Research Article

Existence of Positive Solutions of Fourth-Order Problems with Integral Boundary Conditions

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We study the existence of positive solutions of the following fourth-order boundary value problem with integral boundary conditions, $u^{(4)}(t) = f(t, u(t), u''(t))$, $t \in (0, 1)$, $u(0) = \int_0^1 g(s)u(s)ds$, $u(1) = 0$, $u''(0) = \int_0^1 h(s)u''(s)ds$, $u''(1) = 0$, where $f : [0, 1] \times [0, +\infty) \times (-\infty, 0] \rightarrow [0, +\infty)$ is continuous, $g, h \in L^1[0, 1]$ are nonnegative. The proof of our main result is based upon the Krein-Rutman theorem and the global bifurcation techniques.

1. Introduction

The deformations of an elastic beam in an equilibrium state, whose both ends are simple supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous; see Gupta [1, 2]. In the past twenty more years, the existence of solutions and positive solutions of these kinds of problems and the Lidstone problem has been extensively studied; see [3–9] and the references therein. In [3], Ma was concerned with the existence of positive solutions of (1.1) and (1.2) under the assumptions:

(H1) $f : [0, 1] \times [0, +\infty) \times (-\infty, 0] \rightarrow [0, +\infty)$ is continuous and there exist constants $a, b, c, d \in [0, +\infty)$, with $a + b > 0, c + d > 0$ such that

$$f(t, u, p) = au - bp + o(|(u, p)|), \quad \text{as } |(u, p)| \longrightarrow 0 \tag{1.3}$$

uniformly for $t \in [0, 1]$, and

$$f(t, u, p) = cu - dp + o(|u, p|), \quad \text{as } |(u, p)| \longrightarrow \infty \tag{1.4}$$

uniformly for $t \in [0, 1]$, where $|(u, p)| := \sqrt{u^2 + p^2}$;

(H2) $f(t, u, p) > 0$ for $t \in [0, 1]$ and $(u, p) \in ([0, +\infty) \times (-\infty, 0]) \setminus \{(0, 0)\}$;

(H3) there exist constants $a_0, b_0 \in [0, +\infty)$ satisfying $a_0^2 + b_0^2 > 0$ and

$$f(t, u, p) \geq a_0 u - b_0 p, \quad (t, u, p) \in [0, 1] \times [0, +\infty) \times (-\infty, 0]. \tag{1.5}$$

Ma proved the following.

Theorem A (see [3, Theorem 4.1]). *Let (H1), (H2), and (H3) hold. Assume that either*

$$\mu_1(c, d) < 1 < \mu_1(a, b), \tag{1.6}$$

or

$$\mu_1(a, b) < 1 < \mu_1(c, d), \tag{1.7}$$

where $\mu_1(\alpha, \beta)$ denotes the first generalized eigenvalue of the generalized eigenvalue problem

$$\begin{aligned} u^{(4)}(t) &= \mu(\alpha u(t) - \beta u''(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{1.8}$$

Then (1.1) and (1.2) have at least one positive solution.

At the same time, we notice that a class of boundary value problems with integral boundary conditions appeared in heat conduction, chemical engineering underground water flow, thermoelasticity, and plasma physics. Such a kind of problems include two-point, three-point, multipoint and nonlocal boundary value problems as special cases and attracting the attention of a few readers; see [10–13] and the references therein. For example, In particular, Zhang and Ge [10] used Guo-Krasnoselskii fixed-point theorem to study existence and nonexistence of positive solutions of the following fourth-order boundary value problem with integral boundary conditions:

$$\begin{aligned} u^{(4)}(t) &= w(t) f(t, u(t), u''(t)), \quad t \in (0, 1), \\ u(0) &= \int_0^1 g(s) u(s) ds, \quad u(1) = 0, \\ u''(0) &= \int_0^1 h(s) u''(s) ds, \quad u''(1) = 0, \end{aligned} \tag{P}$$

where w may be singular at $t = 0$ and (or) $t = 1$; $f : [0, 1] \times [0, +\infty) \times (-\infty, 0] \rightarrow [0, +\infty)$ is continuous, and $g, h \in L^1[0, 1]$ are nonnegative.

