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Research Article

Multiple Positive Solutions of a Singular Emden-Fowler Type Problem for Second-Order Impulsive Differential Systems

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This paper studies the existence, and multiplicity of positive solutions of a singular boundary value problem for second-order differential systems with impulse effects. By using the upper and lower solutions method and fixed point index arguments, criteria of the multiplicity, existence and nonexistence of positive solutions with respect to parameters given in the system are established.

1. Introduction

In this paper, we consider systems of impulsive differential equations of the form

$$
\begin{gathered}
u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad t \in (0,1),\ t \neq t_1, \\
v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,1),\ t \neq t_1, \\
\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \Delta v|_{t=t_1} = I_v(v(t_1)), \\
\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \qquad \Delta v'\big|_{t=t_1} = N_v(v(t_1)), \\
u(0) = a \geq 0, \quad v(0) = b \geq 0, \qquad u(1) = c \geq 0, \quad v(1) = d \geq 0,
\end{gathered}
\tag{P}
$$

where λ, μ are positive real parameters, $\Delta u|_{t=t_1} = u(t_1^+) - u(t_1)$, and $\Delta u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-)$. Throughout this paper, we assume $f, g \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ with $f(0,0) = 0 = g(0,0)$ and $f(u,v) > 0, g(u,v) > 0$ for all $(u,v) \neq (0,0), I_u, I_v \in C(\mathbb{R}_+, \mathbb{R})$ satisfying $I_u(0) = 0 = I_v(0), N_u, N_v \in C(\mathbb{R}_+, (-\infty, 0])$, and $h_i \in C((0,1),(0,\infty)), i = 1,2$. Here we denote $\mathbb{R}_+ = [0,\infty)$. We note that

h_i may be singular at $t = 0$ and/or 1. Let $J = [0,1], J' = [0,1] \setminus \{0,1,t_1\}, PC[0,1] = \{u \mid u : [0,1] \to \mathbb{R}$ be continuous at $t \neq t_1$, left continuous at $t = t_1$, and its right-hand limit at $t = t_1$ exists $\}$ and $X = PC[0,1] \times PC[0,1]$. Then $PC[0,1]$ and X are Banach spaces with norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$ and $\|(u,v)\| = \|u\| + \|v\|$, respectively. The solution of problem (P) means $(u,v) \in X \cap (C^2(J') \times C^2(J'))$ which satisfies (P).

Recently, several works have been devoted to the study of second-order impulsive differential systems. See, for example [1–6], and references therein. In Particular, E.K. Lee and Y.H. Lee [3] studied problem (P) when f and g satisfy $f(0,0) > 0$ and $g(0,0) > 0$. More precisely, let us consider the following assumptions.

- (D_1) $\int_0^1 s(1-s)h_i(s)ds < \infty$, for $i = 1,2$.
- (D_2) $t_1 N_u(u) \leq I_u(u) \leq -(1-t_1)N_u(u)$ and $t_1 N_v(v) \leq I_v(v) \leq -(1-t_1)N_v(v)$.
- (D_3) $u + I_u(u)$ and $v + I_v(v)$ are nondecreasing.
- (D_4) $N_{u,\infty} = \lim_{u\to\infty}|N_u(u)|/u < 1$ and $N_{v,\infty} = \lim_{v\to\infty}|N_v(v)|/v < 1$.
- (D_5) $f_\infty = \lim_{u+v\to\infty} f(u,v)/u+v = \infty$ and $g_\infty = \lim_{u+v\to\infty} g(u,v)/u+v = \infty$.
- (D_6) f and g are nondecreasing on $\mathbb{R}_+^2$, that is, $f(u_1,v_1) \leq f(u_2,v_2)$ and $g(u_1,v_1) \leq g(u_2,v_2)$ whenever $(u_1,v_1) \leq (u_2,v_2)$, where inequality on $\mathbb{R}_+^2$ can be understood componentwise.

Under the above assumptions, they proved that there exists a continuous curve Γ splitting $\mathbb{R}_+^2 \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that problem (3.20) has at least two positive solutions for $(\lambda,\mu) \in \mathcal{O}_1$, at least one positive solution for $(\lambda,\mu) \in \Gamma$, and no solution for $(\lambda,\mu) \in \mathcal{O}_2$.

The aim of this paper is to study generalized Emden-Fowler-type problem for (P), that is, f and g satisfy $f(0,0) = 0$ and $g(0,0) = 0$, respectively. In this case, we obtain two interesting results. First, for Dirichlet boundary condition, that is, $a = b = c = d = 0$, assuming $(D_1), (D_2)$ and

- (D_4') $N_{u,0} = \lim_{u\to 0}|N_u(u)|/u < 1/2$ and $N_{v,0} = \lim_{v\to 0}|N_v(v)|/v < 1/2$,
- (D_5') $f_\infty = \infty, g_\infty = \infty$ and $f_0 = \lim_{u+v\to 0} f(u,v)/u+v = 0, g_0 = \lim_{u+v\to 0} g(u,v)/u+v = 0$,

we prove that problem (P) has at least one positive solution for all $(\lambda,\mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$. On the other hand, for two-point boundary condition, that is, $c > a$ and $d > b$, assuming $(D_1) \sim (D_6)$, we prove that there exists a continuous curve Γ_0 splitting $\mathbb{R}_+^2 \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_{0,1}$ and $\mathcal{O}_{0,2}$ and there exists a subset $\mathcal{O} \subset \mathcal{O}_{0,1}$ such that problem (P) has at least two positive solutions for $(\lambda,\mu) \in \mathcal{O}$, at least one positive solution for $(\lambda,\mu) \in (\mathcal{O}_{0,1} \setminus \mathcal{O}) \cup \Gamma_0$, and no solution for $(\lambda,\mu) \in \mathcal{O}_{0,2}$.

Our technique of proofs is mainly employed by the upper and lower solutions method and several fixed point index theorems.

The paper is organized as follows: in Section 2, we introduce and prove two types of upper and lower solutions and related theorems, one for singular systems with no impulse effect and the other for singular impulsive systems and then introduce several fixed point index theorems for later use. In Section 3, we prove an existence result for Dirichlet boundary value problems and existence and nonexistence part of the result for two-point boundary value problems. In Section 4, we prove the existence of the second positive solution for two point boundary value problems. Finally, in Section 5, we apply main results to prove some theorems of existence, nonexistence, and multiplicity of positive radial solutions for impulsive semilinear elliptic problems.

2. Preliminary

In this section, we introduce two types of fundamental theorems of upper and lower solutions method for a singular system with no impulse effect and an impulsive system and then introduce several well-known fixed point index theorems. We first give definition s of somewhat general type of upper and lower solutions for the following singular system:

$$\begin{gathered} u''(t)+F(t,u(t),v(t))=0, \quad t\in(0,1), \\ v''(t)+G(t,u(t),v(t))=0, \quad t\in(0,1), \\ u(0)=A, \qquad u(1)=C, \qquad v(0)=B, \qquad v(1)=D, \end{gathered} \tag{H}$$

where $F,G\ :(0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous.

Definition 2.1. We say that $(\alpha_u,\alpha_v)\in C[0,1]\times C[0,1]$ is a G-lower solution of (H) if $(\alpha_u,\alpha_v)\in C^2(0,1)\times C^2(0,1)$ except at finite points $\tau_1,\dots,\tau_n$ with $0<\tau_1<\ \dots<\tau_n<1$ such that

(L_1) at each τ_i, there exist $(\alpha'_u(\tau_i^-),\alpha'_v(\tau_i^-)),(\alpha'_u(\tau_i^+),\alpha'_v(\tau_i^+))$ such that $\alpha'_u(\tau_i^-)\le\alpha'_u(\tau_i^+),\alpha'_v(\tau_i^-)\le\alpha'_v(\tau_i^+)$, and

(L_2)

$$\begin{gathered} \alpha''_u(t)+F(t,\alpha_u(t),\alpha_v(t))\ge 0, \\ \alpha''_v(t)+G(t,\alpha_u(t),\alpha_v(t))\ge 0, \quad t\in\frac{(0,1)}{\{\tau_1,\dots,\tau_n\}}, \\ \alpha_u(0)\le A, \qquad \alpha_u(1)\le C, \\ \alpha_v(0)\le B, \qquad \alpha_v(1)\le D. \end{gathered} \tag{2.1}$$

We also say that $(\beta_u,\beta_v)\in C[0,1]\times C[0,1]$ is a G-upper solution of the problem (H) if $(\beta_u,\beta_v)\in C^2(0,1)\times C^2(0,1)$ except at finite points $\sigma_1,\dots,\sigma_m$ with $0<\sigma_1<\cdots<\sigma_m<1$ such that

(U_1) at each σ_i, there exist $(\beta'_u(\sigma_j^-),\beta'_v(\sigma_j^-)),(\beta'_u(\sigma_j^+),\beta'_v(\sigma_j^+))$ such that $\beta'_u(\sigma_j^-)\ge\beta'_u(\sigma_j^+),\beta'_v(\sigma_j^-)\ge\beta'_v(\sigma_j^+)$, and

(U_2)

$$\begin{gathered} \beta''_u(t)+F\bigl(t,\beta_u(t),\beta_v(t)\bigr)\le 0, \\ \beta''_v(t)+G\bigl(t,\beta_u(t),\beta_v(t)\bigr)\le 0, \quad t\in\frac{(0,1)}{\{\sigma_1,\dots,\sigma_n\}}, \\ \beta_u(0)\ge A, \qquad \beta_u(1)\ge C, \\ \beta_v(0)\ge B, \qquad \beta_v(1)\ge D. \end{gathered} \tag{2.2}$$

For the proof of the fundamental theorem on G-upper and G-lower solutions for problem (H), we need the following lemma. One may refer to [7] for the proof.

