\|^2_{L^2(\Omega_2)} + \|\nabla\theta_t\|^2_{L^2(\Omega_2)}\right) \\
&+ \varepsilon\|\operatorname{div} v_t\|^2_{L^2(\Gamma_2)} + \varepsilon\|v_{tt}\|^2_{L^2(\Gamma_2)} \\
&- \left[\frac{\delta m_1\tau_1(2\mu_1+\lambda_1)}{2\kappa_1} + \frac{\tilde{\delta} m_2\tau_2(2\mu_2+\lambda_2)}{2\kappa_2}\frac{\rho_1}{\rho_2}\right]\int_{\Gamma_2} v_t\cdot\frac{\partial v_t}{\partial\nu_{\Gamma_2}}d\Gamma_2.
\end{aligned}
\tag{4.68}
$$

Hence, taking $\delta_2 C \le \delta m_1(2\mu_1+\lambda_1)^2/16\rho_1\kappa_1$, $\delta_2' C' \le (\tilde{\delta} m_2(2\mu_2+\lambda_2)^2/16\rho_2\kappa_2)(\rho_1/\rho_2)$, and $\varepsilon < (\delta r_0\rho_1)/4$ we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\mathcal{E}_4(t) + \frac{m_1(2\mu_1+\lambda_1)}{2\kappa_1}\mathcal{E}_3(t) + \frac{m_2(2\mu_2+\lambda_2)}{2\kappa_2}\widetilde{\mathcal{E}}_3(t) + \delta_2 I_2(t)\right] \\
&\quad\le -\frac{\kappa_1}{4}(2\mu_1+\lambda_1)\left\|\Delta\tilde{\theta}\right\|^2_{L^2(\Omega_1)} - \frac{\kappa_2}{4}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)} \\
&\qquad - \left[C_1 - \delta_2 C - \frac{\delta C_p C}{2}\right]\|\nabla u_t\|^2_{L^2(\Omega_1)} - C_2\|\nabla v_t\|^2_{L^2(\Omega_2)} \\
&\qquad - \frac{\delta m_1(2\mu_1+\lambda_1)^2}{16\rho_1\kappa_1}\|\Delta u\|^2_{L^2(\Omega_1)} - \frac{\delta m_2(2\mu_2+\lambda_2)^2}{16\rho_2\kappa_2}\frac{\rho_1}{\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)} \\
&\qquad + \widetilde{C}_\varepsilon\left(\left\|\nabla\tilde{\theta}\right\|^2_{L^2(\Omega_1)} + \left\|\nabla\tilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right) + C_\varepsilon\left(\|\nabla\theta\|^2_{L^2(\Omega_2)} + \|\nabla\theta_t\|^2_{L^2(\Omega_2)}\right) \\
&\qquad + 2\varepsilon\|\operatorname{div} v_t\|^2_{L^2(\Gamma_2)} - \frac{r_0\rho_1}{4}\|v_{tt}\|^2_{L^2(\Gamma_2)} + \frac{\delta_2(2\mu_1+\lambda_1)}{2}\|v_t\|^2_{L^2(\Gamma_2)} \\
&\qquad - \left[\frac{\delta_2 r_0(2\mu_1+\lambda_1)}{2} - \frac{\delta C}{2} + C\right]\left\|\frac{\partial v_t}{\partial\nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)},
\end{aligned}
\tag{4.69}
$$

where we have used

$$
\|u_t\|^2_{L^2(\Gamma_2)} \le C_\rho\|\nabla u_t\|^2_{L^2(\Omega_2)}, \quad C_\rho > 0. \tag{4.70}
$$

Using (1.10), we have that

$$
\left\|\frac{\partial v_t}{\partial\nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)} = \|\operatorname{div} v_t\|^2_{L^2(\Gamma_2)}. \tag{4.71}
$$

