Research Article

Multiple Solutions of p-Laplacian with Neumann and Robin Boundary Conditions for Both Resonance and Oscillation Problem

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We discuss Neumann and Robin problems driven by the p-Laplacian with jumping nonlinearities. Using sub-sup solution method, Fucík spectrum, mountain pass theorem, degree theorem together with suitable truncation techniques, we show that the Neumann problem has infinitely many nonconstant solutions and the Robin problem has at least four nontrivial solutions. Furthermore, we study oscillating equations with Robin boundary and obtain infinitely many nontrivial solutions.

1. Introduction

Let Ω be a bounded domain of R^n with smooth boundary $\partial\Omega$, we consider the following problems:

(i) Neumann problem:

$$-\Delta_p u + \alpha|u|^{p-2}u = f(x, u), \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \tag{p_1}$$

(ii) Robin problem:

$$-\Delta_p u + \alpha|u|^{p-2}u = f(x, u), \quad \text{in } \Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + b(x)|u|^{p-2}u = 0, \quad \text{on } \partial\Omega, \tag{p_2}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator of u with $1 < p < \infty$, $\alpha > 0$, $b(x) \in L^\infty(\partial\Omega)$, $b(x) \geq 0$, and $b(x) \neq 0$ on $\partial\Omega$, $f(x,0) = 0$ for a.e. $x \in \Omega$, and $\partial u/\partial v$ denotes the outer normal derivative of u with respect to $\partial\Omega$. Our purpose is to show the multiplicity of solutions to (p_1) and (p_2).

It is known that (p_1) and (p_2) are the Euler-Lagrange equations of the functionals

$$J_1(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha}{p}\int_\Omega |u|^p dx - \int_\Omega F(x,u)dx,$$

$$J_2(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha}{p}\int_\Omega |u|^p dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p ds - \int_\Omega F(x,u)dx, \tag{1.1}$$

respectively, defined on the Sobolev space $W^{1,p}(\Omega)$, where $F(x,u) = \int_0^u f(x,s)ds$. The critical points of functionals correspond to the weak solutions of problems. In Li [1] and Zhang et al. [2], the authors study the existence and multiple solutions of (p_1) and (p_2) using the critical points theory for the semilinear case $p = 2$. There also have been some papers dealing with the quasilinear case $p \neq 2$ using the critical point theory, and some existence results of solutions have been generalized to this case in the work of Perera [3], Zhang et al. [4], and Zhang-Li [5]. Most of these papers use the minimax arguments, and nontrivial solutions are obtained with the assumption that the nonlinearity is superlinear at 0. In this paper, we give the nontrivial solutions of (p_1) and (p_2) with a jumping nonlinearity when the asymptotic limits of the nonlinearity fall in the regions formed by the curves of the Fucík spectrum. Our technique is based on mountain pass theorem, computing the critical groups and Fucík spectrum.

Our general assumptions are the following.

(f_1) There is constant $C > 0$ such that $f(x,t)$ satisfies the following subcritical conditions:

$$|f(x,t)| \leq C(|t|^q + 1) \quad \text{for every } x \in \Omega, \ t \in R, \tag{1.2}$$

with $p - 1 < q < p^* - 1$, where $p^* = np/(n-p)$ if $n > p$, and $p^* = \infty$ if $n = 1, 2, \ldots, p$.

(f_2) $\exists$ sequence $\{a_i\}$ and $\{b_i\}$, where $a_i, b_i \in R$, $i = 1, 2, \ldots$, which satisfy $a_i > 0$, $b_i < 0$ and $a_i \nearrow +\infty$, $b_i \searrow -\infty$ as $i \to \infty$. And at the same time $\{a_i\}$, $\{b_i\}$ satisfy

$$f(x,a_i) = \alpha a_i^{p-1}, \quad f(x,b_i) = -\alpha|b_i|^{p-1}, \quad \text{for every } x \in \Omega \tag{1.3}$$

which means that $\{a_i\}$, $\{b_i\}$ are constant solution sequences of (p_1).

Let $a_0 = b_0 = 0$, $f(x,t) < \alpha t^{p-1}$ if $t \in (a_i, a_{i+1})$, where i is an odd number, $i \geq 1$; $f(x,t) > \alpha t^{p-1}$ if $t \in (a_i, a_{i+1})$, where i is an even number, $i \geq 0$; $f(x,t) < -\alpha|t|^{p-1}$ if $t \in (b_{i+1}, b_i)$, where i is an even number, $i \geq 0$; $f(x,t) > -\alpha|t|^{p-1}$ if $t \in (b_{i+1}, b_i)$, where i is an odd number, $i \geq 1$, for every $x \in \Omega$.

(f_3) For all $t \neq a_i, b_i$, f is C^1; $f'_-(x,a_i) \neq f'_+(x,a_i)$, $f'_-(x,b_i) \neq f'_+(x,b_i)$, where i is an even number, $i \geq 2$, $f'_-(x,t)$, $f'_+(x,t)$ denote the left and the right derivatives of f at t, respectively.

(f_4) Let $(a, b) = (f'_+(x, a_i) - \alpha, f'_-(x, a_i) - \alpha)$ for i is an even number, $i \geq 2$. For $(a, b) \in R^2$, the problem

$$-\Delta_p u = a\big[(u - c)^+\big]^{p-1} - b\big[(u - c)^-\big]^{p-1}, \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega,$$

(1.4)

only has constant solution c, where $(u-c)^\pm(x) = \max\{\pm(u-c), 0\}$ and c is a constant.

And $f'_{i-}(x, a_i) - \alpha > \lambda_2$, $f'_{i+}(x, a_i) - \alpha > \lambda_2$ for i is an even number, $i \geq 2$, where

$$f_i(x, t) = \begin{cases} 0, & t < 0, \\ f(x, t), & 0 \leq t \leq a_i, \\ f(x, a_i), & t > a_i, \end{cases}$$

(1.5)

and $f'_{i-}(x, a_i), f'_{i+}(x, a_i)$ denote the left and the right derivatives of f_i at a_i, respectively, and λ_2 is the second of the eigenvalue problems with Neumann boundary value condition.

(f_5) $\exists m > \alpha$, such that $f(x, t) + m|t|^{p-2}t$ is increasing in t.

In particular, from (f_2), we know that (p_1) has infinitely many constant solutions, a.e., $\{a_i\}, \{b_i\}, i = 0, 1, 2, \ldots$. In this paper, we mainly discuss whether it has many nonconstant solutions and what their locations are.

Then we have the main results of this paper.

Theorem 1.1. *Assume that (f_1)–(f_5) hold. Then (p_1) has infinitely many nonconstant solutions. Moreover, if one chooses some order intervals which have two pairs of strict constant sub-sup solutions, then (p_1) has at least two nonconstant solutions in some order intervals.*

Furthermore, if we assume that $f'_-(x, 0) \neq f'_+(x, 0)$ under the same conditions as in Theorem 1.1, we can have at least one sign-changing solution which is of mountain pass type from the mountain pass theorem in order interval. When we discuss multiple solutions of (p_1), we notice that there may be infinitely many sign-changing solutions under stronger assumptions. In fact, if we give more assumptions, we can obtain infinitely many sign-changing solutions.