Lemma 2.2. *Let $F, G : D \to \mathbb{R}$ be continuous functions and $D \subset (0,1)\times\mathbb{R}\times\mathbb{R}$. Assume that there exist $h_F, h_G \in C((0,1),\mathbb{R}^+)$ such that*

$$|F(t,u,v)| \le h_F(t), \qquad |G(t,u,v)| \le h_G(t), \tag{2.3}$$

for all $(t,u,v) \in (0,1)\times\mathbb{R}\times\mathbb{R}$, and

$$\int_0^1 s(1-s)h_F(s)ds + \int_0^1 s(1-s)h_G(s)ds < \infty. \tag{2.4}$$

Then problem (H) has a solution.

Let $D_\alpha^\beta = \{(t,u,v) \mid (\alpha_u(t),\alpha_v(t)) \le (u,v) \le (\beta_u(t),\beta_v(t)), t \in [0,1]\}$. Then the fundamental theorem of G-upper and G-lower solutions for singular problem (H) is given as follows.

Theorem 2.3. *Let (α_u,α_v) and (β_u,β_v) be a G-lower solution and a G-upper solution of problem (H), respectively, such that*

(a_1) $(\alpha_u(t),\alpha_v(t)) \le (\beta_u(t),\beta_v(t))$ *for all* $t \in [0,1]$.

Assume also that there exist $h_F, h_G \in C((0,1),\mathbb{R}^+)$ such that

(a_2) $|F(t,u,v)| \le h_F(t)$ *and* $|G(t,u,v)| \le h_G(t)$ *for all* $(t,u,v) \in D_\alpha^\beta$;

(a_3) $\int_0^1 s(1-s)h_F(s)ds + \int_0^1 s(1-s)h_G(s)ds < \infty$;

(a_4) $F(t,u,v_1) \le F(t,u,v_2)$, *whenever* $v_1 \le v_2$ *and* $G(t,u_1,v) \le G(t,u_2,v)$, *whenever* $u_1 \le u_2$.

Then problem (H) has at least one solution (u,v) such that

$$(\alpha_u(t),\alpha_v(t)) \le (u(t),v(t)) \le \left(\beta_u(t),\beta_v(t)\right), \quad \forall\, t \in [0,1]. \tag{2.5}$$

Proof. Define a modified function of F as follows:

$$F_*(t,u,v) = \begin{cases} F(t,\beta_u(t),v) - \dfrac{u-\beta_u(t)}{1} + u^2 & \text{if } u > \beta_u(t), \\ F(t,u,v) & \text{if } \alpha_u(t) \le u \le \beta_u(t), \\ F(t,\alpha_u(t),v) - \dfrac{u-\alpha_u(t)}{1} + u^2 & \text{if } u < \alpha_u(t), \end{cases}$$

$$F^*(t,u,v) = \begin{cases} F_*(t,u,\beta_v(t)) & \text{if } v > \beta_v(t), \\ F_*(t,u,v) & \text{if } \alpha_v(t) \le v \le \beta_v(t), \\ F_*(t,u,\alpha_v(t)) & \text{if } v < \alpha_v(t), \end{cases}$$

$$G_*(t,u,v) = \begin{cases} G(t,\beta_u(t),v) & \text{if } u > \beta_u(t), \\ G(t,u,v) & \text{if } \alpha_u(t) \le u \le \beta_u(t), \\ G(t,\alpha_u(t),v) & \text{if } \ u < \alpha_u(t), \end{cases} \tag{2.6}$$

$$G^*(t,u,v) = \begin{cases} G_*(t,u,\beta_v(t),) - \dfrac{v-\beta_v(t)}{1} + v^2 & \text{if } v > \beta_v(t), \\ G_*(t,u,v) & \text{if } \alpha_v(t) \le v \le \beta_v(t), \\ G_*(t,u,\alpha_v(t)) - \dfrac{v-\alpha_v(t)}{1} + v^2 & \text{if } v < \alpha_v(t). \end{cases} \tag{2.7}$$

Then $F^*, G^* : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and

$$\begin{aligned} |F^*(t,u,v)| &\le m(\alpha_u,\beta_u) + h_F(t), \\ |G^*(t,u,v)| &\le m(\alpha_v,\beta_v) + h_G(t), \end{aligned} \tag{2.8}$$

for all $(t,u,v) \in (0,1) \times \mathbb{R} \times \mathbb{R}$, where $m(\alpha,\beta) = \|\alpha\| + \|\beta\| + 1$. For the problem

$$\begin{gathered} u''(t) + F^*(t,u(t),v(t)) = 0, \\ v''(t) + G^*(t,u(t),v(t)) = 0, \quad t \in (0,1) \\ u(0) = A, \qquad u(1) = C, \qquad v(0) = B, \qquad v(1) = D, \end{gathered} \tag{M}$$

Lemma 2.2 guarantees the existence of solutions of problem (M) and thus it is enough to prove that any solution (u,v) of problem (M) satisfies

$$(\alpha_u(t),\alpha_v(t)) \le (u(t),v(t)) \le (\beta_u(t),\beta_v(t)), \quad \forall t \in [0,1]. \tag{2.9}$$

Suppose, on the contrary, $(\alpha_u,\alpha_v) \not\le (u,v)$, so we consider the case $\alpha_u \not\le u$. Let $(\alpha_u - u)(t_0) = \max_{t\in[0,1]}(\alpha_u - u)(t) > 0$. If $t_0 \in (0,1) \setminus \{\tau_1,\dots,\tau_n\}$, then $(\alpha_u - u)''(t_0) \le 0$. We consider two cases. First, if $\alpha_v(t_0) \le v(t_0)$, then by $\alpha_u(t_0) > u(t_0)$ and condition (a_4),

$$\begin{aligned}
0 &\ge (\alpha_u - u)''(t_0) = \alpha_u''(t_0) + F^*(t_0, u(t_0), v(t_0)) \\
&= \alpha_u''(t_0) + F(t_0, \alpha_u(t_0), v(t_0)) - \frac{u(t_0) - \alpha_u(t_0)}{1 + u^2(t_0)} \\
&\ge \alpha_u''(t_0) + F(t_0, \alpha_u(t_0), \alpha_v(t_0)) - \frac{u(t_0) - \alpha_u(t_0)}{1 + u^2(t_0)} \\
&\ge \frac{\alpha_u(t_0) - u(t_0)}{1 + u^2(t_0)} > 0,
\end{aligned} \tag{2.10}$$

which is a contradiction. Next, if $\alpha_v(t_0) > v(t_0)$, then by the definition of F^*,

$$\begin{aligned}
0 &\ge (\alpha_u - u)''(t_0) = \alpha_u''(t_0) + F^*(t_0, u(t_0), v(t_0)) \\
&= \alpha_u''(t_0) + F(t_0, \alpha_u(t_0), \alpha_v(t_0)) - \frac{u(t_0) - \alpha_u(t_0)}{1 + u^2(t_0)} > 0,
\end{aligned} \tag{2.11}$$

which is also a contradiction. If $t_0 = \tau_i$ for some $i = 1, \dots, n$, then since $\alpha_u - u$ attains its positive maximum at τ_i,

$$(\alpha_u - u)'(\tau_i^-) \ge 0, \qquad (\alpha_u - u)'(\tau_i^+) \le 0. \tag{2.12}$$

If $(\alpha_u - u)'(\tau_i^-) > 0$, then

$$0 < (\alpha_u - u)'(\tau_i^-) - (\alpha_u - u)'(\tau_i^+) = \alpha_u'(\tau_i^-) - \alpha_{u'}(\tau_i^+). \tag{2.13}$$

This leads a contradiction to the definition of G-lower solution. If $(\alpha_u - u)'(\tau_i^-) = 0$, then there exists $\delta > 0$ such that for all $t \in (\tau_i - \delta, \tau_i)$,

$$(\alpha_u - u)(t) > 0, \qquad (\alpha_u - u)'(t) \ge 0, \qquad (\alpha_u - u)''(t) \le 0. \tag{2.14}$$

For $t \in (\tau_i - \delta, \tau_i)$, if $\alpha_v(t) \le v(t)$, then by $\alpha_u(t) > u(t)$ and condition (a_4),

$$\begin{aligned}
0 &\ge (\alpha_u - u)''(t) = \alpha_u''(t) + F^*(t, u(t), v(t)) \\
&= \alpha_u''(t) + F(t, \alpha_u(t), v(t)) - \frac{u(t) - \alpha_u(t)}{1 + u^2(t)} \\
&\ge \alpha_u''(t) + F(t, \alpha_u(t), \alpha_v(t)) - \frac{u(t) - \alpha_u(t)}{1 + u^2(t)} \\
&\ge \frac{\alpha_u(t) - u(t)}{1 + u^2(t)} > 0,
\end{aligned} \tag{2.15}$$

which is a contradiction. If $\alpha_v(t) > v(t)$, then by definition of F^*,

$$0 \geq (\alpha_u - u)''(t) = \alpha_u''(t) + F^*(t, u(t), v(t))$$
$$= \alpha_u''(t) + F(t, \alpha_u(t), \alpha_v(t)) - \frac{u(t) - \alpha_u(t)}{1 + u^2(t)} > 0, \tag{2.16}$$

which is a contradiction. If $t_0 = 0$ or 1, then

$$0 < (\alpha_u - u)(0) = \alpha_u(0) - A \leq 0,$$
$$0 < (\alpha_u - u)(1) = \alpha_u(1) - C \leq 0, \tag{2.17}$$

which is a contradiction. Similarly, we get contradictions for the case $\alpha_v \nleq v$. The proof for $(u, v) \leq (\beta_u, \beta_v)$ can be done by similar fashion. □

Now we introduce definition and fundamental theorem of upper and lower solutions for impulsive differential systems of the form

$$u''(t) + F(t, u(t), v(t)) = 0, \quad t \neq t_1, \ t \in (0,1),$$
$$u''(t) + G(t, u(t), v(t)) = 0, \quad t \neq t_1, \ t \in (0,1),$$
$$\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \Delta v|_{t=t_1} = I_v(v(t_1)), \tag{S}$$
$$\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \qquad \Delta v'\big|_{t=t_1} = N_v(v(t_1)),$$
$$u(0) = a, \qquad v(0) = b, \qquad u(1) = c, \qquad v(1) = d,$$

where $F, G \in C((0,1) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_u, I_v \in C(\mathbb{R}_+, \mathbb{R})$ satisfying $I_u(0) = 0 = I_v(0)$ and $N_u, N_v \in C(\mathbb{R}_+, (-\infty, 0])$.