Motivated by [3, 10], in this paper, we consider the existence of positive solutions of the following fourth-order boundary value problem with integral boundary conditions:

$$
\begin{aligned}
u^{(4)}(t) &= f(t, u(t), u''(t)), \quad t \in (0, 1), \\
u(0) &= \int_0^1 g(s)u(s)ds, \quad u(1) = 0, \\
u''(0) &= \int_0^1 h(s)u''(s)ds, \quad u''(1) = 0,
\end{aligned}
\tag{1.9}
$$

under the assumption

(H4) $g, h \in L^1[0, 1]$ are nonnegative, and $\|g\|_1 := \int_0^1 g(s)ds \leq \pi/4, \nu := \int_0^1 (1-s)h(s)ds < 1$.
The main result of this paper is the following.

Theorem 1.1. *Let (H1), (H2), (H3), and (H4) hold. Assume that either*

$$
\lambda_1(c, d) < 1 < \lambda_1(a, b)
\tag{1.10}
$$

or

$$
\lambda_1(a, b) < 1 < \lambda_1(c, d),
\tag{1.11}
$$

where

$$
u^{(4)}(t) = \lambda\big(\alpha u(t) - \beta u''(t)\big), \quad t \in (0, 1),
\tag{1.12}
$$

$$
u(0) = \int_0^1 g(s)u(s)ds, \quad u(1) = 0,
\tag{1.13}
$$

$$
u''(0) = \int_0^1 h(s)u''(s)ds, \quad u''(1) = 0.
\tag{1.14}
$$

Then (1.9) has at least one positive solution.

Remark 1.2. Theorem 1.1 generalizes [3, Theorem 4.1] where the special case $g = 0$ and $h = 0$ was treated.

Remark 1.3. Zhang and Ge [10] proved existence and nonexistence of positive solutions via Guo-Krasnoselskii fixed-point theorem under some conditions which do not involve the eigenvalues of (1.12)–(1.14). While our Theorem 1.1 is established under (1.10) or (1.11) which is related to the eigenvalues of (1.12)–(1.14). Moreover, (1.10) and (1.11) are optimal. Let us consider the problem

$$
u^{(4)}(x) = \pi^4 u(x) + 1, \quad x \in (0, 1),
\tag{1.15}
$$

$$
u(0) = u(1) = u''(0) = u''(1) = 0.
\tag{1.16}
$$

In this case, $\lambda_1(\pi^4, 0) = 1$ and the corresponding eigenfunction is $\varphi_1 = \sin \pi x$. However, (1.15) and (1.16) has no positive solution. (In fact, suppose on the contrary that (1.15) and (1.16) has a positive solution u. Multiplying (1.15) with $\sin \pi x$ and integrating from 0 to 1, we get a desired contradiction!).

Suppose that E is a real Banach space with norm $\|\cdot\|$. Let K be a cone in E. A nonlinear mapping $A : [0, \infty) \times K \to E$ is said to be *positive* if $A([0, \infty) \times K) \subseteq K$. It is said to be *K-completely continuous* if A is continuous and maps bounded subsets of $[0, \infty) \times K$ to precompact subset of E. Finally, a positive linear operator V on E is said to be a *linear minorant* for A if $A(\lambda, u) \geq \lambda V(x)$ for $(\lambda, u) \in [0, \infty) \times K$. If B is a continuous linear operator on E, denote $r(B)$ the spectral radius of B. Define

$$c_K(B) = \{\lambda \in [0, \infty) : \text{there exists } x \in K \text{ with } \|x\| = 1, \ x = \lambda B x\}. \tag{1.17}$$

The following lemma will play a very important role in the proof of our main results, which is essentially a consequence of Dancer [14, Theorem 2].

Lemma 1.4. *Assume that*
   (i) K has nonempty interior and $E = \overline{K - K}$;
   (ii) $A : [0, \infty) \times K \to E$ is K-completely continuous and positive, $A(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$, $A(0, u) = 0$ for $u \in K$ and

$$A(\lambda, u) = \lambda B u + F(\lambda, u), \tag{1.18}$$

where $B : E \to E$ is a strongly positive linear compact operator on E with the spectral radius $r(B) > 0$, $F : [0, \infty) \times K \to E$ satisfies $\|F(\lambda, u)\| = \circ(\|u\|)$ as $\|u\| \to 0$ locally uniformly in λ.
   Then there exists an unbounded connected subset C of