Thus

$$
\begin{aligned}
&\frac{d}{dt}\left[\mathcal{E}_4(t)+\frac{m_1(2\mu_1+\lambda_1)}{2\kappa_1}\mathcal{E}_3(t)+\frac{m_2(2\mu_2+\lambda_2)}{2\kappa_2}\widetilde{\mathcal{E}}_3(t)+\delta_1 I_1(t)+\delta_2 I_2(t)\right]\\
&\quad\le -\frac{\kappa_1}{4}(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}-\frac{\kappa_2}{4}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}\\
&\qquad-\left[C_1-\delta_1 C_{\kappa_0}-\delta_2 C-\frac{\delta C_p C}{2}\right]\|\nabla u_t\|^2_{L^2(\Omega_1)}-C_2\|\nabla v_t\|^2_{L^2(\Omega_2)}\\
&\qquad-\frac{\delta m_1(2\mu_1+\lambda_1)^2}{32\rho_1\kappa_1}\|\Delta u\|^2_{L^2(\Omega_1)}-\frac{\delta m_2(2\mu_2+\lambda_2)^2}{32\rho_2\kappa_2}\frac{\rho_1}{\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)}\\
&\qquad+\widetilde{C}_\varepsilon\left(\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}+\left\|\nabla\widetilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right)+C_\varepsilon\left(\|\nabla\theta\|^2_{L^2(\Omega_2)}+\|\nabla\theta_t\|^2_{L^2(\Omega_2)}\right)\\
&\qquad-\frac{r_0\rho_1}{4}\|v_{tt}\|^2_{L^2(\Gamma_2)}+\frac{\delta_2(2\mu_1+\lambda_1)}{2}\|v_t\|^2_{L^2(\Gamma_2)}\\
&\qquad-\left[\frac{\delta_2 r_0(2\mu_1+\lambda_1)}{2}-\frac{\delta C}{2}+C+\frac{\delta_1\kappa_0}{4}\right]\left\|\frac{\partial v_t}{\partial\nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)},
\end{aligned}
\tag{4.72}
$$

where $\varepsilon < k_0\delta_1/2$. □

We define the functional

$$
\mathcal{L}(t)=N\mathcal{E}_1(t)+N\mathcal{E}_2(t)+\mathcal{M}(t)+\varepsilon_0\Phi(t),
\tag{4.73}
$$

where N and ε_0 are positive constants.

Theorem 4.7. *Let us suppose that* $(u,\widetilde{\theta},v,\theta)$ *is a strong solution of the system* (1.1)–(1.13). *Then there exist positive constants* C_0 *and* γ *such that*

$$
\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_4(t)\le C_0(\mathcal{E}_1(0)+\mathcal{E}_2(0)+\mathcal{E}_4(0))e^{-\gamma t}.
\tag{4.74}
$$

Proof. We will assume that the initial data is 3-regular. The conclusion will follow by standard density arguments. Using Lemmas 4.3 and 4.5 and considering boundary conditions, we find that

$$
\begin{aligned}
&\frac{d}{dt}[\mathcal{M}(t)+\varepsilon_0\Phi(t)]\\
&\quad\le -\frac{\kappa_1}{4}(2\mu_1+\lambda_1)\left\|\Delta\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}-\frac{\kappa_2}{4}\frac{\rho_1}{\rho_2}(2\mu_2+\lambda_2)\|\Delta\theta\|^2_{L^2(\Omega_2)}
\end{aligned}
$$

$$
\begin{aligned}
&-\left[C_1-\delta_1 C_{\kappa_0}-\delta_2 C-\frac{(\delta+\varepsilon_0)\ C_p C}{2}\right]\|\nabla u_t\|^2_{L^2(\Omega_1)}-C_2\|\nabla v_t\|^2_{L^2(\Omega_2)}\\
&-\frac{\delta m_1(2\mu_1+\lambda_1)^2}{32\rho_1\kappa_1}\|\Delta u\|^2_{L^2(\Omega_1)}-\frac{\delta m_2(2\mu_2+\lambda_2)^2}{32\rho_2\kappa_2}\frac{\rho_1}{\rho_2}\|\Delta v\|^2_{L^2(\Omega_2)}\\
&+\widetilde{C}_\varepsilon\left(\left\|\nabla\widetilde{\theta}\right\|^2_{L^2(\Omega_1)}+\left\|\nabla\widetilde{\theta}_t\right\|^2_{L^2(\Omega_1)}\right)+C_\varepsilon\left(\|\nabla\theta\|^2_{L^2(\Omega_2)}+\|\nabla\theta_t\|^2_{L^2(\Omega_2)}\right)\\
&-\frac{r_0\rho_1}{4}\|v_{tt}\|^2_{L^2(\Gamma_2)}+\frac{\delta_2(2\mu_1+\lambda_1)}{2}\|v_t\|^2_{L^2(\Gamma_2)}\\
&-\frac{\varepsilon_0}{2}\int_{\Omega_2}\left[(2\mu_2+\lambda_2)|\nabla v_t|^2+\rho_0|v_{tt}|^2\right]d\Omega_2-\frac{\varepsilon_0(n-1)(2\mu_2+\lambda_2)}{2}\|v_t\|_{L^2(\Gamma_2)}\\
&-\left[\frac{\delta_2 r_0(2\mu_1+\lambda_1)}{2}-\frac{\delta C}{2}+C+\frac{\delta_1\kappa_0}{4}-\varepsilon_0\right]\left\|\frac{\partial v_t}{\partial\nu_{\Gamma_2}}\right\|^2_{L^2(\Gamma_2)}.
\end{aligned}
\tag{4.75}
$$