Definition 2.4. $(\alpha_u, \alpha_v) \in X \cap (C^2(J') \times C^2(J'))$ is called a lower solution of problem (S) if

$$\alpha_u''(t) + F(t, \alpha_u(t), \alpha_v(t)) \geq 0, \quad t \neq t_1,$$
$$\alpha_v''(t) + G(t, \alpha_u(t), \alpha_v(t)) \geq 0, \quad t \neq t_1,$$
$$\Delta \alpha_u|_{t=t_1} = I_u(\alpha_u(t_1)), \qquad \Delta \alpha_v|_{t=t_1} = I_v(\alpha_v(t_1)), \tag{2.18}$$
$$\Delta \alpha_u'\big|_{t=t_1} \geq N_u(\alpha_u(t_1)), \qquad \Delta \alpha_v'\big|_{t=t_1} \geq N_v(\alpha_v(t_1)),$$
$$\alpha_u(0) \leq a, \qquad \alpha_v(0) \leq b, \qquad \alpha_u(1) \leq c, \qquad \alpha_v(1) \leq d.$$

We also define an upper solution $(\beta_u, \beta_v) \in X \cap (C^2(J') \times C^2(J'))$ if (β_u, β_v) satisfies the reverses of the above inequalities.

The following existence theorem for upper and lower solutions method is proved in [3].

Theorem 2.5. *Let (α_u, α_v) and (β_u, β_v) be lower and upper solutions of problem (S), respectively, satisfying (a_1). Moreover, we assume $(a_2) \sim (a_4)$ and (D_3). Then problem (S) has at least one solution (u, v) such that*

$$(\alpha_u(t), \alpha_v(t)) \le (u(t), v(t)) \le (\beta_u(t), \beta_v(t)), \quad \forall\, t \in [0,1]. \tag{2.19}$$

The following theorems are well known cone theoretic fixed point theorems. See Lakshmikantham ([8]) for proofs and details.

Theorem 2.6. *Let X be a Banach space and $\mathcal{K}$ a cone in X. Assume that Ω_1 and Ω_2 are bounded open subsets in X with $0 \in \Omega_1$ and $\Omega_1 \subset \Omega_2$. Let $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous such that either*

(i) $\|Tu\| \le \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$ or

(ii) $\|Tu\| \ge \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 2.7. *Let X be a Banach space, $\mathcal{K}$ a cone in X and Ω bounded open in X. Let $0 \in \Omega$ and $T : \mathcal{K} \cap \overline{\Omega} \to \mathcal{K}$ be condensing. Suppose that $Tx \ne \nu x$, for all $x \in \mathcal{K} \cap \partial\Omega$ and all $\nu \ge 1$. Then*

$$i(T, \mathcal{K} \cap \Omega, \mathcal{K}) = 1. \tag{2.20}$$

3. Existence

In this section, we prove an existence theorem of positive solutions for problem (P) with Dirichlet boundary condition and the existence and nonexistence part of the result for problem (P) with two-point boundary condition. Let us consider the following second-order impulsive differential systems.

$$\begin{gathered}
u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad t \in (0,1),\ t \ne t_1, \\
v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,1),\ t \ne t_1, \\
\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \Delta v|_{t=t_1} = I_v(v(t_1)) \\
\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \qquad \Delta v'\big|_{t=t_1} = N_v(v(t_1)), \\
u(0) = a \ge 0, \qquad v(0) = b \ge 0, \qquad u(1) = c \ge 0, \qquad v(1) = d \ge 0,
\end{gathered} \tag{P}$$

where λ, μ are positive real parameters, $f, g \in C(\mathbb{R}_+^2, [0,\infty))$ with $f(0,0) = 0, g(0,0) = 0$, and $f(u,v) > 0, g(u,v) > 0$ for all $(u,v) \ne (0,0), I_u, I_v \in C(\mathbb{R}_+, \mathbb{R})$ satisfying $I_u(0) = 0 = I_v(0), N_u, N_v \in C(\mathbb{R}_+, (-\infty, 0])$, and $h_1, h_2 \in C((0,1), (0,\infty))$ may be singular at $t = 0$ and/or 1.

We first set up an equivalent operator equatio for problem (P). Let us define $A_\lambda : X \rightarrow PC[0,1]$ and $B_\mu : X \rightarrow PC[0,1]$ by taking

$$\begin{aligned} A_\lambda(u,v)(t) &\triangleq a + (c-a)t + \lambda \int_0^1 K(t,s)h_1(s)f(u(s),v(s))ds + W_u(t,u), \\ B_\mu(u,v)(t) &\triangleq b + (d-b)t + \mu \int_0^1 K(t,s)h_2(s)g(u(s),v(s))ds + W_v(t,v), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} K(t,s) &= \begin{cases} t(-I_v(v(t_1)) - (1-t_1)N_v(v(t_1))), & 0 \le t \le t_1, \\ (1-t)(I_v(v(t_1)) - t_1 N_v(v(t_1))), & t_1 < t \le 1. \end{cases} \\ W_u(t,u)(t) &= \begin{cases} t(-I_u(u(t_1)) - (1-t_1)N_u(u(t_1))), & 0 \le t \le t_1, \\ (1-t)(I_u(u(t_1)) - t_1 N_u(u(t_1))), & t_1 < t \le 1, \end{cases} \\ W_v(t,u)(t) &= \begin{cases} t(-I_v(v(t_1)) - (1-t_1)N_v(v(t_1))), & 0 \le t \le t_1, \\ (1-t)(I_v(v(t_1)) - t_1 N_v(v(t_1))), & t_1 < t \le 1. \end{cases} \end{aligned} \tag{3.2}$$

Also define

$$T_{\lambda,\mu}(u,v) \triangleq \left(A_\lambda(u,v), B_\mu(u,v)\right). \tag{3.3}$$

Then $T_{\lambda,\mu} : X \rightarrow X$ is well defined on X and problem (P) is equivalent to the fixed-point equation

$$T_{\lambda,\mu}(u,v) = (u,v) \quad \text{in X}. \tag{3.4}$$

Mainly due to $(D_1), T_{\lambda,\mu}$ is completely continuous (see [3] for the proof). Let $\|u\|_0 = \sup_{t\in[0,t_1]}|u(t)|$, $\|u\|_1 = \sup_{t\in[t_1,1]}|u(t)|$, $S_0 = [t_1/4, 3t_1/4]$, $S_1 = [3t_1+1/4, t_1+3/4]$, $\mathcal{P} = \{(u,v) \in X \mid u,v \ge 0\}$, and $\mathcal{K} = \{(u,v) \in \mathcal{P} \mid \min_{t\in S_0}(u(t)+v(t)) \ge t_1/4(\|u\|_0 + \|v\|_0), \min_{t\in S_1}(u(t) + v(t)) \ge 1 - t_1/4(\|u\|_1 + \|v\|_1)\}$. Then $\|u\| = \max\{\|u\|_0, \|u\|_1\}$ and $\mathcal{P}, \mathcal{K}$ are cones in X. By using concavity of $T_{\lambda,\mu}$(u) with $u \in \mathcal{P}$, we can easily show that $T_{\lambda,\mu}(\mathcal{P}) \subset \mathcal{K}$.