We assume the following.

(F) $F(x, t) > ((\lambda_2 + \alpha + \varepsilon_0)/p)t^p$, $|t| \geq M$, M is large enough, where λ_2 is the second eigenvalue of Neumann problem of $-\Delta_p$ and $\varepsilon_0 > 0$.

Corollary 1.2. *Under the same conditions as in Theorem 1.1, (F) and $f'_-(x, 0) \neq f'_+(x, 0)$, then one can get infinitely many sign-changing solutions for (p_1) which are of mountain pass type or not mountain pass type but with positive local degree.*

For the Robin problem, if $\exists M_1 > 0, M_2 > 0$ such that $f(x, M_1) = 0, f(x, -M_2) = 0$ for a.e. $x \in \Omega$, then we give the following assumptions:

(g_1) $f \in C^1(\overline{\Omega} \times R^1 \setminus \{0\})$, $f'_-(x,0) \neq f'_+(x,0)$, and $\min\{f'_+(x,0), f'_-(x,0)\} > \lambda_1 + \alpha$ for a.e. $x \in \Omega$, where $f'_-(x,0)$, $f'_+(x,0)$ denote the left and the right derivatives of f at 0, respectively, and λ_1 is the first eigenvalue of Robin problem of $-\Delta_p$;

(g_2) let $(a,b) = (f'_+(x,0) - \alpha, f'_-(x,0) - \alpha)$. For $(a,b) \in R^2$, the problem

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1}, \quad \text{in } \Omega,$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + b(x)|u|^{p-2}u = 0, \quad \text{on } \partial\Omega,$$

(1.6)

only has trivial solution 0, where $u^\pm(x) = \max\{\pm u, 0\}$.

In this case, we have the following.

Theorem 1.3. *Assume that (f_1), (f_5), (g_1), (g_2) hold. Then one has at least four nontrivial solutions of problem (p_2).*

Furthermore, we give the following stronger assumption:

(F') $F(x,t) > [(\lambda_2 + \alpha + \varepsilon_0)/p + \tilde{C}]t^p$, $|t| \geq M$, $F(x,u) = \int_0^u f(x,s)ds$, $u \in E_2$, where $E_2 = \{u \in W^{1,p}(\Omega) : u = k\varphi_1 + t\varphi_2\}$, $\tilde{C} \gg (C^2/2)\|b(x)\|_{L^\infty(\partial\Omega)}$. Here C is the imbedding constant of Sobolev Trace Theorem (see [6]), M is large enough, ε_0 is small enough, λ_2 is the second of the eigenvalue problems with Robin boundary value condition, and φ_1, φ_2 are the first and the second eigenfunction, respectively.

Then we have the following.

Corollary 1.4. *Assume that f is satisfied as in Theorem 1.3 and (F'), then one can have infinitely many sign-changing solutions for (p_2) which are of mountain pass type or not mountain pass type but with positive local degree.*

In the oscillating problems of Robin boundary, a.e., (f_2) holds. We make the following assumption.

(F'') $\int_\Omega F(x,t\varphi_1)dx \geq ((\lambda_1 + \alpha + \varepsilon_0)/p + \tilde{C})t^p \int_\Omega \varphi_1^p dx$, $|t| \geq M$, where φ_1 is the first eigenvalue of the Robin problem and $\int_\Omega \varphi_1^p dx = 1$.

Then we have the following.

Theorem 1.5. *Assume that f is satisfied as in Theorem 1.3 and (f_2), (F''), one can get infinitely many nontrivial solutions of problem (p_2). Some of them are minimum points; others are mountain pass points.*

2. Preliminaries

Now we recall the notion of critical groups of an isolated critical point u of a C^1 functional J briefly. Let $U \subset M$ be an isolated neighborhood of u such that there are no critical points of J in $U \setminus \{u\}$; M is a Banach space. The critical groups of u are defined as

$$C_q(J,u) = H_q(J^c \cap U, (J^c \setminus \{u\}) \cap U; G), \quad q = 0, 1, 2, \ldots,$$

(2.1)

where $c = J(u)$ and $J^c = \{u \in M | J(u) \leq c\}$ is a level set of J and $H_q(X, Y; G)$ are singular relative homology groups with a Abelian coefficient group G, $Y \subset X$, $q = 0, 1, 2, \ldots$. They are independent of the choices of U, hence are well defined. Use $H^q(X; G)$ to stand for the qth singular cohomology group with an Abelian coefficient group G; from now on we denote it by $H^q(X)$. Assume that $J \in C^2(M, R)$, and a critical point u is called nondegenerate if the Hessian $J''(u)$ at this point has a bounded inverse. Let u be a nondegenerate critical point of J; we call the dimension of the negative space corresponding to the spectral decomposition of $J''(u)$, that is, the dimension of the subspace of negative eigenvectors of $J''(u)$, the Morse index of u, and denote it by $\text{ind}(J''(u))$. If $C_1(J, u) \neq 0$, then we call an isolated critical point u of J as a mountain pass point. For the details, we refer to [7].

We have the following basic facts on the critical groups for an isolated critical point of J.

(a) Let u be is an isolated minimum point of J, then $C_q(J, u) = \delta_{q0} G$.

(b) If $J \in C^2(M, R)$ and u is a nondegenerate critical point of J with Morse index j, then $C_q(J, u) = \delta_{qj} G$.

Definition 2.1. If any sequence $\{u_k\} \subset M$ which satisfies $J(u_k) \to c$ and $J'(u_k) \to 0$ ($k \to \infty$) has a convergent subsequence, one says that J satisfies the $(PS)_c$ condition. If J satisfies $(PS)_c$ condition for all $c \in R$, one says that J satisfies the (PS) condition.

Lemma 2.2 (see [8]). *Assume that $\underline{u}$ and $\overline{u}$ are, respectively, lower and upper solutions for the problem*

$$-\Delta_p u = g(x, u), \quad in\ \Omega,$$

$$|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + b(x)|u|^{p-2} u = 0, \quad on\ \partial\Omega, \tag{2.2}$$

with $\underline{u} \leq \overline{u}$ a.e. in Ω, where $g(x, s)$ is a Carathéodory function on $\Omega \times R$ with the property that, for any $s_0 > 0$, there exists a constant A such that $|g(x, s)| \leq A$ for a.e. $x \in \Omega$ and all $s \in [-s_0, s_0]$. Consider the associated functional

$$\Phi(u) := \frac{1}{p} \int_\Omega |\nabla u|^p - \int_\Omega G(x, u), \tag{2.3}$$

where $G(x, u) := \int_0^s g(x, t) dt$ and the interval $M := \{u \in W^{1,P}(\Omega) : \underline{u} \leq u \leq \overline{u}$ a.e. in $\Omega\}$. Then the infimum of Φ on M is achieved at some u, and such a u is a solution of the above problem.