$$\mathfrak{D}_K(A) = \{(\lambda, u) \in [0, \infty) \times K : u = A(\lambda, u), \ u \neq 0\} \cup \left\{\left(r(B)^{-1}, 0\right)\right\} \tag{1.19}$$

such that $(r(B)^{-1}, 0) \in C$.
   Moreover, if A has a linear minorant V and there exists a

$$(\mu, y) \in (0, \infty) \times K \tag{1.20}$$

such that $\|y\| = 1$ and $\mu V y \geq y$, then C can be chosen in

$$\mathfrak{D}_K(A) \cap \left([0, \mu] \times K\right). \tag{1.21}$$

Proof. Since B is a strongly positive compact endomorphism of E and P has nonempty interior, we have from Amann [15, Theorem 3.2] that the set $c_K(B)$ in [14, Theorem 2] reduces to a single point $(r(B)^{-1}, 0)$. Now the desired result is a consequence of Dancer [14, Theorem 2]. $\square$

The rest of the paper is arranged as follows. In Section 2, we state and prove some preliminary results about the spectrum of (1.12)–(1.14). Finally, in Section 3, we proved our main result.

2. Generalized Eigenvalues

Lemma 2.1 (see [10]). *Assume that (H4) holds. Then for any* $y \in C[0,1]$, *the boundary value problem*

$$-u''(t) = y(t), \quad t \in (0,1),$$

$$u(0) = \int_0^1 g(s)u(s)ds, \quad u(1) = 0 \tag{2.1}$$

has a unique solution u which is given by

$$u(t) = \int_0^1 H_1(t,s)y(s)ds, \quad t \in [0,1], \tag{2.2}$$

where

$$H_1(t,s) = G(t,s) + \frac{1-t}{1-\mu}\int_0^1 G(s,\tau)g(\tau)d\tau, \quad \mu := \int_0^1 (1-s)g(s)ds,$$

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Lemma 2.2 (see [10]). *Assume that (H4) holds. Then for any* $y \in C[0,1]$, *the boundary value problem*

$$u^{(4)}(t) = y(t), \quad t \in (0,1),$$

$$u(0) = \int_0^1 g(s)u(s)ds, \quad u(1) = 0,$$

$$u''(0) = \int_0^1 h(s)u''(s)ds, \quad u''(1) = 0 \tag{2.4}$$

has a unique solution u which is given by

$$u(t) = \iint_0^1 H_1(t,x)H_2(x,s)y(s)ds\,dx, \quad t \in [0,1], \tag{2.5}$$

where

$$H_2(x,s) = G(x,s) + \frac{1-x}{1-v} \int_0^1 G(s,\tau)h(\tau)d\tau. \tag{2.6}$$

Lemma 2.3 (see [10]). *Assume that (H4) holds. Then one has*

$$H_1(t,s) > 0, \quad H_2(t,s) > 0, \quad G(t,s) > 0, \quad t,s \in (0,1),$$
$$H_1(t,s) \geq 0, \quad H_2(t,s) \geq 0, \quad G(t,s) \geq 0, \quad t,s \in [0,1]. \tag{2.7}$$

Let

(H5) $(\alpha,\beta) \in [0,+\infty) \times [0,+\infty)$ *be two given constants with* $\alpha + \beta > 0$.

Definition 2.4. One says that λ *is a generalized eigenvalue of linear problem*

$$u^{(4)}(t) = \lambda(\alpha u(t) - \beta u''(t)), \quad t \in (0,1), \tag{2.8}$$

$$u(0) = \int_0^1 g(s)u(s)ds, \quad u(1) = 0,$$
$$u''(0) = \int_0^1 h(s)u''(s)ds, \quad u''(1) = 0, \tag{2.9}$$

if (2.8) and (2.9) have nontrivial solutions.

Let

$$e(t) := \cos\frac{\pi}{2}t, \quad t \in [0,1]. \tag{2.10}$$

Let $Y = C[0,1]$ with the norm $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$. Let $Z = L^1[0,1]$ with the norm $\|u\|_1 = \int_0^1 |u(s)|ds$.