We now prove the existence theorem of positive solutions for Dirichlet boundary value problem

$$
\begin{gathered}
u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad t \in (0,1),\ t \neq t_1, \\
v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,1)\ t \neq t_1, \\
\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \Delta v|_{t=t_1} = I_v(v(t_1)), \\
\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \qquad \Delta v'\big|_{t=t_1} = N_v(v(t_1)), \\
u(0) = 0, \qquad v(0) = 0, \qquad u(1) = 0, \qquad v(1) = 0.
\end{gathered}
\tag{P_D}
$$

Theorem 3.1. *Assume* $(D_1), (D_2), (D_4')$*, and* (D_5')*. Then problem* (P_D) *has at least one positive solution for all* $(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$*.*

Proof. First, we consider case $\lambda > 0$ and $\mu > 0$. By the fact $N_{u,0} < 1/2$ and $N_{v,0} < 1/2$, we may choose $c_1, m_1 > 0$ such that $\max\{N_{u,0}, N_{v,0}\} < c_1 < 1/2, |N_u(u)| \leq c_1 u$ for $u \leq m_1$ and $|N_v(v)| \leq c_1 v$ for $v \leq m_1$. Also choose η_λ and η_μ satisfying $0 < \eta_\lambda < (1 - 2c_1)/2\lambda \int_0^1 s(1-s) h_1(s) ds$ and $0 < \eta_\mu < (1 - 2c_1)/2\mu \int_0^1 s(1-s) h_2(s) ds$. Since $f_0 = 0$ and $g_0 = 0$, there exist $m_2, m_3 > 0$ such that $f(u,v) \leq \eta_\lambda (u + v)$ for $u + v \leq m_2$ and $g(u,v) \leq \eta_\mu (u+v)$ for $u + v \leq m_3$. Let $\Omega_1 = B_{M_1} = \{(u,v) \in X \mid \|(u,v)\| < M_1\}$ with $M_1 = \min\{m_1, m_2, m_3\}$. Then for $(u,v) \in \mathcal{K} \cap \partial\Omega_1$, we obtain by using (D_2)

$$
\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_0^1 K(t,s) h_1(s) f(u(s), v(s)) ds + W_u(t,u) \\
&\leq \lambda \eta_\lambda \int_0^1 s(1-s) h_1(s)(u(s) + v(s)) ds + |N_u(u(t_1))| \\
&\leq \left(\lambda \eta_\lambda \int_0^1 s(1-s) h_1(s) ds + c_1 \right) \|(u,v)\| \\
&\leq \frac{1}{2} \|u,v\|,
\end{aligned}
\tag{3.5}
$$

for all $t \in [0,1]$. Similarly, we obtain

$$
B_\mu(u,v)(t) \leq \frac{1}{2\|(u,v)\|} \tag{3.6}
$$

for all $t \in [0,1]$. Thus

$$
\|T_{\lambda,\,\mu}(u,v)\| = \|A_\lambda(u,v)\| + \|B_\mu(u,v)\| \leq \|(u,v)\|. \tag{3.7}
$$

On the other hand, let us choose η_1 and η_2 such that

$$\frac{1}{\eta_2} < \mu \min\left\{ \frac{t_1}{8} \min_{t\in S_0} \int_{S_0} K(t,s)h_2(s)ds, \frac{1-t_1}{8} \min_{t\in S_1} \int_{S_1} K(t,s)h_2(s)ds \right\}.$$
$$\frac{1}{\eta_2} < \mu \min\left\{ \frac{t_1}{8} \min_{t\in S_0} \int_{S_0} K(t,s)h_2(s)ds, \frac{1-t_1}{8} \min_{t\in S_1} \int_{S_1} K(t,s)h_2(s)ds \right\}. \tag{3.8}$$

Also by (D_5'), we may choose R_f and R_g such that $f(u,v) \geq \eta_1(u+v)$ for $u+v \geq R_f$ and $g(u,v) \geq \eta_2(u+v)$ for $u+v \geq R_g$. Let $\Omega_2 = \{(u,\mathrm{v}) \in X \mid \|(u,v)\| < M_2\}$, where $M_2 = \max\{8R_f/t_1, 8R_f/(1-t_1), 8R_g/t_1, 8R_g/(1-t_1), M_1+1\}$. Then $\Omega_1 \subset \Omega_2$. Let $(u,v) \in \mathcal{K} \cap \partial\Omega_2$, then we have the following four cases: $\|u\| \geq \|v\|$ and $\|u\| = \|u\|_0$, $\|u\| \geq \|v\|$ and $\|u\| = \|u\|_1$, $\|u\| \leq \|v\|$ and $\|v\| = \|v\|_0$, and $\|u\| \leq \|v\|$ and $\|v\| = \|v\|_1$. We consider the first case, the rest of them can be considered in a similar way. So let $\|u\| \geq \|v\|$ and $\|u\| = \|u\|_0$; then for $t \in S_0$, we have

$$u(t)+v(t) \geq u(t) \geq \frac{t_1}{8}(2\|u\|_0) \geq \frac{t_1}{8}(\|u\|+\|v\|) = \frac{t_1}{8}\|(u,v)\| \geq R_f. \tag{3.9}$$

Thus $f(u(t),v(t)) \geq \eta_1(u(t)+v(t))$ for $t \in S_0$. Since $W_u(t,u) \geq 0$, we get for $t \in S_0$,

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_0^1 K(t,s)h_1(s)f(u(s),v(s))ds + W_u(t,u) \\
&\geq \lambda \int_{S_0} K(t,s)h_1(s)f(u(s),v(s))ds \\
&\geq \lambda\eta_1 \int_{S_0} K(t,s)h_1(s)(u(s)+v(s))ds \\
&\geq \lambda\eta_1 \frac{t_1}{8} \int_{S_0} K(t,s)h_1(s)ds(2\|u\|_0 + 2\|v\|_0) \\
&\geq \lambda\eta_1 \frac{t_1}{8} \min_{t\in S_0} \int_{S_0} K(t,s)h_1(s)ds\|(u,v)\| > \|(u,v)\|.
\end{aligned} \tag{3.10}$$

Therefore,

$$\left\|T_{\lambda,\mu}(u,v)\right\| \geq \|A_\lambda(u,v)\| > \|(u,v)\|, \tag{3.11}$$

and by Theorem 2.6, $T_{\lambda,\mu}$ has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Second, consider case $\lambda > 0$ and $\mu = 0$. Taking c_1, η_λ, m_1, and m_2 as above and using the same computation, we may show

$$\|A_\lambda(u,v)\| \leq \frac{1}{2}\|(u,v)\|, \tag{3.12}$$

for all $(u,v) \in \mathcal{K} \cap \partial\Omega_1$, where $\Omega_1 = B_{M_1}$ with $M_1 = \min\{m_1, m_2\}$. Since $\mu = 0$,

$$B_\mu(u,v)(\mathrm{t}) = W_v(t,v) \le |N_v(v(t_1))| \le c_1\|(u,v)\| \le \frac{1}{2}\|(u,v)\|, \tag{3.13}$$

for all $t \in [0,1]$. Thus

$$\left\|T_{\lambda,\mu}(u,v)\right\| \le \|A_\lambda(u,v)\| + \left\|B_\mu(u,v)\right\| \le \|(u,v)\|, \tag{3.14}$$

for $(u,v) \in \mathcal{K} \cap \partial\Omega_1$. Now, let us choose η_1 and R_f as above and let $\Omega_2 = \{(u,v) \in X \mid \|(u,v)\| < M_2\}$, where $M_2 = \max\{8R_f/t_1, 8R_f/1 - t_1, M_1 + 1\}$. Then $\Omega_1 \subset \Omega_2$ and we can show by the same computation as above,

$$\left\|T_{\lambda,\mu}(u,v)\right\| \ge \|A_\lambda(u,v)\| > \|(u,v)\|, \tag{3.15}$$

for $(u,v) \in \mathcal{K} \cap \partial\Omega_2$ and thus $T_{\lambda,\mu}$ has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Finally, consider case $\lambda = 0$ and $\mu > 0$. Taking $c_{1,}\eta_\mu, m_1$, and m_3 as the first case, we may show by similar argument,

$$\left\|B_\mu(u,v)\right\| \le \frac{1}{2}\|(u,v)\|, \qquad \|A_\lambda(u,v)\| \le \frac{1}{2}\|(u,v)\|, \tag{3.16}$$

for all $(u,v) \in \mathcal{K} \cap \partial\Omega_1$, where $\Omega_1 = B_{M_1}$ with $M_1 = \min\{m_1, m_3\}$. Thus

$$\left\|T_{\lambda,\mu}(u,v)\right\| \le \|(u,v)\|, \tag{3.17}$$

for $(u,v) \in \mathcal{K} \cap \partial\Omega_1$. Now, let us choose η_2 and R_g as the first case and let $\Omega_2 = \{(u,v) \in X \mid \|(u,v)\| < M_2\}$, where $M_2 = \max\{8R_g/t_1, 8R_g/1 - t_1, M_1 + 1\}$. Then $\Omega_1 \subset \Omega_2$ and we also show similarly, as before,

$$\left\|T_{\lambda,\mu}(u,v)\right\| \ge \left\|B_\mu(u,v)\right\| > \|(u,v)\|, \tag{3.18}$$

for $(u,v) \in \mathcal{K} \cap \partial\Omega_2$. Therefore, $T_{\lambda,\mu}$ has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and this completes the proof. □

Now let us consider two point boundary value problems given as follows:

$$\begin{gathered}
u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad t \in (0,1),\ \mathrm{t} \neq t_1, \\
v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,1),\ t \neq t_1, \\
\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \Delta v|_{t=t_1} = I_v(v(t_1)), \\
\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \qquad \Delta v'\Delta v\big|_{t=t_1} = N_v(v(t_1)), \\
u(0) = a \ge 0, \qquad v(0) = b \ge 0, \qquad u(1) = c > a, \qquad v(1) = d > b.
\end{gathered} \tag{P_T}$$

Lemma 3.2. *Assume* (D_5). *Let* $\mathcal{R}$ *be a compact subset of* $\mathbb{R}_+^2 \setminus \{(0,0)\}$. *Then there exists a constant* $b_{\mathcal{R}} > 0$ *such that for all* $(\lambda, \mu) \in \mathcal{R}$ *for possible positive solutions* (u, v) *of problem* (3.20) *at* (λ, μ), *one has* $\|(u, v)\| < b_{\mathcal{R}}$.