In what follows, we set $X = W^{1,p}(\Omega)$ which is is uniformly convex ($1 < p < \infty$) and equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^p dx + (m+\alpha) \int_\Omega |u|^p dx)^{1/p}$. Let E be a Hilbert space and $P_E \subset E$ a closed convex cone such that X is densely embedded in E. Assume that $P = X \cap P_E$, P has nonempty interior $\mathring{P}$ and any order interval is bounded. It is well known that (PS) condition implies the compactness of the critical set at each level $c \in R$, on the case of the above condition. Then we assume the following:

(J_1) $J \in C^2(E, R)$ and satisfies (PS) condition in E and deformation property in X;

(J_2) $\nabla J = id - K_E$, where $K_E : E \to E$ is compact. $K_E(X) \subset X$ and the restriction $K = K_E|_X : X \to X$ is of class C^1 and strongly preserving, that is, $u \gg v \Leftrightarrow u - v \in \dot{P}$;

(J_3) J is bounded from below on any order interval in X.

Lemma 2.3 (Mountain pass theorem in half-order intervals, sup-solutions case (see [9])). *Suppose that J satisfies (J_1)–(J_3). $v_1 < v_2$ is a pair of strict supersolution of $\nabla J = 0$. $v_0 < v_1$ is a subsolution of $\nabla J = 0$. Suppose that $[v_0, v_1]$ and $[v_0, v_2]$ are admissible invariant sets for J. If J has a local strict minimizer w in $[v_0, v_2] \setminus [v_0, v_1]$. Then J has mountain pass points u_0 in $[v_0, v_2] \setminus [v_0, v_1]$.*

Lemma 2.4 (Mountain pass theorem in order intervals (see [10])). *Suppose that J satisfies (J_1)–(J_3) and $\{v_1, v_2\}, \{w_1, w_2\}$ are two pairs of strict sub-sup solutions of $\nabla J = 0$ in X with $v_1 < w_2$, $[v_1, v_2] \cap [w_1, w_2] = \emptyset$. Then J has a mountain pass point $u_0, u_0 \in [v_1, w_2] \setminus ([v_1, v_2] \cup [w_1, w_2])$. More precisely, let v_0 be the maximal minimizer of J in $[v_1, v_2]$ and w_0 the minimal minimizer of J in $[w_1, w_2]$. Then $v_0 \ll u_0 \ll w_0$. Moreover, $C_1(J, u_0)$, the critical group of J at u_0, is nontrivial.*

Remark 2.5. (a) Lemma 2.4 still holds if $J \in C^1(E, R)$, K is of class C^0 (see [10]).

(b) For $X = W^{1,p}(\Omega)$, we define $g_p(t) := |t|^{p-2}t$. From assumption (f_5), there exists $m > \alpha$ such that $f(x, u) - \alpha|u|^{p-2}u + mg_p(u)$ is strictly increasing in u. The assumption is not essential but is assumed for simplicity. If such m does not exist then we can approximate f by a sequence of functions so that m as above exists, and obtain the solutions by passing to limits. For $m > \alpha$, we need the operator

$$A_m : X \longrightarrow X, \quad A_m(u) = \left(-\Delta_p + mg_p(\cdot)\right)^{-1}\left((x, u) + mg_p(u)\right). \tag{2.4}$$

From [11], we know that A_m is compact, that is, it is continuous and maps bounded subsets of X into relatively compact subsets of X. Since $-\Delta_p u + mg_p(u)$ is a positive operator,

$$K := \left(-\Delta_p u + mg_p(u)\right)^{-1}\left(f(x, u) - \alpha|u|^{p-2}u + mg_p(u)\right) \tag{2.5}$$

is strongly orderpreserving. From the above discussion, we have the mountain pass theorem in order intervals of J_1 and J_2.

Next, let us recall some notions and known results on Fucík spectrum.

The Fucík spectrum of p-Laplacian on $W^{1,p}(\Omega)$ is defined as the set Σ_p of those points $(a, b) \in R^2$ for which the problem

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1}, \quad u \in W^{1,p}(\Omega) \tag{2.6}$$

has nontrivial solutions. Here $u^{\pm}(x) = \max\{\pm u, 0\}$.

For the semilinear case $p = 2$, it is known that Σ_2 consists, at least locally, of curves emanating from the points (λ_l, λ_l) where $\{\lambda_l\}_{l \in N}$ are the distinct eigenvalues of $-\Delta$ (see, e.g., [12]). It was shown in Schechter [13] that Σ_2 contains continuous and strictly decreasing curves C_{l_1}, C_{l_2} through (λ_l, λ_l) such that the points in the square $Q_l = (\lambda_{l-1}, \lambda_{l+1})^2$ that are either below the lower curve C_{l_1} or above the upper curve C_{l_2} are free of Σ_2, while the points on the curves are in Σ_2 when they do not coincide. The points in the region between the curves may or may not belong to Σ_2.

As shown in Lindqvist [14] that the first eigenvalue λ_1 of $-\Delta_p$ is positive, simple and admits a strictly positive eigenfunction φ_1, so Σ_p contains the two lines $\lambda_1 \times R$ and $R \times \lambda_1$. This generalized notion of spectrum was introduced for the semilinear case $p = 2$ in the 1970s by Fucík [12] in connection with jumping nonlinearities. A first nontrivial curve C_2 in Σ_p through (λ_2, λ_2) that is continuous, strictly decreasing, and asymptotic to $\lambda_1 \times R$ and $R \times \lambda_1$ at infinity was constructed and variationally characterized by a mountain-pass procedure in Cuesta et al. [15].

Consider the problem

$$-\Delta_p u = a\big[(u-c)^+\big]^{p-1} - b\big[(u-c)^-\big]^{p-1}, \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial v} = 0, \quad \text{on } \partial\Omega,$$

(2.7)

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1}, \quad \text{in } \Omega,$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial v} + b(x)|u|^{p-2}u = 0, \quad \text{on } \partial\Omega,$$

(2.8)

from the variational point of view; solutions of (2.7) and (2.8) are the critical points of the functional

$$I_1(u) = I_1(u, a, b) = \int_\Omega \left(|\nabla u|^p - a\big[(u-c)^+\big]^p - b\big[(u-c)^-\big]^p\right)dx,$$

$$I_2(u) = I_2(u, a, b) = \int_\Omega \left(|\nabla u|^p - a(u^+)^p - b(u^-)^p\right)dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p ds,$$

(2.9)

respectively, where c is a constant.