Let

$$X = \left\{u \in C^2[0,1] \mid u \text{ satisfies } (2.9), \exists \, \varepsilon > 0, \text{s. t.} - \varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t), \, t \in [0,1]\right\}. \tag{2.11}$$

For $u \in X$, from Lemma 2.1, it follows that

$$u(t) = \int_0^1 H_1(t,s)(-u''(s))ds, \quad t \in [0,1]. \tag{2.12}$$

By simple calculations, we have

$$
\begin{aligned}
\int_0^1 H_1(t,s)e(s)ds &= \int_0^1 G(t,s)e(s)ds + \frac{1-t}{1-\mu}\int_0^1\left[\int_0^1 G(s,\tau)g(\tau)d\tau\right]e(s)ds \\
&= \frac{4}{\pi^2}e(t) - \frac{4(1-t)}{\pi^2} + \frac{1-t}{1-\mu}\int_0^1\left[\int_0^1 G(s,\tau)g(\tau)d\tau\right]e(s)ds \\
&\leq \frac{4}{\pi^2}e(t) - \frac{4(1-t)}{\pi^2} + \frac{1-t}{1-\mu}\int_0^1 G(s,s)e(s)ds\int_0^1 g(\tau)d\tau \\
&= \frac{4}{\pi^2}e(t) - \frac{4(1-t)}{\pi^2} + \frac{4(1-t)}{\pi^2}\frac{\|g\|_1}{1-\mu}\left(\frac{4}{\pi}-1\right).
\end{aligned}
\tag{2.13}
$$

Combining this with (H4), we conclude that

$$
\int_0^1 H_1(t,s)e(s)ds \leq \frac{4}{\pi^2}e(t), \quad t\in[0,1].
\tag{2.14}
$$

This together with (2.12) and the fact that $-\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t)$ imply that

$$
-\frac{4}{\pi^2}\varepsilon e(t) \leq u(t) \leq \frac{4}{\pi^2}\varepsilon e(t), \quad t\in[0,1].
\tag{2.15}
$$

Since $(4/\pi^2)\varepsilon < \varepsilon$, we may define the norm of $u\in X$ by

$$
\|u\|_X := \inf\{\varepsilon \mid -\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t), t\in[0,1]\}.
\tag{2.16}
$$

We claim that $(X,\|\cdot\|_X)$ is a Banach space.

In fact, let $\{y_n\}\subset X$ be a Cauchy sequence, that is, $\|y_n - y_m\|_X \to 0$ as $n,m\to\infty$. From the definition of $\|\cdot\|_X$, it follows that

$$
\|y_n - y_m\|_{C^2} \leq c^*\|y_n - y_m\|_X, \quad \text{for some constant } c^* > 0,
\tag{2.17}
$$

where $\|\cdot\|_{C^2}$ is a normal in X defined by $\|y\|_{C^2} := \max\{|-y''(t)| \mid t\in[0,1]\}$. Thus,

$$
\|y_n - y_m\|_{C^2} \longrightarrow 0, \quad \text{as } m, n\longrightarrow\infty.
\tag{2.18}
$$

By the completeness of $(C^2[0,1],\|\cdot\|_{C^2})$, there exists $y^* \in C^2[0,1]$, such that

$$
\|y_n - y^*\|_{C^2} \longrightarrow 0, \quad \text{as } n\longrightarrow\infty.
\tag{2.19}
$$

From the fact that $\|y_n - y_m\|_X \to 0$, we have that for arbitrary $\epsilon > 0$, there exists $N > 0$, such that

$$
\|y_n - y_m\|_X \leq \epsilon, \quad \text{whenever } m, n > N;
\tag{2.20}
$$

and subsequently,

$$-\epsilon e(t) \leq y_n(t) - y_m(t) \leq \epsilon e(t), \quad \text{whenever } m, \ n > N. \tag{2.21}$$

Fixed n and let $m \to \infty$, we get

$$-\epsilon e(t) \leq y_n(t) - y^*(t) \leq \epsilon e(t), \quad \text{whenever } n > N. \tag{2.22}$$

This is,

$$\|y_n - y^*\|_X \leq \epsilon, \quad \text{whenever } n > N. \tag{2.23}$$

Therefore, $(X, \|\cdot\|_X)$ is a Banach space.

    Let

$$P := \{u \in X \mid u(t) \geq 0, \ u''(t) \leq 0, \ t \in [0,1]\}. \tag{2.24}$$

Then the cone P is normal and nonempty interior P^0 and $X = \overline{P - P}$.