Proof. Suppose on the contrary that there is a sequence (u_n, v_n) of positive solutions of (3.20) at (λ_n, μ_n) such that $(\lambda_n, \mu_n) \in \mathcal{R}$ for all n and $\|(u_n, v_n)\| \to \infty$. Since $(0,0) \notin \mathcal{R}$, there is a subsequence, say again $\{(\lambda_n, \mu_n)\}$, such that $\alpha \triangleq \min\{\lambda_n\} > 0$ or $\beta \triangleq \min\{\mu_n\} > 0$. First, we assume $\alpha > 0$. From $\|(u_n, v_n)\| \to \infty$, we know $\|u_n\|_0 + \|v_n\|_0 \to \infty$ or $\|u_n\|_1 + \|v_n\|_1 \to \infty$. Suppose $\|u_n\|_0 + \|v_n\|_0 \to \infty$. Then by the concavity of u_n and v_n, we have

$$u_n(t) + v_n(t) \geq \frac{t_1}{4}\left(\|u_n\|_0 + \|v_n\|_0\right), \tag{3.19}$$

for $t \in S_0$. Let us choose $\eta_1 > (2\pi)^2/t_1^2\alpha\overline{h}_1$, where $\overline{h}_1 = \min_{t\in S_0} h_1(t)$. Then by (D_5), there exists $R_f > 0$ such that

$$f(u, v) \geq \eta_1(u + v) \quad \forall u + v \geq R_f. \tag{3.20}$$

Since $\|u_n\|_0 + \|v_n\|_0 > (4/t_1)R_f$ for sufficiently large n, (3.19) implies $u_n(t) + v_n(t) > R_f$ for $t \in S_0$. Thus for $t \in S_0$,

$$f(u_n(t), v_n(t)) > \eta_1(u_n(t) + v_n(t)) \geq \eta_1 u_n(t). \tag{3.21}$$

Hence we have for $t \in S_0$,

$$0 = u_n''(t) + \lambda_n h_1(t) f(u_n(t), v_n(t)) > u_n''(t) + \alpha\overline{h}_1\eta_1 u_n(t). \tag{3.22}$$

If we multiply by $\phi(t) = \sin(2\pi/t_1)(t-(t_1/4))$ both sides in the above inequality and integrate on S_0, then by the facts $\phi'(t_1/4) > 0, \phi'(3t_1/4) < 0$ and integration by part, we obtain

$$\begin{aligned} 0 &> \int_{t_1/4}^{3t_1/4} u_n''(t)\phi(t)dt + \alpha\overline{h}_1\eta_1 \int_{t_1/4}^{3t_1/4} u_n(t)\phi(t)dt \\ &\geq -\left(\frac{2\pi}{t_1}\right)^2 \int_{t_1/4}^{3t_1/4} u_n(t)\phi(t)dt + \alpha\overline{h}_1\eta_1 \int_{t_1/4}^{3t_1/4} u_n(t)\phi(t)dt. \end{aligned} \tag{3.23}$$

Thus $(2\pi/t_1)^2/\alpha\overline{h}_1 \geq \eta_1$ which is a contradiction to the choice of η_1. Suppose $\|u_n\|_1 + \|v_n\|_1 \to \infty$, then we also get a contradiction by a similar calculation with $\eta_2 > (2\pi)^2/(1-t_1)^2\ \alpha\widetilde{h}_1$, where $\widetilde{h}_1 = \min_{t\in S_1} h_1(t)$. Finally, the case $\beta > 0$ can also be proved by similar way using the condition $g_\infty = \infty$. □

Lemma 3.3. *Assume* (D_1), (D_3), *and*

(Q) $f(u, v_1) \leq f(u, v_2)$, *whenever* $v_1 \leq v_2$, $g(u_1, v) \leq g(u_2, v)$, *whenever* $u_1 \leq u_2$.

If problem (3.20) *has a positive solution at* $(\overline{\lambda},\overline{\mu})$. *Then the problem also has a positive solution at* (λ,μ) *for all* $(\lambda,\mu)\le(\overline{\lambda},\overline{\mu})$.

Proof. Let $(\overline{u},\overline{v})$ be a positive solution of problem (3.20) at $(\overline{\lambda},\overline{\mu})$ and let $(\lambda,\mu)\in\mathbb{R}_+^2\setminus\{(0,0)\}$ with $(\lambda,\mu)\le(\overline{\lambda},\overline{\mu})$. Then $(\overline{u},\overline{v})$ is an upper solution of (3.20) at (λ,μ). Define (α_u,α_v) by

$$\alpha_u(t)=\begin{cases}0, & t\in[0,t_1],\\ \dfrac{c}{1-t_1}(t-t_1), & t\in(t_1,1],\end{cases}$$
$$\alpha_v(t)=\begin{cases}0, & t\in[0,t_1],\\ \dfrac{d}{1-t_1}(t-t_1), & t\in(t_1,1].\end{cases}\tag{3.24}$$

Then (α_u,α_v) is a lower solution of problem (3.20) at (λ,μ). By the concavity of $(\overline{u},\overline{v})$, $(\overline{u},\overline{v})\ge(\alpha_u,\alpha_v)$. Therefore, Theorem 2.5 implies that problem (3.20) has a positive solution at (λ,μ). □

Lemma 3.4. *Assume* $(D_1)\sim(D_4)$ *and* (Q). *Then there exists* $(\lambda^*,\mu^*)>(0,0)$ *such that problem* (3.20) *has a positive solution for all* $(\lambda,\mu)\le(\lambda^*,\mu^*)$.

Proof. It is not hard to see that the following problem:

$$\begin{gathered}u''(t)+h_1(t)=0,\quad t\in(0,1),\ t\neq t_1,\\ v''(t)+h_2(t)=0,\quad t\in(0,1),\ t\neq t_1,\\ \Delta u|_{t=t_1}=I_u(u(t_1)),\qquad \Delta v|_{t=t_1}=I_v(v(t_1)),\\ \Delta u'\big|_{t=t_1}=N_u(u(t_1)),\qquad \Delta v'\big|_{t=t_1}=N_v(v(t_1)),\\ u(0)=a\ge0,\qquad v(0)=b\ge0,\qquad u(1)=c>a,\qquad v(1)=d>b\end{gathered}\tag{3.25}$$

has a positive solution so let (β_u,β_v) be a positive solution. Let $M_f=\sup_{t\in[0,1]}f(\beta_u(t),\beta_v(t))$ and $M_g=\sup_{t\in[0,1]}g(\beta_u(t),\beta_v(t))$. Then $M_f,M_g>0$ and for $(\lambda^*,\mu^*)=(1/M_f,1/M_g)$, we get

$$\begin{gathered}\beta_u''+\lambda^*h_1(t)f(\beta_u(t),\beta_v(t))=h_1(t)(\lambda^*f(\beta_u(t),\beta_v(t))-1)\le0,\\ \beta_v''+\mu^*h_2(t)g(\beta_u(t),\beta_v(t))=h_2(t)(\mu^*g(\beta_u(t),\beta_v(t))-1)\le0.\end{gathered}\tag{3.26}$$

This shows that (β_u,β_v) is an upper solution of (3.20) at (λ^*,μ^*). On the other hand, (α_u,α_v) given in Lemma 3.3 is obviously a lower solution and $(\alpha_u,\alpha_v)\le(\beta_u,\beta_v)$. Thus by Theorem 2.5, (3.20) has a positive solution at (λ^*,μ^*) and the proof is done by Lemma 3.3. □

We introduce a known existence result for a singular boundary value problem with no impulse effect.

Lemma 3.5 (see[9]). *Consider,* $(D_1), (D_5)$ *and* (Q). *For problem*

$$\begin{aligned} &u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \\ &v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,1), \\ &u(0) = a \geq 0, \qquad u(1) = c > a, \\ &v(0) = b \geq 0, \qquad v(1) = d > b, \end{aligned} \tag{U_T}$$

let $\mathcal{A}_T = \{(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\} \mid (3.27)$ *has a positive solution at* $(\lambda, \mu)\}$. *Then* $(\mathcal{A}_T, \leq)$ *is bounded above.*

Define $\mathcal{A} = \{(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\} \mid (3.20)$ *has a positive solution at* $(\lambda, \mu)\}$. *Then* $\mathcal{A} \neq \emptyset$ *by Lemma 3.4 and* $(\mathcal{A}, \leq)$ *is a partially ordered set.*

Lemma 3.6. *Assume* $(D_1) \sim (D_6)$. *Then* $(\mathcal{A}, \leq)$ *is bounded above.*