If (a, b) does not belong to Σ_p, c is the constant solution of (2.7), that is, c is an isolated critical point of I_1; 0 is the trivial solution of (2.8), that is, 0 is an isolated critical point of I_2, then from the definition of critical group, we have the $C_q(I_1, c)$ and $C_q(I_2, 0)$ defined, $q = 0, 1, 2, \ldots$. Now, we give some results relative to the computation of the critical groups which are the results of Dancer and Perera [16]. Let $C_{11} = ((-\infty, \lambda_1] \times \lambda_1) \cup (\lambda_1 \times (-\infty, \lambda_1))$ and $C_{12} = (\lambda_1 \times [\lambda_1, +\infty)) \cup ((\lambda_1, +\infty) \times \lambda_1)$.

Lemma 2.6. (i) *If (a, b) lies below C_{11}, then $C_q(I, c) = \delta_{q0} Z$.*
     (ii) *If (a, b) lies between C_{11} and C_{12}, then $C_q(I, c) = 0$ for all q.*
     (iii) *If (a, b) lies between C_{12} and C_2, then $C_q(I, c) = \delta_{q1} Z$.*
     (iv) *If (a, b) does not belong to Σ_p, but lies above C_2, then $C_q(I, c) = 0$ for $q = 0, 1$.*

Denote

$$I_s(u) = \int_\Omega |\nabla u|^p - s|(u - c)^+|^p, \quad u \in X, \tag{2.10}$$

and $\widetilde{I}_s$ is the restriction of I_s to the C^1 manifold

$$S = \left\{ u \in X : \int_\Omega |u - c|^p = 1 \right\}. \tag{2.11}$$

As noted in [16], the critical groups of I are related to the homology groups of sublevel sets of $\widetilde{I}_{a-b}$. We have that

$$I|_S = \widetilde{I}_{a-b} - b, \tag{2.12}$$

so the sublevel sets

$$I^d = \{ u \in X : I(u) \le d \}, \qquad \widetilde{I}_s^d = \left\{ u \in S : \widetilde{I}_s \le d \right\} \tag{2.13}$$

are related by

$$I^d \cap S = \widetilde{I}_{a-b}^{d+b}. \tag{2.14}$$

Lemma 2.7. *If (a, b) does not belong to Σ_p, then*

$$C_q(I, c) \cong \begin{cases} \delta_{q0} Z, & \text{if } \widetilde{I}_{a-b}^b = \emptyset, \\ \widetilde{H}_{q-1}\left(\widetilde{I}_{a-b}^b \right), & \text{otherwise,} \end{cases} \tag{2.15}$$

where $\widetilde{H}_q$ denote reduced homology groups. It also holds with $\widetilde{I}_{a-b}^b$ replaced by $\bigcirc_b = \{ u \in S : \widetilde{I}_{a-b}(u) > b \}$.

3. The Proof of the Main Results

Let

$$
f_i(x,t) = \begin{cases} 0, & t < 0, \\ f(x,t), & 0 \le t \le a_i, \\ f(x,a_i), & t > a_i. \end{cases} \qquad F_i(x,t) = \int_0^t f_i(x,s)ds
$$

$$
J_{1i}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha}{p}\int_\Omega |u|^p dx - \int_\Omega F_i(x,u)dx.
$$

(3.1)

It is well known that critical points of J_{1i} correspond to weak solutions of the following equation:

$$
\begin{aligned}
-\Delta_p u + \alpha|u|^{p-2}u &= f_i(x,u), \quad \text{in } \Omega, \\
\frac{\partial u}{\partial v} &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
$$

(p')

We have that $f_i(x,t) \in C^0(R,R)$ and $J_{1i} \in C^1(E,R)$. We can discuss similar case for b_i.

Next, we give the relation of the solutions of (p') and the solutions of (p_1), that is, Lemma 3.2 below. In order to prove Lemma 3.2, we firstly give the comparison principle.

Let

$$
L_p := -\Delta_p + a(x)|u|^{p-2}u,
$$

$$
\lambda_{1,p}(a) = \inf\left\{\int_\Omega [|\nabla u|^p + a(x)|u|^p]dx, u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p dx = 1\right\}.
$$

(3.2)

Lemma 3.1 (comparison principle (see [17])). *Assume* $a \in L^\infty(\Omega)$, $\lambda_{1,p}(a) > 0$. *The* $L_p u \in L^\infty(\Omega)$ *with* $u|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$, *and* $L_p u \le 0$ *with* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, *then* $u \le 0$.

Lemma 3.2. *If* $u_i(x)$ *is a solution of* (p'), *then* $u_i(x)$ *is also a solution of* (p_1) *and satisfies* $0 \le u_i(x) \le a_i$, $i = 1, 2, \ldots$.

Proof. Suppose that the conclusion is false. Now, consider the domain $U_i = \{x \in \Omega \mid u_i(x) > a_i\}$, then we have

$$
\begin{aligned}
-\Delta_p u &= f_i(x,u) - \alpha|u|^{p-2}u \le 0, \quad \text{in } U_i, \\
u &= a_i, \quad \text{on } \partial U_i,
\end{aligned}
$$

(3.3)

where $-\Delta_p u = f_i(x,u) - \alpha|u|^{p-2}u = f(x,a_i) - \alpha|u|^{p-2}u \le f(x,a_i) - \alpha a_i^{p-1} = 0$, $x \in U_i$ by the definition of $f_i(x,u)$. By the comparison principle, we can conclude that $u_i(x) \le 0$ in U_i. It is a contradiction, so we have that $U_i = \emptyset$, that is, $u_i(x) \le a_i$.

Similarly, we consider $V_i = \{x \in \Omega \mid u_i(x) < 0\}$, by the comparison principle; we also get the contradiction, so we have that $V_i = \emptyset$, that is, $u_i(x) \ge 0$. From the above discussion, we

have that $0 \leq u_i(x) \leq a_i$, $i = 1, 2, \ldots$ and $f_i(x, u) = f(x, u_i)$, so $u_i(x)$ is a solution of (p_1). This completes the proof of the lemma.   □

Remark 3.3. From the above discussion, by applying Lemma 3.2, we know that solutions of (p') are also the solutions of (p_1) if we want to prove Theorem 1.1, we only need to prove that (p') has infinitely many nonconstant solutions under the assumptions as in Theorem 1.1 and (p') has two nonconstant solutions in every order interval.