    In fact, for any $u \in X$, it follows from the definition of X that

    (1) there exist real number $a > 0$, such that

$$|u''(t)| \leq ae(t), \quad t \in [0,1]; \tag{2.25}$$

    (2) $u(1) = 0, u(0) = \int_0^1 g(s)u(s)ds$.

From $u(0) = \int_0^1 g(s)u(s)ds$ and (H4), we obtain that $u'(t^*) = 0$ for some $t^* \in (0,1)$. Moreover,

$$|u'(t)| = \left| u'(t^*) + \int_{t^*}^t u''(s)ds \right| \leq a, \tag{2.26}$$

and subsequently,

$$|u(t)| = \left| u(1) - \int_t^1 u'(s)ds \right| \leq |u(1)| + \int_t^1 |u'(s)| ds \leq a(1-t) \leq \hat{a}e(t), \quad t \in [0,1], \tag{2.27}$$

for some $\hat{a} > 0$. We may take $\hat{a}$ satisfying

$$\hat{a} > \frac{4a}{\pi^2}. \tag{2.28}$$

Now, let us define

$$
\begin{aligned}
y(t) &= \hat{a}e(t), \\
z(t) &= u(t) + y(t).
\end{aligned}
\tag{2.29}
$$

Then $y, z \in P$, and

$$
u = z(t) - y(t).
\tag{2.30}
$$

Thus, $X \subseteq \overline{P - P}$. Obviously, $\overline{P - P} \subseteq X$.

Lemma 2.5. *Assume that* (H4) *holds. Then for any* $u \in X$, *one has*

$$
\|u\|_\infty < \|u''\|_\infty \leq \|u\|_X.
\tag{2.31}
$$

Proof. In fact, for $u \in X$, we have that

$$
\begin{aligned}
u(t) &= \int_0^1 H_1(t,s)\big(-u''(s)\big)\,ds \\[2mm]
&\leq \|u''\|_\infty \int_0^1 H_1(t,s)\,ds \\[2mm]
&= \|u''\|_\infty \int_0^1 \left[G(t,s) + \frac{1-t}{1-\mu}\int_0^1 G(s,\tau)g(\tau)\,d\tau \right] ds \\[2mm]
&\leq \|u''\|_\infty \left[\int_0^1 G(s,s)\,ds + \frac{1}{1-\mu}\int_0^1 G(s,s)\,ds \int_0^1 g(\tau)\,d\tau \right] \\[2mm]
&= \|u''\|_\infty \left[\frac{1}{6} + \frac{1}{6}\frac{\|g\|_1}{1-\mu} \right].
\end{aligned}
\tag{2.32}
$$

From $\mu \leq \|g\|_1 \leq \pi/4$, we have that $1/6 + (1/6)(\|g\|_1/(1-\mu)) < 1$, and so $|u(t)| < \|u''\|_\infty$, and accordingly $\|u\|_\infty < \|u''\|_\infty$.

We have from the fact that $-\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t)$, $t \in [0,1]$, that

$$
|u''(t)| \leq \varepsilon e(t) \leq \varepsilon,
\tag{2.33}
$$

which implies that $\|u''\|_\infty \leq \varepsilon$, and consequently $\|u''\|_\infty \leq \|u\|_X$. □

For $u \in X$, define a linear operator $T : X \to Y$ by

$$
Tu(t) = \iint_0^1 H_1(t,x)H_2(x,s)\big[\alpha u(s) - \beta u''(s)\big]\,ds\,dx, \quad t \in [0,1].
\tag{2.34}
$$

Theorem 2.6. *Assume that (H4) and (H5) hold. Let $r(T)$ be the spectral radius of T. Then (2.8) and (2.9) has an algebraically simple eigenvalue, $\lambda_1(\alpha, \beta) = (r(T))^{-1}$, with a positive eigenfunction $\varphi_1(\cdot) \in P^0$. Moreover, there is no other eigenvalue with a positive eigenfunction.*

Remark 2.7. If $g = h \equiv 0$, then $\lambda_1(\alpha, \beta)$ can be explicitly given by

$$\lambda_1(\alpha, \beta) = \frac{\pi^4}{\alpha + \beta \pi^2}, \tag{2.35}$$

and the corresponding eigenfunction $\varphi_1(t) = \sin \pi t, t \in [0, 1]$.