Proof. Suppose on the contrary that there exists a sequence $(\lambda_n, \mu_n) \in \mathcal{A}$ such that $|(\lambda_n, \mu_n)| \to \infty$. Let (u_n, v_n) be a positive solution of problem (3.20) at (λ_n, μ_n). By condition (D_2), we may choose sequences $(s_n), (t_n)$ in $[0, t_1) \cup (t_1, 1]$ such that if $I_u(u_n(t_1)) > 0$, then $t_n \in (t_1, 1]$ and

$$\begin{aligned} &I_u(u_n(t_1)) + (t_n - t_1) N_u(u_n(t_1)) = 0, \\ &I_u(u_n(t_1)) + (t - t_1) N_u(u_n(t_1)) > 0, \quad \text{on } [t_1, t_n), \\ &I_u(u_n(t_1)) + (t - t_1) N_u(u_n(t_1)) < 0, \quad \text{on } (t_n, 1]; \end{aligned} \tag{3.27}$$

if $I_u(u_n(t_1)) < 0$, then $t_n \in [0, t_1)$ and

$$\begin{aligned} &I_u(u_n(t_1)) + (t_n - t_1) N_u(u_n(t_1)) = 0, \\ &I_u(u_n(t_1)) + (t - t_1) N_u(u_n(t_1)) > 0, \quad \text{on } [0, t_n), \\ &I_u(u_n(t_1)) + (t - t_1) N_u(u_n(t_1)) < 0, \quad \text{on } (t_n, t_1]; \end{aligned} \tag{3.28}$$

if $I_v(v_n(t_1)) > 0$, then $s_n \in (t_1, 1]$ and

$$\begin{aligned} &I_v(v_n(t_1)) + (s_n - t_1) N_v(v_n(t_1)) = 0, \\ &I_v(v_n(t_1)) + (t - t_1) N_v(v_n(t_1)) > 0, \quad \text{on } [t_1, s_n), \\ &I_v(v_n(t_1)) + (t - t_1) N_v(v_n(t_1)) < 0, \quad \text{on } (s_n, 1]; \end{aligned} \tag{3.29}$$

if $I_v(v_n(t_1)) < 0$, then $s_n \in [0, t_1)$ and

$$\begin{aligned} &I_v(v_n(t_1)) + (s_n - t_1) N_v(v_n(t_1)) = 0, \\ &I_v(v_n(t_1)) + (t - t_1) N_v(v_n(t_1)) > 0, \quad \text{on } [0, s_n), \\ &I_v(v_n(t_1)) + (t - t_1) N_v(v_n(t_1)) < 0, \quad \text{on } (s_n, t_1]. \end{aligned} \tag{3.30}$$

If $I_u(u_n(t_1)) > 0$, define

$$\widetilde{u}_n(t) = \begin{cases} u_n(t), & \text{on } [0, t_1], \\ u_n(t) - (I_u(u_n(t_1)) + (t - t_1)N_u(u_n(t_1))), & \text{on } (t_1, t_n), \\ u_n(t), & \text{on } [t_n, 1], \end{cases} \tag{3.31}$$

and if $I_u(u_n(t_1)) < 0$, define

$$\widetilde{u}_n(t) = \begin{cases} u_n(t), & \text{on } [0, t_n], \\ u_n(t) + (I_u(u_n(t_1)) + (t - t_1)N_u(u_n(t_1))), & \text{on } (t_n, t_1], \\ u_n(t), & \text{on } (t_1, 1]. \end{cases} \tag{3.32}$$

Moreover, if $I_v(v_n(t_1)) > 0$, define

$$\widetilde{v}_n(t) = \begin{cases} v_n(t), & \text{on } [0, t_1], \\ v_n(t) - (I_v(v_n(t_1)) + (t - t_1)N_v(v_n(t_1))), & \text{on } (t_1, s_n), \\ v_n(t), & \text{on } [s_n, 1], \end{cases} \tag{3.33}$$

and if $I_v(v_n(t_1)) < 0$, define

$$\widetilde{v}_n(t) = \begin{cases} v_n(t), & \text{on } [0, s_n], \\ v_n(t) + (I_v(v_n(t_1)) + (t - t_1)N_v(v_n(t_1))), & \text{on } (s_n, t_1], \\ v_n(t), & \text{on } (t_1, 1]. \end{cases} \tag{3.34}$$

Then we can easily see that $(\widetilde{u}_n, \widetilde{v}_n) \in (C[0,1] \times C[0,1]) \cap (C^2(0,1) \times C^2(0,1))$ except t_1, t_n, s_n. Furthermore, $(\widetilde{u}'_n(t_1^-), \widetilde{v}'_n(t_1^-)) = (\widetilde{u}'_n(t_1^+), \widetilde{v}'_n(t_1^+)), (\widetilde{u}'_n(t_n^-), \widetilde{v}'_n(t_n^-)) \geq (\widetilde{u}'_n(t_n^+), \widetilde{v}'_n(t_n^+))$ and $(\widetilde{u}'_n(s_n^-), \widetilde{v}'_n(s_n^-)) \geq (\widetilde{u}'_n(s_n^+), \widetilde{v}'_n(s_n^+))$. We also see $(u_n(t), v_n(t)) \geq (\widetilde{u}_n(t), \widetilde{v}_n(t))$ on $[0,1]$. Thus by (D_6), we get

$$\begin{aligned} \widetilde{u}''_n(t) + \lambda_n h_1(t) f(\widetilde{u}_n(t), \widetilde{v}_n(t)) &= u''_n(t) + \lambda_n h_1(t) f(\widetilde{u}_n(t), \widetilde{v}_n(t)) \\ &= \lambda_n h_1(t) \big(f(\widetilde{u}_n(t), \widetilde{v}_n(t)) - f(u_n(t), v_n(t)) \big) \leq 0, \\ \widetilde{v}''_n(t) + \mu_n h_2(t) g(\widetilde{u}_n(t), \widetilde{v}_n(t)) &= v''_n(t) + \mu_n h_2(t) g(\widetilde{u}_n(t), \widetilde{v}_n(t)) \\ &= \mu_n h_2(t) \big(g(\widetilde{u}_n(t), \widetilde{v}_n(t)) - g(u_n(t), v_n(t)) \big) \leq 0. \end{aligned} \tag{3.35}$$

We also get $\widetilde{u}_n(0) = u_n(0) = a, \widetilde{u}_n(1) = u_n(1) = c, \widetilde{v}_n(0) = v_n(0) = b$, and $\widetilde{v}_n(1) = v_n(1) = d$. Thus $(\widetilde{u}_n, \widetilde{v}_n)$ is a G-upper solution of problem (U_T) at (λ_n, μ_n). If $I_u(u_n(t_1)) = 0$ or $I_v(v_n(t_1)) = 0$, then we consider $\widetilde{u}_n = u_n$ or $\widetilde{v}_n = v_n$ as a G-upper solution. Let $(\widetilde{\alpha}_u(t), \widetilde{\alpha}_v(t)) = ((c - a)t + a, (d - b)t + b)$, then $(\widetilde{\alpha}_u, \widetilde{\alpha}_v)$ is the G-lower solution of (3.27) at (λ_n, μ_n). Therefore, by Theorem 2.3, problem (3.27) has a positive solution for all (λ_n, μ_n). This contradicts to Lemma 3.5 and the proof is done. □

Lemma 3.7. *Assume* $(D_1) \sim (D_6)$. *Then every nonempty chain in* $\mathcal{A}$ *has a unique supremum in* $\mathcal{A}$.

Proof. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Without loss of generality, we may choose a distinct sequence $\{(\lambda_n, \mu_n)\} \subset \mathcal{C}$ such that $(\lambda_n, \mu_n) \leq (\lambda_{n+1}, \mu_{n+1})$. By Lemma 3.6, there exists $(\lambda_{\mathcal{C}}, \mu_{\mathcal{C}})$ such that $(\lambda_n, \mu_n) \to (\lambda_{\mathcal{C}}, \mu_{\mathcal{C}})$. If we show $(\lambda_{\mathcal{C}}, \mu_{\mathcal{C}}) \in \mathcal{A}$, then the proof is done. Since $\{(\lambda_n, \mu_n)\}$ is bounded, Lemma 3.2 implies that there is a constant B such that $\|(u_n, v_n)\| < B$, where (u_n, v_n) is a solution corresponding to (λ_n, μ_n). By the compactness of $T_{\lambda,\mu}$, $\{(u_n, v_n)\}$ has a convergent subsequence converging to say, $(u_{\mathcal{C}}, v_{\mathcal{C}})$. By Lebesgue Convergence theorem, we see that $(u_{\mathcal{C}}, v_{\mathcal{C}})$ is a solution of (3.20) at $(\lambda_{\mathcal{C}}, \mu_{\mathcal{C}})$. Thus $(\lambda_{\mathcal{C}}, \mu_{\mathcal{C}}) \in \mathcal{A}$. □

Theorem 3.8. *Assume* $(D_1) \sim (D_6.)$ *Then there exists a continuous curve* Γ *splitting* $\mathbb{R}^2_+ \setminus \{(0,0)\}$ *into two disjoint subsets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *such that problem* (P_T) *has at least one positive solution for* $(\lambda, \mu) \in \mathcal{O}_1 \cup \Gamma$ *and no solution for* $(\lambda, \mu) \in \mathcal{O}_2$.