Theorem 3.4. *There are infinitely many nonconstant solutions of (p'). Moreover, if there exists some order intervals which have two pairs of strict constant sub-sup solutions, then there are at least two nonconstant solutions in these order intervals.*

Proof. We treat the case of a_i; the other case of b_i is proved by a similar argument.
    If (f_2) holds, then

$$-\Delta_p a_i = 0 = f_i(x, a_i) - \alpha a_i^{p-1}, \quad \text{for a.e. } x \in \Omega, \tag{3.4}$$

so $\{a_i\}$ are all positive constant solutions of (p'). Assuming that i is large enough and i is an even number, we also infer that $\{a_{2k-1}\}$ are local minimums, $k = 1, 2, \ldots, i/2$. So we get $\underline{u}_{2k-1}$ and $\overline{u}_{2k-1}$ a strict subsolution and sup-solution pair for (p'), satisfying $\underline{u}_{2k-1} < a_{2k-1} < \overline{u}_{2k-1}$ for each k, $k = 1, 2, \ldots, i/2$.
    Now, we study the order interval $[\underline{u}_1, \overline{u}_3]$ in X which includes two suborder intervals $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_3, \overline{u}_3]$, $a_2 \in [\underline{u}_1, \overline{u}_3]$.
    We infer that $J_{1i}(u)$ satisfies deformation properties and is bounded from below on $[\underline{u}_1, \overline{u}_3]$ and so we get a mountain pass point $u_1 \in [\underline{u}_1, \overline{u}_3] \setminus ([\underline{u}_1, \overline{u}_1] \cup [\underline{u}_3, \overline{u}_3])$ according to mountain pass theorem in order interval, we have that $C_1(J_{1i}, u_1)$ is nontrivial.
    From assumption (f_3), we know that the left and the right derivatives of f_i at a_2 are different; we consider the problem

$$-\Delta_p u = f_i(x, u) - \alpha |u|^{p-2} u, \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega, \tag{3.5}$$

where $f_i \in C(\overline{\Omega} \times R)$ and as $u \to a_2$ we have

$$f_i(x, u) - \alpha |u|^{p-2} u = (f'_{i+}(x, a_2) - \alpha) \left[(u - a_2)^+ \right]^{p-1}$$
$$- (f'_{i-}(x, a_2) - \alpha) \left[(u - a_2)^- \right]^{p-1} + \circ \left(|u - a_2|^{p-1} \right). \tag{3.6}$$

We take $a = f'_{i+}(x, a_2) - \alpha$, $b = f'_{i-}(x, a_2) - \alpha$, then from assumption (f_4) and the definition of Σ_p, we know that (a, b) does not belong to Σ_p. So, we have the following.

    (1) If (a, b) does not belong to Σ_p, but lies above C_2, then

$$C_q(J_{1i}, a_2) = 0 \quad \text{for } q = 0, 1 \tag{3.7}$$

by Lemma 2.6(iv). In this case, $C_1(J_{1i}, a_2) = 0$, so $C_q(J_{1i}, a_2) \ncong C_q(J_{1i}, u_1)$, and we have $u_1 \neq a_2$.

(2) Denote

$$J_{a-b}(u) = \int_\Omega |\nabla u|^p - (a-b)|(u-a_2)^+|^p, \quad u \in X, \tag{3.8}$$

and $\widetilde{J}_{a-b}$ is the restriction of J_{a-b} to the C^1 manifold

$$S = \left\{ u \in X : \int_\Omega |u - a_2|^p = 1 \right\}, \tag{3.9}$$

where $a = f'_{i+}(x, a_2) - \alpha, b = f'_{i-}(x, a_2) - \alpha$ as shown above.

From (f_4), we know that (a, b) does not belong to Σ_p, and if $\widetilde{J}_{a-b}(u) > b$, a.e. $\widetilde{J}^b_{a-b} = \emptyset$, then

$$C_q(J_{1i}, a_2) = \delta_{q0} Z \tag{3.10}$$

by Lemma 2.7. In this case, $C_1(J_{1i}, a_2) = 0$, so $C_q(J_{1i}, a_2) \ncong C_q(J_{1i}, u_1)$, and we have $u_1 \neq a_2$.

Similarly, applying the mountain pass theorem in order interval to $[\underline{u}_3, \overline{u}_5]$ which contain two sub-order intervals $[\underline{u}_3, \overline{u}_3]$ and $[\underline{u}_5, \overline{u}_5]$, we get a mountain pass point u_2 and prove that $C_q(J_{1i}, a_4) \ncong C_q(J_{1i}, u_2)$, so $u_2 \neq a_4$ from Lemmas 2.6 and 2.7.

We let the procedure go on. So $i/2 - 1$ mountain pass points are available which are nonconstant solutions of (p'), where i is large enough and i is an even number. Then we have infinitely many nonconstant positive solutions of (p') by the arbitrary of i.

We can discuss the similar case for b_i and get infinitely many nonconstant negative solutions.

Now, we discuss the solutions in $[\underline{u}_1, \overline{u}_3]$ more deeply. Since u_1 is a mountain pass point, for the Leray-Schauder degree of $id - K^i$, we have the computing formular

$$\deg\left(id - K^i, B(u_1, r), 0\right) = -1, \tag{3.11}$$

where $r > 0$ is small enough, $K^i = (-\Delta_p + (m+\alpha)g_p(\cdot))^{-1} f_i^*|_X : X \to X$ is of class C^0 and strongly preserving, $f_i^*(x, u) = f_i(x, u) + mg_p(u)$ (see Remark 2.5(b)). Then according to Poincaré-Hopf formular for C^1 case and the computation of $C_q(J_{1i}, a_2)$, we have

$$\text{index}(J_{1i}, a_2) = (-1)^l. \tag{3.12}$$

Furthermore, for minimum points a_1, a_3,

$$C_q(J_{1i}, a_1) \cong \delta_{q0} G, \qquad C_q(J_{1i}, a_3) \cong \delta_{q0} G. \tag{3.13}$$

From the additivity of Leray-Schauder degree and Theorem 1.1 in [10], we can get

$$
\begin{aligned}
1 &= \deg\left(id - K^i, [\underline{u}_1, \overline{u}_3], 0\right) \\
&= \deg\left(id - K^i, [\underline{u}_1, \overline{u}_1], 0\right) + \deg\left(id - K^i, [\underline{u}_3, \overline{u}_3], 0\right) + \deg\left(id - K^i, B(a_2, r), 0\right) \\
&\quad + \deg\left(id - K^i, B(u_1, r), 0\right) \\
&= 1 + 1 + (-1)^l + (-1).
\end{aligned}
\tag{3.14}
$$

So we have $(-1)^l = 1$. It is impossible. From the above discussion, we conclude that there must exist another critical point $u_1^* \in [\underline{u}_1, \overline{u}_3]$, which satisfies $u_1^* \neq u_1$ and is nonconstant.

Similarly, we can discuss the order interval $[\underline{u}_3, \overline{u}_5]$, and we get another critical point $u_2^* \neq u_2$. We let the procedure go on.

This completes the proof of Theorem 3.4.   □

Thus, we prove that the conclusion of Theorem 1.1 holds.

The Proof of Corollaries 1.2 and 1.4.

Proof. See Theorem 3.5 of Li [1].   □

Proof of Theorem 1.3. From the variational point of view, solutions of (p_2) are the critical points of the functional

$$
J_2(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{\alpha}{p} \int_\Omega |u|^p dx + \frac{1}{p} \int_{\partial\Omega} b(x)|u|^p ds - \int_\Omega F(x, u) dx,
\tag{3.15}
$$

defined on $X := W^{1,p}(\Omega)$, where $F(x, u) = \int_0^u f(x, s) ds$.