From (4.1), (4.2), and (4.75) we have that

$$
\frac{d\mathcal{L}(t)}{dt}\le -C_0\mathcal{N}(t),
\tag{4.76}
$$

where

$$
\begin{aligned}
\mathcal{N}=&\int_{\Omega_1}\left(|\nabla u_t|^2+|\Delta u|^2+|\Delta u_t|^2+|u_{tt}|^2+\left|\nabla\widetilde{\theta}\right|^2+\left|\nabla\widetilde{\theta}_t\right|^2\right)d\Omega_1\\
&+\int_{\Omega_2}\left(|\nabla v_t|^2+|\Delta v|^2+|\Delta v_t|^2+|v_{tt}|^2+|\nabla\theta_t|^2\right)d\Omega_2.
\end{aligned}
\tag{4.77}
$$

Using the Cauchy inequality, we see that there exist positive constants $C,\gamma>0$ such that

$$
\mathcal{L}(t)\le C\mathcal{N}(t),\qquad \frac{d\mathcal{L}(t)}{dt}\le-\gamma\mathcal{L}(t).
\tag{4.78}
$$

Then $\mathcal{L}(t)\le\mathcal{L}(0)e^{-\gamma t}$. Note that for N large enough we have that

$$
C_1(\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_4(t))\le\mathcal{L}(t)\le C_2(\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_4(t)).
\tag{4.79}
$$

From the above two inequalities our conclusion follows. □

Appendix

We introduce the following functional:

$$\Phi(\varphi, t) = \rho \int_{\Omega} \varphi_t (h \cdot \nabla)\varphi \, d\Omega, \tag{A.1}$$

where Ω is a symmetric set of $\mathbb{R}^n$.

Lemma A.1. *Let Ω be a radially symmetric set of $\mathbb{R}^n$. Suppose that $f \in H^1([0,T] : L^2(\Omega))$ and $h \in [C^2(\overline{\Omega})]^n$. Then for any function $\varphi \in H^2([0,T] : L^2(\Omega)) \cap H^1([0,T] : H^2(\Omega))$ satisfying*

$$\rho\varphi_{tt} - b\Delta\varphi = f, \tag{A.2}$$

where ρ and b are positive constants, one has that

$$\begin{aligned}\frac{d\Phi(\varphi,t)}{dt} &= b\int_{\Gamma} (h \cdot \nabla)\varphi \frac{\partial \varphi}{\partial \nu_{\Gamma}} d\Gamma \\ &\quad + \frac{1}{2}\rho \int_{\Gamma} |\varphi_t|^2 \left(\sum_{i=1}^{n} h_i \nu_{i_{\Gamma}} \right) d\Gamma - \frac{1}{2} b \int_{\Gamma} |\nabla\varphi|^2 \left(\sum_{i=1}^{n} h_i \nu_{i_{\Gamma}} \right) d\Gamma \\ &\quad - \frac{1}{2} \int_{\Omega} \left(\rho|\varphi_t|^2 - b|\nabla\varphi|^2 \right) \left(\sum_{1=1}^{n} \frac{\partial h_i}{\partial x_i} \right) d\Omega \\ &\quad - b\int_{\Omega} \nabla\varphi \left(\sum_{i=1}^{n} \nabla h_i \frac{\partial \varphi}{\partial x_i} \right) d\Omega + \int_{\Omega} f(h \cdot \nabla\varphi) d\Omega,\end{aligned} \tag{A.3}$$

where $\Gamma = \partial\Omega$.

Proof. We consider

$$\begin{aligned}\frac{d\Phi(\varphi,t)}{dt} &= \rho\int_{\Omega} \varphi_{tt}(h \cdot \nabla)\varphi \, d\Omega + \rho\int_{\Omega} \varphi_t (h \cdot \nabla)\varphi_t d\Omega \\ &= \int_{\Omega} f(h \cdot \nabla)\varphi \, d\Omega + b\int_{\Omega} \Delta\varphi (h \cdot \nabla)\varphi \, d\Omega + \rho\int_{\Omega} \varphi_t (h \cdot \nabla)\varphi_t d\Omega.\end{aligned} \tag{A.4}$$