Proof. (λ^*, μ^*) is given in Lemma 3.4. We know from Lemma 3.4 that (3.20) has a positive solution at $(0, s)$ for all $0 < s \leq \mu^*$. Thus $\{(0, s) \mid s > 0\} \cap \mathcal{A}$ is a nonempty chain in $\mathcal{A}$ and by Lemma 3.7, it has unique supremum of the form $(0, s^*)$ in $\mathcal{A}$. This implies that (3.20) has a positive solution at $(0, s)$ for all $0 < s \leq s^*$ and no solution at $(0, s)$ for all $s > s^*$. Similarly, there is $r^* \geq \lambda^*$ such that (3.20) has a positive solution at $(r, 0)$ for all $0 < r \leq r^*$ and no solution at $(r, 0)$ for all $r > r^*$. Define $L : \mathbb{R} \to \mathbb{R}^2$ by taking $L(t) = \{(r, s) \mid s = r + t\}$. Then for $t \in [-r^*, s^*]$, $L(t) \cap \mathcal{A}$ is a nonempty chain in $\mathcal{A}$. Define $\Gamma(t)$ as the unique supremum of $L(t) \cap \mathcal{A}$. Then Γ is well defined on $[-r^*, s^*]$ and as a consequence of Lemma 3.3, we see that Γ is continuous on $[-r^*, s^*]$, $\Gamma(-r^*) = (r^*, 0)$, and $\Gamma(s^*) = (0, s^*)$. Therefore, the curve $\Gamma = \Gamma[-r^*, s^*]$ separates $\mathbb{R}^2_+ \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$, where $\mathcal{O}_1$ is bounded and $\mathcal{O}_2$ is unbounded and we get the conclusion of this theorem for $\Gamma, \mathcal{O}_1$, and $\mathcal{O}_2$. □

4. Multiplicity

In this section, we study existence of the second positive solution for two point boundary value problem (3.20) with (λ, μ) in certain region of $\mathcal{O}_1$ appeared in Theorem 3.8. For the computation of fixed point index, we need to consider problems of the form

$$\begin{gathered}
u''(t) + \lambda h_1(t) f(u(t), v(t)) = 0, \quad t \in (0,1), \ t \neq t_1, \\
v''(t) + \mu h_2(t) g(u(t), v(t)) = 0, \quad t \in (0,1), \ t \neq t_1, \\
\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \Delta v|_{t=t_1} = I_v(v(t_1)), \\
\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \quad \Delta v'\big|_{t=t_1} = N_v(v(t_1)), \\
u(0) = a + \varepsilon, \qquad u(1) = c + \varepsilon, \\
v(0) = b + \varepsilon, \qquad v(1) = d + \varepsilon,
\end{gathered} \tag{P_T^ε}$$

where $\varepsilon > 0, c > a \geq 0$ and $d > b \geq 0$. Theorem 3.8 implies that there exists a continuous curve Γ_ε splitting $\mathbb{R}^2_+ \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_{\varepsilon,1}$ and $\mathcal{O}_{\varepsilon,2}$ such that the problem (4.1) has at least one positive solution for $(\lambda, \mu) \in \mathcal{O}_{\varepsilon,1} \cup \Gamma_\varepsilon$ and no solution for $(\lambda, \mu) \in \mathcal{O}_{\varepsilon,2}$. Using upper and lower solutions argument, we can easily show that if $0 < \bar{\varepsilon} < \varepsilon$, then $\mathcal{O}_{\varepsilon,1} \cup \Gamma_\varepsilon \subset \mathcal{O}_{\bar{\varepsilon},1} \cup \Gamma_{\bar{\varepsilon}}$. Let $\mathcal{O} = \cup_{\varepsilon>0}(\mathcal{O}_{\varepsilon,1} \cup \Gamma_\varepsilon)$, then $\mathcal{O} \subset \mathcal{O}_{0,1}$. We state the main theorem for two point boundary value problem (3.20) as follows.

Theorem 4.1. *Assume* $(D_1) \sim (D_6)$. *Then there exists a continuous curve* Γ_0 *splitting* $\mathbb{R}^2_+ \setminus \{(0,0)\}$ *into two disjoint subsets* $\mathcal{O}_{0,1}$ *and,* $\mathcal{O}_{0,2}$ *and there exists a subset* $\mathcal{O} \subset \mathcal{O}_{0,1}$ *such that problem* (3.20) *has at least two positive solutions for* $(\lambda,\mu) \in \mathcal{O}$, *at least one positive solution for* $(\lambda,\mu) \in (\mathcal{O}_{0,1} \setminus \mathcal{O}) \cup \Gamma_0$, *and no solution for* $(\lambda,\mu) \in \mathcal{O}_{0,2}$.

Proof. Let $\mathcal{O} = \cup_{\varepsilon>0}(\mathcal{O}_{\varepsilon,1} \cup \Gamma_\varepsilon)$ and let $(\lambda,\mu) \in \mathcal{O}$. It is enough to prove that problem (3.20) has the second solution at (λ,μ). By the definition of $\mathcal{O}$, there exists $\varepsilon > 0$ such that $(\lambda,\mu) \in \mathcal{O}_{\varepsilon,1} \cup \Gamma_\varepsilon$. That is (4.1) has a positive solution at (λ,μ). So let $(u_\varepsilon, v_\varepsilon)$ be a positive solution of problem (4.1) at (λ,μ) and let $\Omega = \{(u,v) \in X \mid -\varepsilon < u(t) < u_\varepsilon(t), -\varepsilon < v(t) < v_\varepsilon(t) \text{ for } t \in [0,1], u(t_1^+) < u_\varepsilon(t_1^+), v(t_1^+) < v_\varepsilon(t_1^+)\}$. Then Ω is bounded open in X, $0 \in \Omega$. Furthermore, $T_{\lambda,\mu} : \mathcal{K} \cap \overline{\Omega} \to \mathcal{K}$ is condensing, since it is completely continuous. We show that $T_{\lambda,\mu}(u,v) \neq \nu(u,v)$ for all $(u,v) \in \mathcal{K} \cap \partial\Omega$ and all $\nu \geq 1$. If it is not true, then there exist $(u,v) \in \mathcal{K} \cap \partial\Omega$ and $\nu_0 \geq 1$ such that $T_{\lambda,\mu}(u,v) = \nu_0(u,v)$. Thus (u,v) is a positive solution of the following equation

$$\begin{gathered}
\nu_0 u''(t) + \lambda h_1(t) f(u(t),v(t)) = 0, \quad t \in (0,1),\ t \neq t_1, \\
\nu_0 v''(t) + \mu h_2(t) g(u(t),v(t)) = 0, \quad t \in (0,1),\ t \neq t_1, \\
\nu_0 \,\Delta u|_{t=t_1} = I_u(u(t_1)), \qquad \nu_0 \,\Delta v|_{t=t_1} = I_v(v(t_1)), \\
\nu_0 \,\Delta u'\big|_{t=t_1} = N_u(u(t_1)), \qquad \nu_0 \,\Delta v'\big|_{t=t_1} = N_v(v(t_1)), \\
\nu_0 u(0) = a, \qquad \nu_0 v(0) = b, \qquad \nu_0 u(1) = c, \qquad \nu_0 v(1) = d,
\end{gathered} \tag{4.1}$$

and we can consider the following two cases. The first case is $u(t_0) = u_\varepsilon(t_0)$ or $v(t_0) = v_\varepsilon(t_0)$ for some $t_0 \in (0,1)$. The second case is $u(t_1^+) = u_\varepsilon(t_1^+)$ or $v(t_1^+) = v_\varepsilon(t_1^+)$. First, let us consider case $u(t_0) = u_\varepsilon(t_0)$ for some $t_0 \in (0,1)$. One may prove similarly for case $v(t_0) = v_\varepsilon(t_0)$. If $t_0 \in J'$, that is, $t_0 \neq t_1$, let $m(t) = (u - u_\varepsilon)(t)$, then $m''(t) \geq 0$ on J', $m(0) < 0$, $m(1) < 0$, and $m(t_1) \leq 0$. Thus on one of intervals $(0,t_1)$ or $(t_1,1)$ containing t_0, maximum principle implies $m \equiv 0$ and this contradicts to the facts of $m(0) < 0$ and $m(1) < 0$. If $t_0 = t_1$, then $u(t_1) = u_\varepsilon(t_1)$, $u(t) \leq u_\varepsilon(t)$ and $v(t) \leq v_\varepsilon(t)$ on $[0,1]$. Thus by (D_6) and (D_2), we get the following contradiction:

$$\begin{aligned}
u(t_1) = \frac{1}{\nu_0} A_\lambda(u,v)(t_1) &= \frac{a + (b-a)t_1}{\nu_0} + \frac{\lambda}{\nu_0}\int_0^1 K(t_1,s) h_1(s) f(u(s),v(s))\,ds \\
&\quad + \frac{-t_1\big(I_u(u(t_1)) + (1-t_1)N_u(u(t_1))\big)}{\nu_0} \\
&< a + \varepsilon + (b-a)t_1 + \lambda \int_0^1 K(t_1,s) h_1(s) f(u_\varepsilon(s),v_\varepsilon(s))\,ds \\
&\quad - t_1\big(I_u(u_\varepsilon(t_1)) + (1-t_1)N_u(u_\varepsilon(t_1))\big) \\
&= u_\varepsilon(t_1) = u(t_1).
\end{aligned} \tag{4.2}$$

Second, let us consider $u(t_1^+) = u_\varepsilon(t_1^+)$. Since $u(t) \le u_\varepsilon(t)$ and $v(t) \le v_\varepsilon(t)$, we get

$$\begin{aligned} u(t_1^+) = \frac{1}{\nu_0} A_\lambda(u,v)(t_1^+) &= \frac{a+(b-a)t_1}{\nu_0} + \frac{\lambda}{\nu_0}\int_0^1 K(t_1,s)h_1(s)f(u(s),v(s))ds \\ &\quad + \frac{(1-t_1)(I_u(u(t_1)) - t_1 N_u(u(t_1)))}{\nu_0} \\ &< a+\varepsilon+(b-a)t_1 + \lambda\int_0^1 K(t_1,s)h_1(s)f(u_\varepsilon(s),v_\varepsilon(s))ds \\ &\quad + (1-t_1)(I_u(u_\varepsilon(t_1)) - t_1 N_u(u_\varepsilon(t_1))) \\ &= u_\varepsilon(t_1^+) = u(t_1^+). \end{aligned} \tag{4.3}$$