We show that J_2 belongs to $C^1(X, R)$. In fact, we set

$$
J_{21}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{\alpha}{p} \int_\Omega |u|^p dx - \int_\Omega F(x, u) dx, \qquad J_{22}(u) = \frac{1}{p} \int_{\partial\Omega} b(x)|u|^p ds.
\tag{3.16}
$$

Under the condition (f_1), it is well known that J_{21} is a C^1-functional. Next, we consider J_{22}. If we let $u, v \in X$, $0 < |t| < 1$,

$$
\begin{aligned}
\frac{[J_{22}(u + tv) - J_{22}(u)]}{t} &= \int_{\partial\Omega} b(x)|u|^{p-2}uv\, ds + \sum_{q \geq 2} \frac{C_p^q}{p} t^{q-1} \int_{\partial\Omega} b(x)|u|^{p-q}|v|^q ds \\
&\longrightarrow \int_{\partial\Omega} b(x)|u|^{p-2}uv\, ds, \quad (t \longrightarrow 0).
\end{aligned}
\tag{3.17}
$$

So we have that J_{22} has a Gateaux derivative and $\langle J_{22}'(u), v \rangle = \int_{\partial\Omega} b(x)|u|^{p-2}uv\, ds$.

Let $u_n \to u$ in X; now, by Hölder's and Sobolev's inequalities we can estimate

$$
\begin{aligned}
\left| \langle J'_{22}(u_n) - J'_{22}(u), v \rangle \right| &= \left| \int_{\partial\Omega} b(x) \left(|u_n|^{p-2} u_n - |u|^{p-2} u \right) v \, ds \right| \\[2mm]
&\leq \|b\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \left| \left(|u_n|^{p-2} u_n - |u|^{p-2} u \right) v \right| ds \\[2mm]
&\leq \begin{cases} c\|b\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} (|u_n| + |u|)^{p-2} |u_n - u||v| ds & \text{if } p \geq 2, \\[3mm] c\|b\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u_n - u|^{p-1} |v| ds & \text{if } p < 2, \end{cases} \\[3mm]
&\leq \begin{cases} c\|b\|_{L^\infty(\partial\Omega)} \|T(u_n - u)\|_{L^{p'}(\partial\Omega)} \|Tv\|_{L^p(\partial\Omega)} & \text{if } p \geq 2, \\[3mm] c\|b\|_{L^\infty(\partial\Omega)} \|T(u_n - u)\|^{p-1}_{L^p(\partial\Omega)} \|Tv\|_{L^p(\partial\Omega)} & \text{if } p < 2, \end{cases}
\end{aligned}
\tag{3.18}
$$

but, when $p \geq 2$, $p' = p/(p-1) \leq p$, we have $\|u\|_{L^{p'}} \leq \|u\|_{L^p}$, then we have

$$
\begin{aligned}
\left| \langle J'_{22}(u_n) - J'_{22}(u), v \rangle \right| &\leq \begin{cases} c\|b\|_{L^\infty(\partial\Omega)} \|T(u_n - u)\|_{L^p(\partial\Omega)} \|Tv\|_{L^p(\partial\Omega)} & \text{if } p \geq 2, \\[3mm] c\|b\|_{L^\infty(\partial\Omega)} \|T(u_n - u)\|^{p-1}_{L^p(\partial\Omega)} \|Tv\|_{L^p(\partial\Omega)} & \text{if } p < 2, \end{cases} \\[3mm]
&\leq \begin{cases} c\|b\|_{L^\infty(\partial\Omega)} \|u_n - u\|_{W^{1,p}(\Omega)} \|v\|_{W^{1,p}(\Omega)} & \text{if } p \geq 2, \\[3mm] c\|b\|_{L^\infty(\partial\Omega)} \|u_n - u\|^{p-1}_{W^{1,p}(\Omega)} \|v\|_{W^{1,p}(\Omega)} & \text{if } p < 2, \end{cases}
\end{aligned}
\tag{3.19}
$$

where $1/p + 1/p' = 1$, $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is trace operator, and $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for all $u \in W^{1,p}(\Omega)$ with the constant C depending on Ω by Sobolev Trace Theorem (see [6]).

To get (3.18), we have used the following well-known inequalities:

$$
\left| |u|^{p-2} u - |v|^{p-2} v \right| \leq \begin{cases} c(|u| + |v|)^{p-2} |u - v| & \text{if } p \geq 2, \\[3mm] c|u - v|^{p-1} & \text{if } p < 2, \end{cases}
\tag{3.20}
$$

which hold for a convenient $c > 0$, $u, v \in R^n$. So

$$
\left\| J'_{22}(u_n) - J'_{22}(u) \right\| \leq \begin{cases} c\|b\|_{L^\infty(\partial\Omega)} \|u_n - u\|_{W^{1,p}(\Omega)} & \text{if } p \geq 2, \\[3mm] c\|b\|_{L^\infty(\partial\Omega)} \|u_n - u\|^{p-1}_{W^{1,p}(\Omega)} & \text{if } p < 2. \end{cases} \longrightarrow 0, \quad (n \longrightarrow \infty).
\tag{3.21}
$$

So $J'_{22}(u)$ is continuous and $J_2 \in C^1(X, R)$.

Consider the truncated functions

$$\widetilde{f}(x,t) = \begin{cases} 0, & t \le -M_2, \\ f(x,t), & -M_2 \le t \le M_1, \\ 0, & t \ge M_1 \end{cases} \tag{3.22}$$

and the corresponding functional

$$\widetilde{J}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha}{p}\int_\Omega |u|^p dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p ds - \int_\Omega \widetilde{F}(x,u)dx, \tag{3.23}$$

$\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds$.

From Perera [18] we have that $\widetilde{J}$ satisfies (PS) condition. From the deformation theorem, we know that $\widetilde{J}$ satisfies deformation property when $\widetilde{J}$ satisfies (PS) condition. By a similar discussion as in Theorem 1.1, we only need to discuss the critical points of $\widetilde{J}$.

Now, we construct the sub-sup solutions of (p_2). It is easy to see that M_1 is a constant sup-solution of (p_2) and $-M_2$ is a constant subsolution. Moreover, we consider $\varepsilon\varphi_1$ for all $\varepsilon > 0$ small enough. From [14] we know that $\varphi_1(x) > 0$, $x \in \overline{\Omega}$. In fact, with $\underline{u} := \varepsilon\varphi_1$, by (g_1) we have

$$-\Delta_p\underline{u} + \alpha|\underline{u}|^{p-2}\underline{u} - f(x,\underline{u}) = \varepsilon^{p-1}\varphi_1^{p-1}(x)\left[(\lambda_1 + \alpha) - \frac{f(x,\varepsilon\varphi_1)}{\varepsilon^{p-1}\varphi_1^{p-1}}\right] \le 0. \tag{3.24}$$

Furthermore, $\varphi_1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $-\Delta_p\varphi_1 = \lambda\varphi_1^{p-1}$ in the weak sense, then the regularity theory for the p-Laplacian (e.g., [19]) implies $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha = \alpha(n,p) \in (0,1)$. Moreover $\varphi_1 \ge 0$. In addition, by the strong maximum principle of [20] and $\varphi_1 \ne 0$, then $\varphi_1 > 0$ in Ω and $\partial\varphi_1/\partial\nu < 0$ on $\partial\Omega$. So if $b(x)$ is small enough at some point $x_0 \in \partial\Omega$, we can have $[|\nabla\underline{u}|^{p-2}(\partial\underline{u}/\partial\nu) + b(x)|\underline{u}|^{p-2}\underline{u}]|_{x_0} \le 0$. From the above discussion, we have a sub-solution of (p_2), a.e., $\varepsilon\varphi_1(x_0)$. With $\varepsilon\varphi_1 := \varepsilon\varphi_1(x_0)$.