Moreover

$$\int_{\Omega} \varphi_t (h \cdot \nabla)\varphi_t d\Omega = \frac{1}{2}\int_{\Gamma} \varphi_t^2 h \cdot \nu_{\Gamma} d\Gamma - \frac{1}{2}\int_{\Omega} |\varphi_t|^2 \left(\sum_{i=1}^{n} \frac{\partial h_i}{\partial x_i} \right) d\Omega. \tag{A.5}$$

Hence

$$\begin{aligned}\frac{d\Phi(\varphi,t)}{dt} &= \int_\Omega f(h\cdot\nabla)\varphi\, d\Omega + b\int_\Omega \Delta\varphi(h\cdot\nabla)\varphi\, d\Omega \\ &\quad + \frac{1}{2}\rho\int_\Gamma \varphi_t^2\left(\sum_{i=1}^n h_i\cdot\nu_{i_\Gamma}\right)d\Gamma - \frac{1}{2}\rho\int_\Omega |\varphi_t|^2\left(\sum_{i=1}^n \frac{\partial h_i}{\partial x_i}\right)d\Omega.\end{aligned} \tag{A.6}$$

On the other hand,

$$\int_\Omega \Delta\varphi(h\cdot\nabla)\varphi\, d\Omega = \int_\Gamma (h\cdot\nabla\varphi)(\nabla\varphi\cdot\nu_\Gamma)d\Gamma - \int_\Omega \nabla\varphi\cdot\nabla(h\cdot\nabla\varphi)d\Omega. \tag{A.7}$$

Using

$$\begin{aligned} h\cdot\nabla\varphi &= \sum_{i=1}^n h_i\,\frac{\partial\varphi}{\partial x_i}, \qquad \nabla(h\cdot\nabla\varphi) = \sum_{i=1}^n\left(\nabla h_i\frac{\partial\varphi}{\partial x_i} + h_i\frac{\partial\nabla\varphi}{\partial x_i}\right), \\ \nabla\varphi\cdot\nabla(h\cdot\nabla\varphi) &= \nabla\varphi\sum_{i=1}^n\nabla h_i\frac{\partial\varphi}{\partial x_i} + \frac{1}{2}\sum_{i=1}^n h_i\frac{\partial|\nabla\varphi|^2}{\partial x_i}\end{aligned} \tag{A.8}$$

we obtain

$$\begin{aligned}\int_\Omega \Delta\varphi(h\cdot\nabla)\varphi\, d\Omega &= \int_\Gamma (h\cdot\nabla\varphi)(\nabla\varphi\cdot\nu_\Gamma)d\Gamma - \sum_{i=1}^n\int_\Omega\left[\nabla\varphi\nabla h_i\frac{\partial\varphi}{\partial x_i} + \frac{1}{2}h_i\frac{\partial|\nabla\varphi|^2}{\partial x_i}\right]d\Omega \\ &= \int_\Gamma (h\cdot\nabla\varphi)\nabla\varphi\cdot\nu_\Gamma d\Gamma - \int_\Omega\nabla\varphi\left(\sum_{i=1}^n\nabla h_i\frac{\partial\varphi}{\partial x_i}\right)d\Omega - \frac{1}{2}\int_\Omega\sum_{i=1}^n h_i\frac{\partial|\nabla\varphi|^2}{\partial x_i}d\Omega \\ &= \int_\Gamma (h\cdot\nabla\varphi)\nabla\varphi\cdot\nu_\Gamma d\Gamma - \int_\Omega\nabla\varphi\left(\sum_{i=1}^n\nabla h_i\frac{\partial\varphi}{\partial x_i}\right)d\Omega \\ &\quad + \frac{1}{2}\int_\Omega|\nabla\varphi|^2\left(\sum_{i=1}^n\frac{\partial h_i}{\partial x_i}\right)d\Omega - \frac{1}{2}\int_\Gamma|\nabla\varphi|^2\left(\sum_{i=1}^n h_i\cdot\nu_{i_\Gamma}\right)d\Gamma \\ &= \int_\Gamma (h\cdot\nabla\varphi)\nabla\varphi\cdot\nu_\Gamma d\Gamma - \frac{1}{2}\int_\Gamma|\nabla\varphi|^2\left(\sum_{i=1}^n h_i\cdot\nu_{i_\Gamma}\right)d\Gamma \\ &\quad - \int_\Omega\nabla\varphi\left(\sum_{i=1}^n\nabla h_i\frac{\partial\varphi}{\partial x_i}\right)d\Omega + \frac{1}{2}\int_\Omega|\nabla\varphi|^2\left(\sum_{i=1}^n\frac{\partial h_i}{\partial x_i}\right)d\Omega.\end{aligned} \tag{A.9}$$

Replacing in (A.6) the result follows. □

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