One may show the contradiction similarly for case $v(t_1^+) = v_\varepsilon(t_1^+)$. This contradiction shows $T_{\lambda,\mu}(u,v) \ne \nu(u,v)$ for all $(u,v) \in \mathcal{K} \cap \partial\Omega$ and $\nu \ge 1$. Therefore, by Theorem 2.7, we obtain

$$i(T_{\lambda,\mu}, \mathcal{K}\cap\Omega, \mathcal{K}) = 1. \tag{4.4}$$

On the other hand, by Lemma 3.6, we know that there is (λ_1,μ_1) such that (3.20) has no positive solution at (λ_1,μ_1).Thus for any open set $\mathcal{U}$ in X, we get

$$i(T_{\lambda_1,\mu_1}, \mathcal{K}\cap\mathcal{U}, \mathcal{K}) = 0. \tag{4.5}$$

Let $\mathcal{R}$ be a compact rectangle containing (λ,μ) and (λ_1,μ_1). By Lemma 3.2, for all $(\overline{\lambda},\overline{\mu}) \in \mathcal{R}$, there exists $b_{\mathcal{R}} > 0$ such that all possible solutions (u,v) of (P_T) at $(\overline{\lambda},\overline{\mu})$ satisfy $\|(u,v)\| < b_{\mathcal{R}}$ and $\Omega \subset B_{b_{\mathcal{R}}}$. Define $h : [0,1] \times (\overline{B}_{b_{\mathcal{R}}} \cap \mathcal{K}) \to \mathcal{K}$ by

$$h(\tau,(u,v)) = T_{\tau\lambda_1+(1-\tau)\lambda,\tau\mu_1+(1-\tau)\mu}(u,v). \tag{4.6}$$

Then $h(0,(u,v)) = T_{\lambda,\mu}(u,v), h(1,(u,v)) = T_{\lambda_1,\mu_1}(u,v), h$ is completely continuous on $[0,1] \times \mathcal{K}$, and $h(\tau,(u,v)) \ne (u,v)$ for all $(\tau,(u,v)) \in [0,1] \times (\partial B_{b_{\mathcal{R}}} \cap \mathcal{K})$. By the property of homotopy invariance and (4.5), we have

$$i(T_{\lambda,\mu}, B_{b_{\mathcal{R}}}\cap\mathcal{K}, \mathcal{K}) = i(T_{\lambda_1,\mu_1}, B_{b_{\mathcal{R}}}\cap\mathcal{K}, \mathcal{K}) = 0. \tag{4.7}$$

By the additive property and (4.4), (4.7), we have

$$i\left(T_{\lambda,\mu}, \left(\frac{B_{b_{\mathcal{R}}}}{\overline{\Omega}}\right)\cap\mathcal{K}, \mathcal{K}\right) = -1. \tag{4.8}$$

Therefore, (3.20) has another positive solution in $(B_{b_{\mathcal{R}}}\setminus\overline{\Omega})\cap\mathcal{K}$ at $(\lambda,\mu) \in \mathcal{O}$ and this completes the proof. □

5. Applications

In this section, we apply the results in previous sections to study the existence and multiplicity theorems of positive radial solutions for impulsive semilinear elliptic problems.

5.1. On an Annular Domain

Let us consider

$$\Delta u + \lambda k_1(|x|) f(u,v) = 0,$$
$$\Delta v + \mu k_2(|x|) g(u,v) = 0, \quad \text{in } \Omega(l_1,l_2), |x| \neq r_1,$$
$$\Delta u|_{|x|=r_1} = I_u\left(u|_{|x|=r_1}\right), \quad \Delta v|_{|x|=r_1} = I_v\left(u|_{|x|=r_1}\right),$$
$$\left.\frac{\Delta \partial u}{\partial r}\right|_{|x|=r_1} = -\frac{r_1^{1-n} N_u\left(u|_{|x|=r_1}\right)}{m}, \qquad \left.\frac{\Delta \partial v}{\partial r}\right|_{|x|=r_1} = -\frac{r_1^{1-n} N_v\left(v|_{|x|=r_1}\right)}{m}, \qquad (P_A)$$
$$u(x) = a \geq 0, \qquad v(x) = b \geq 0 \quad \text{if } |x| = l_1,$$
$$u(x) = c > a, \qquad v(x) = d > b \quad \text{if } |x| = l_2,$$

where $f(0,0) = 0, g(0,0) = 0, \Delta$ is the Laplacian of $u, 0 < l_1 < r_1 < l_2$, and $\Omega(l_1,l_2) = \{x \in \mathbf{R}^n |\ l_1 < |x| < l_2\}$ with $n > 2$. $\partial u/\partial r$ denotes the differentiation in the radial direction, $\Delta u|_{|x|=r_1} = u(r_1^+) - u(r_1), \Delta(\partial u/\partial r)|_{|x|=r_1} = (\partial u/\partial r)(r_1^+) - (\partial u/\partial r)(r_1^-)$ and $m = -\int_{l_1}^{l_2} t^{1-n} dt$. Applying consecutive changes of variables, $r = |x|, s = -\int_r^{l_2} t^{1-n} dt$ and $t = m - s/m$, we may transform problem (5.1) into problem (3.20), where $t_1 = (r_1^{2-n} - l_1^{2-n})/(l_2^{2-n} - l_1^{2-n})$ and h_i can be written as

$$h_i(t) = m^2 [r(m(1-t))]^{2(n-1)} k_i(r(m(1-t))). \tag{5.1}$$

If $k_i : [l_1,l_2] \to (0,\infty)$ are continuous, then $h_i : [0,1] \to (0,\infty)$ are also continuous and satisfies (D_1). We may apply Theorem 4.1 to obtain the following result.

Corollary 5.1. *Assume $(D_2) \sim (D_6)$. Let $k_i \in C([l_1,l_2],(0,\infty)), i = 1,2$. Then there exists a continuous curve Γ_0 splitting $\mathbb{R}_+^2 \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_{0,1}$ and $\mathcal{O}_{0,2}$ and there exists a subset $\mathcal{O} \subset \mathcal{O}_{0,1}$ such that problem (5.1) has at least two positive solutions for $(\lambda,\mu) \in \mathcal{O}$, at least one positive solution for $(\lambda,\mu) \in (\mathcal{O}_{0,1} \setminus \mathcal{O}) \cup \Gamma_0$, and no solution for $(\lambda,\mu) \in \mathcal{O}_{0,2}$.*

If $k_i : (l_1,l_2) \to (0,\infty)$ are continuous and singular at $r = l_1$ and/or l_2, then h_i are also singular at $t = 0$ and/or 1. In this case, we assume

$$\int_{l_1}^{l_2} \left(r^{(2-n)} - l_1^{(2-n)}\right)\left(l_2^{(2-n)} - r^{(2-n)}\right) k_i(r) dr < \infty, \tag{5.2}$$

then we can easily check that both h_i satisfy (D_1) and apply Theorem 4.1 to obtain the following corollary.

Corollary 5.2. *Assume* $(D_2) \sim (D_6)$*. If both* $k_i \in C((l_1, l_2), (0, \infty))$ *satisfy*

$$\int_{l_1}^{l_2} \left(r^{2-n} - l_1^{2-n}\right)\left(l_2^{2-n} - r^{2-n}\right) k_i(r)\,dr < \infty. \tag{5.3}$$

Then the conclusion of Corollary 5.1 is valid.

5.2. On an Exterior Domain

Let us consider

$$\begin{gathered}
\Delta u + \lambda k_1(|x|) f(u, v) = 0, \\
\Delta v + \mu k_2(|x|) g(u, v) = 0, \qquad |x| > r_0, \quad |x| \neq r_1, \\
\Delta u|_{|x|=r_1} = I_u\left(u|_{|x|=r_1}\right), \qquad \Delta v|_{|x|=r_1} = I_v\left(v|_{|x|=r_1}\right), \\
\left.\frac{\Delta \partial u}{\partial r}\right|_{|x|=r_1} = \frac{n-2}{r_0}\left(\frac{r_1}{r_0}\right)^{1-n} N_u\left(u|_{|x|=r_1}\right), \\
\left.\frac{\Delta \partial v}{\partial r}\right|_{|x|=r_1} = \frac{n-2}{r_0}\left(\frac{r_1}{r_0}\right)^{1-n} N_v\left(v|_{|x|=r_1}\right), \\
u(x) = a \geq 0, \qquad v(x) = b \geq 0, \quad \text{if } |x| = r_0, \\
u(x) \longrightarrow c > a, \qquad v(x) \longrightarrow d > b, \quad \text{as } |x| \longrightarrow \infty,
\end{gathered} \tag{P_E}$$

where $f(0,0) = 0, g(0,0) = 0, 0 < r_0 < r_1$, and $n > 2$. Assume that both $k_i : [r_0, \infty) \to (0, \infty)$ are continuous. Applying changes of variables, $r = |x|$ and $t = 1 - (r/r_0)^{2-n}$, we may transform problem (5.4) into problem (3.20), where $t_1 = 1 - (r_0/r_1)^{n-2}$ and h_i are written as

$$h_i(t) = \frac{r_0^{2(n-1)}}{(n-2)^2}(1-t)^{-2(n-1)/(n-2)} k_i\left(r_0(1-t)^{-1/(n-2)}\right). \tag{5.4}$$

We know that h_i are singular at $t = 1$ and can easily check that h_i satisfy (D_1) if k_i satisfy $\int_{r_0}^{\infty} r k_i(r)\,dr < \infty$ for $i = 1, 2$. Thus by Theorem 4.1, we obtain the following result.

Corollary 5.3. *Assume* $(D_2) \sim (D_6)$*. If both* $k_i \in C([r_0, \infty), (0, \infty))$ *satisfy*

$$\int_{r_0}^{\infty} r k_i(r)\,dr < \infty, \tag{5.5}$$

then the conclusion of Corollary 5.1 is valid for problem (5.4).

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