By a similar argument we can find that $[-M_2, -\varepsilon\varphi_1]$ is a pair of strict sub-sup solutions.

Now we study the order interval $[-M_2, M_1]$ in X which includes two suborder intervals $[-M_2, -\varepsilon\varphi_1]$ and $[\varepsilon\varphi_1, M_1]$. By Lemma 2.2, there exists weak solutions of (p_2) (relative minimum points) u_2, u_3 in $[-M_2, -\varepsilon\varphi_1]$ and $[\varepsilon\varphi_1, M_1]$, respectively. We can infer that $\widetilde{J}(u)$ is bounded from below on $[-M_2, M_1]$, so we get a mountain pass point $u_1 \in [-M_2, M_1] \setminus ([-M_2, -\varepsilon\varphi_1] \cup [\varepsilon\varphi_1, M_1])$ according to mountain pass theorem in order interval. From the definition of mountain pass point, we have that $C_1(\widetilde{J}, u_1)$ is nontrivial.

From assumption (g_2), we know that the left and the right derivatives of $\widetilde{f}$ at 0 are different, we consider the problem

$$-\Delta_p u = \widetilde{f}(x,u) - \alpha|u|^{p-2}u, \quad \text{in } \Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial\nu} + b(x)|u|^{p-2}u = 0, \quad \text{on } \partial\Omega, \tag{3.25}$$

where $\tilde{f} \in C(\overline{\Omega} \times R)$, and as $u \to 0$ we have

$$\tilde{f}(x,u) - \alpha|u|^{p-2}u = \left(\tilde{f}'_+(x,0) - \alpha\right)(u^+)^{p-1} - \left(\tilde{f}'_-(x,0) - \alpha\right)(u^-)^{p-1} + \circ\left(|u|^{p-1}\right). \quad (3.26)$$

We take $a = \tilde{f}'_+(x,0) - \alpha$, $b = \tilde{f}'_-(x,0) - \alpha$; then also from assumption (g_2) and the definition of Σ_p, we know that $(a,b) \notin \Sigma_p$.

Then we consider the following cases.

(1) If (a,b) does not belong to Σ_p, but lies above C_2, then

$$C_q\left(\tilde{J},0\right) = 0 \quad \text{for } q = 0,1 \quad (3.27)$$

by Lemma 2.6(iv). In this case, $C_1(\tilde{J},0) = 0$, so $C_q(\tilde{J},0) \not\cong C_q(\tilde{J},u_1)$, we have $u_1 \neq 0$.

(2) Denote

$$J_{a-b}(u) = \int_\Omega |\nabla u|^p - (a-b)(u^+)^p + \int_{\partial\Omega} b(x)|u|^p ds, \quad u \in X, \quad (3.28)$$

and $\tilde{J}_{a-b}$ is the restriction of J_{a-b} to the C^1 manifold

$$S = \left\{ u \in X : \int_\Omega |u|^p = 1 \right\}, \quad (3.29)$$

where $a = \tilde{f}'_+(x,0) - \alpha$, $b = \tilde{f}'_-(x,0) - \alpha$ as shown above.

From (g_2), we know that (a,b) does not belong to Σ_p, if $\tilde{J}_{a-b}(u) > b$, a.e. $\tilde{J}^b_{a-b} = \emptyset$, then

$$C_q\left(\tilde{J},0\right) = \delta_{q0}Z \quad (3.30)$$

by Lemma 2.7. In this case, $C_1(\tilde{J},0) = 0$, so $C_q(\tilde{J},0) \not\cong C_q(\tilde{J},u_1)$, and we have $u_1 \neq 0$.

Now, we discuss the solutions in $[-M_2, M_1]$ more deeply. We already have four solutions $0, u_1, u_2, u_3$, where u_1 is the mountain pass point and u_2, u_3 are the local minimum points of J_2. For the minimum points u_2, u_3, we have

$$C_q\left(\tilde{J},u_2\right) \cong \delta_{q0}G, \qquad C_q\left(\tilde{J},u_3\right) \cong \delta_{q0}G. \quad (3.31)$$

Since u_1 is a mountain pass point, for the Leray-Schauder degree of $id - \tilde{K}$, we have the computing formular

$$\deg\left(id - \tilde{K}, B(u_1,r),0\right) = -1, \quad (3.32)$$

where $r > 0$ is small enough, $\tilde{K} = (-\Delta_p + (m+\alpha)g_p(\cdot))^{-1}f^*|_X : X \to X$ is of class C^0 and

strongly order preserving, $f^*(x, u) = f(x, u) + mg_p(u)$ (see Remark 2.5(b)). Then according to Poincaré-Hopf formula for C^1 case and the computation of $C_q(\tilde{J}, 0)$, we have

$$\text{index}\left(\tilde{J}, 0\right) = (-1)^{d_{l-1}}. \tag{3.33}$$

From the additivity of Leray-Schauder degree and Theorem 1.1 in [10], we can get

$$\begin{aligned}
1 &= \deg\left(id - \tilde{K}, [-M, M], 0\right) \\
&= \deg\left(id - \tilde{K}, [-M, -\varepsilon\varphi_1], 0\right) + \deg\left(id - \tilde{K}, [\varepsilon\varphi_1, M], 0\right) \\
&\quad + \deg\left(id - \tilde{K}, B(0, r), 0\right) + \deg\left(id - \tilde{K}, B(u_1, r), 0\right) \\
&= 1 + 1 + (-1)^{d_{l-1}} + (-1).
\end{aligned} \tag{3.34}$$

It is impossible. From the above discussion, we conclude that there must exist another critical point $u^* \in [-M_2, M_1]$, which satisfies $u^* \neq u_1$ and is nontrivial.

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.5. Consider the truncated function

$$f_i(x, t) = \begin{cases} 0, & t < 0, \\ f(x, t), & 0 \leq t \leq a_i, \\ f(x, a_i), & t > a_i. \end{cases} \tag{3.35}$$

Corresponding functional is

$$J_i(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{\alpha}{p} \int_\Omega |u|^p dx + \frac{1}{p} \int_{\partial\Omega} b(x)|u|^p ds - \int_\Omega F_i(x, u)dx, \tag{3.36}$$

where $F_i(x, u) = \int_0^u f_i(x, s)ds$, $i = 1, 2, \ldots$.