Proof of Theorem 2.6. From Lemma 2.2, it is easy to check that (2.8) and (2.9) is equivalent to the integral equation

$$u(t) = \lambda T u(t), \quad t \in [0, 1]. \tag{2.36}$$

We claim that $T : X \to X$.
   In fact, for $u \in X$, we have that

$$\begin{aligned}
\left| -(Tu)''(t) \right| &= \left| \int_0^1 H_2(t, s) \left[\alpha u(s) - \beta u''(s) \right] ds \right| \\
&\leq \left(\alpha \|u\|_\infty + \beta \|u''\|_\infty \right) \int_0^1 H_2(t, s) ds \\
&:= C_0 \int_0^1 H_2(t, s) ds.
\end{aligned} \tag{2.37}$$

Since

$$\begin{aligned}
\int_0^1 H_2(t, s) ds &= \int_0^1 G(t, s) ds + \frac{1-t}{1-v} \iint_0^1 G(s, \tau) h(\tau) d\tau ds \\
&= \frac{t(1-t)}{2} + (1-t) \frac{\iint_0^1 G(s, \tau) h(\tau) d\tau ds}{1-v} \\
&\leq (1-t) \left[\frac{1}{2} + \frac{\iint_0^1 G(s, \tau) h(\tau) d\tau ds}{1-v} \right] \\
&:= \Gamma_0 (1-t),
\end{aligned} \tag{2.38}$$

and for some constant $\rho_1 > 0$, it concludes that

$$\rho_1 e(t) \leq 1 - t \leq e(t), \quad t \in [0, 1]. \tag{2.39}$$

Hence,

$$-\sigma e(t) \leq -(Tu)''(t) \leq \sigma e(t), \quad t \in [0,1], \tag{2.40}$$

where $\sigma := \Gamma_0 C_0$, it follows that $Tu \in X$, and accordingly $T(X) \subseteq X$.

If $u \in P$, then $\alpha u(t) - \beta u''(t) \geq 0$ on $t \in [0,1]$, and accordingly

$$(Tu)''(t) = -\int_0^1 H_2(t,s)\left[\alpha u(s) - \beta u''(s)\right]ds \leq 0, \quad t \in [0,1],$$

$$(Tu)(t) = \int_0^1 H_1(t,s)\left(-(Tu)''(s)\right)ds \geq 0, \quad t \in [0,1]. \tag{2.41}$$

Thus $Tu \in P$, and accordingly $T(P) \subseteq P$.

Now, since $T(X) \subseteq C^4[0,1] \cap X$, and $C^4[0,1] \cap X$ is compactly embedded in X, we have that $T : X \to X$ is compact.

Next, we show that $T : P \to P$ is positive.

For $u \in P \setminus \{0\}$, if $\alpha > 0$, from Lemma 2.3, we have

$$
\begin{aligned}
-(Tu)''(t) &\geq \alpha \int_0^1 H_2(t,s)u(s)ds \\
&= \alpha \left[\int_0^1 G(t,s)ds + \frac{1-t}{1-\nu}\int_0^1 \left(\int_0^1 G(s,\tau)h(\tau)d\tau\right)u(s)ds\right] \\
&\geq \frac{\alpha \int_0^1 \left[\int_0^1 G(s,\tau)h(\tau)d\tau\right]u(s)ds}{1-\nu}(1-t) \\
&:= \Gamma_3(1-t).
\end{aligned}
\tag{2.42}
$$

Combining this with (2.39), there exist $r_2 := \Gamma_3 \rho_1 > 0$ such that

$$-(Tu)''(t) \geq r_2 e(t), \quad t \in [0,1]. \tag{2.43}$$

For $u \in P \setminus \{0\}$, if $\beta > 0$, applying a similar proof process of (2.43), we have

$$
\begin{aligned}
-(Tu)''(t) &\geq \beta \int_0^1 H_2(t,s)\left[-u''(s)\right]ds \\
&\geq \frac{\beta \int_0^1 \left[\int_0^1 G(s,\tau)h(\tau)d\tau\right](-u''(s))ds}{1-\nu}(1-t) \\
&:= \Gamma_4(1-t).
\end{aligned}
\tag{2.44}
$$

Combining this with (2.39), there exist $r_3 := \Gamma_4 \rho_1 > 0$ such that

$$-(Tu)''(t) \ge r_3 e(t), \quad t \in [0,1]. \tag{2.45}$$

This together with (2.9) and (H4) imply $u(t) \ge 0$ on $[0,1]$.