It is known that the solution of (p_2) is also a solution of the following equation as in the same discussion in Lemma 3.2:

$$\begin{aligned}
-\Delta_p u + \alpha|u|^{p-2}u &= f_i(x, u), \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial v} + b(x)|u|^{p-2}u &= 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{3.37}$$

By the standard argument we know that J_i satisfies (J_1)–(J_3) and the order intervals consisted by sub-super-solutions are admissible invariant set of J_i. Taking $v_0 = -M_2$, $v_1 = a_1 > 0$, then $J_i(u)$ has a minimizer $u_1 \in [v_0, v_1]$. By assumption (F'') there exists a $t_1 > 0$ such that

$$
\begin{aligned}
J_2(t_1\varphi_1) &= \frac{t_1^p}{p}\int_\Omega |\nabla\varphi_1|^p dx + \frac{\alpha}{p}t_1^p\int_\Omega \varphi_1^p dx + \frac{t_1^p}{p}\int_{\partial\Omega} b(x)\varphi_1^p ds - \int_\Omega F(x, t_1\varphi_1)dx \\
&\leq \frac{(\lambda_1 + \alpha)t_1^p}{p}\int_\Omega \varphi_1^p dx - \frac{(\lambda_1 + \alpha + \varepsilon_0 + \tilde{C})t_1^p}{p}\int_\Omega \varphi_1^p dx < J_2(u_1).
\end{aligned}
\tag{3.38}
$$

If we take $v_2 = a_{n_1} > t_1\varphi_1$, where $n_1 < i$, then

$$
J_i(t_1\varphi_1) = J_2(t_1\varphi_1) < J_i(u_1)
\tag{3.39}
$$

which implies that $J_i(u)$ has a minimizer $u_2 \in [v_0, v_2] \setminus [v_0, v_1]$ such that $J_i(u_2) < J_i(u_1)$. By Lemma 2.3 we get a mountain pass point u_3. Moreover, $v_0 < u_i < v_2$, $i = 1, 2, 3$, and u_i are positive.

Next, we take $v_1 = a_{n_1}$, $v_0 = \varepsilon\varphi_1$. Then $J_i(u)$ has a minimizer $u_2 \in [v_0, v_1]$. By assumption (F'') there is a $t_2 > 0$ such that

$$
J_2(t_2\varphi_1) < J_2(u_2).
\tag{3.40}
$$

If we take $v_2 = a_{n_2} > t_2\varphi_1$, where $n_2 < i$, then

$$
J_i(t_2\varphi_1) = J_2(t_2\varphi_1) < J_i(u_2)
\tag{3.41}
$$

which implies that $J_i(u)$ has a minimizer $u_4 \in [v_0, v_2] \setminus [v_0, v_1]$ such that $J_i(u_4) < J_i(u_2)$. By Lemma 2.3 we get a mountain pass point u_5. Moreover, $v_0 < u_i < v_2$, $i = 1, 2, 3, 4, 5$, and u_i are all positive. Continue making the procedure we obtain the result.

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.5. *Moreover, (p_1) has infinitely many nonconstant negative energy solutions $\{u_k\}$, which are mountain pass types, if the conditions as in Theorem 1.1 hold and $J_1(a_{2k}) \to -\infty$ or $J_1(b_{2k}) \to -\infty$ as $k \to +\infty$.*

Proof. Assume that $J_1(a_{2k}) \to -\infty$ as $k \to +\infty$. Let $c = \inf_{\gamma \in \Gamma}\max_{\gamma(I) \cap S} J_1(u(t))$, where $\Gamma = \{\gamma \in C(I, W) | \gamma(0) = a_{2k-1}, \gamma(1) = a_{2k+1}\}$, and $I = [0, 1]$, $S = W \setminus (W_1 \cup W_2)$, $W = [\underline{u}_{2k-1}, \overline{u}_{2k+1}]$, $W_1 = [\underline{u}_{2k-1}, \overline{u}_{2k-1}]$, $W_2 = [\underline{u}_{2k+1}, \overline{u}_{2k+1}]$, $c^* = J(a_{2k})$, $k = 1, 2, \ldots$. We discuss the problem in W which have two minimum points a_{2k-1} and a_{2k+1}. We have that a_{2k-1} and a_{2k+1} are in the same radial direction $A = \{ke_1 \mid k \in R\}$, e_1 is the first eigenvalue function of $(-\Delta_p + \alpha)$ with Neumann boundary. In fact, e_1 is a constant. We conclude that $c^* \geq c$ (see Corollary 3.4 of C. Li and S. Li [21]). Furthermore, if (f_3), (f_4) hold, then $c^* > c$. In fact, if $c^* = c$, then $c^* = \max_{u \in \gamma^*(I) \cap S} J(u) = \inf_{\gamma \in \Gamma}\max_{\gamma(I) \cap S} J(u(t)) = J(a_{2k})$, where γ^* is a special path between a_{2k-1} and a_{2k+1}, which is a path of radial direction $A = \{ke_1 \mid k \in R\}$. So a_{2k} is a mountain pass point. But according to assumptions (f_3) and (f_4), we know that $C_1(J_1, a_{2k}) = 0$, $(l \neq 2)$, that is, a_{2k} is not a mountain pass type. This is a contradiction. We draw the conclusion. $\quad\square$

Remark 3.6. In Theorems 1.3 and 3.4, we can deal with the case in which (a, b) lies above C_2, but when (a, b) lies between C_{12} and C_2, then

$$
C_q(J_{1i}, a_2) \cong \begin{cases} Z, & q = 1, \\ 0, & q \neq 1, \end{cases} \cong C_q(J_{1i}, u_1),
$$

$$
C_q(\widetilde{J}, 0) \cong \begin{cases} Z, & q = 1, \\ 0, & q \neq 1, \end{cases} \cong C_q(\widetilde{J}, u_1),
$$

(3.42)

we cannot distinguish u_1 from a_2 and 0, then there may not be nonconstant solutions and nontrivial solutions to (p_1) and (p_2).

Remark 3.7. If we give the assumption

$(\widetilde{F})$ $\int_\Omega F(x, u)dx > ((\mu_2 + \varepsilon_0)/2) \int_\Omega u^2 dx$, as $\|u\| \geq M$, $u \in E_2$, where $E_2 = \{u \in E \mid u = k_1 e_1 + k_2 e_2\}$, e_1, e_2 are the first and the second eigenfunctions of $(-\Delta_p + \alpha)$ with Neumann boundary, respectively, for all $k_1, k_2 \in R$, $\|e_1\| = \|e_2\| = 1$, $\varepsilon_0 > 0$, and M is large enough,

then under (f_1)–(f_5) and $(\widetilde{F})$, we can obtain infinitely many nonconstant positive, negative, and sign-changing solutions of (p_1).

As a matter of fact, we can infer $(\widetilde{F})$ from (F).

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