Therefore, it follows from (2.43) and (2.45) that $Tu \in P^0$.

Now, by the Krein-Rutman theorem ([16, Theorem 7.C]; [17, Theorem 19.3]), T has an algebraically simple eigenvalue $r(T) > 0$ with an eigenfunction $\varphi_1(\cdot) \in P^0$. Moreover, there is no other eigenvalue with a positive eigenfunction. Correspondingly, $\lambda_1(\alpha, \beta) = (r(T))^{-1}$ with a positive eigenfunction of $\varphi_1(\cdot)$, is a simple eigenvalue of (2.8) and (2.9). Moreover, for (2.8) and (2.9), there is no other eigenvalue with a positive eigenfunction. $\qquad \square$

3. The Proof of the Main Result

Before proving Theorem 1.1, we denote $L : D(L) \to Y$ by setting

$$Lu := u'''', \quad u \in D(L), \tag{3.1}$$

where

$$D(L) = \left\{ u \in C^4[0,1] \mid u(0) = \int_0^1 g(s)u(s)ds, \ u''(0) = \int_0^1 h(s)u''(s)ds, u(1) = u''(1) = 0 \right\}. \tag{3.2}$$

It is easy to check that $L^{-1} : Y \to X$ is compact.

Let $\zeta, \xi \in C([0,1] \times [0,+\infty) \times (-\infty,0])$ be such that

$$\begin{aligned} f(t,u,p) &= au - bp + \zeta(t,u,p), \\ f(t,u,p) &= cu - dp + \xi(t,u,p). \end{aligned} \tag{3.3}$$

Obviously,(H1) implies that

$$\lim_{|(u,p)| \to 0} \frac{\zeta(t,u,p)}{|(u,p)|} = 0 \quad \text{uniformly for } t \in [0,1], \tag{3.4}$$

$$\lim_{|(u,p)| \to \infty} \frac{\xi(t,u,p)}{|(u,p)|} = 0 \quad \text{uniformly for } t \in [0,1]. \tag{3.5}$$

Let

$$\widetilde{\xi}(r) = \max\{|\xi(t,u,p)| \mid 0 \le |(u,p)| \le r, \ t \in [0,1]\}, \tag{3.6}$$

then $\widetilde{\xi}$ is nondecreasing and

$$\lim_{r \to \infty} \frac{\widetilde{\xi}(r)}{r} = 0. \tag{3.7}$$

Let us consider

$$Lu = \lambda(au - bu'') + \lambda\zeta(t, u, u''), \quad \lambda > 0 \tag{3.8}$$

as a bifurcation problem from the trivial solution $u \equiv 0$. It is to easy to check that (3.8) can be converted to the equivalent equation

$$u(t) = \lambda \left\{ \iint_0^1 H_1(t,x)H_2(x,s)\big[au(s) - bu''(s)\big]dsdx \right\}$$

$$+ \lambda \left\{ \iint_0^1 H_1(t,x)H_2(x,s)\zeta(s,u(s),u''(s))dsdx \right\} \tag{3.9}$$

$$:= A(\lambda, u)(t).$$

From the proof process of Theorem 2.6, the operator $B : X \to X$,

$$Bu(t) := \iint_0^1 H_1(t,x)H_2(x,s)\big[au(s) - bu''(s)\big]dsdx \tag{3.10}$$

is compact and strongly positive. Define $F : [0, +\infty) \times X \to X$ by

$$F(\lambda, u) := \lambda \iint_0^1 H_1(t,x)H_2(x,s)\zeta(s,u(s),u''(s))dsdx, \tag{3.11}$$

then we have from (3.4) and Lemma 2.5 that

$$\|F(\lambda, u)\|_X = o(\|u\|_X), \quad \text{as } \|u\|_X \longrightarrow 0, \tag{3.12}$$

locally uniformly in λ. From (H2) and Theorem 2.6 (with obvious changes), it follows that if (λ, u) is a nontrivial solution of (3.8) with $\lambda > 0$, then $u \in P^0$. Combining this with Lemma 1.4, we conclude that there exists an unbounded connected subset C of the set

$$\left\{ (\lambda, u) \in (0, \infty) \times P \mid u = A(\lambda, u), u \in P^0 \right\} \cup \{(\lambda_1(a,b), 0)\} \tag{3.13}$$

such that $(\lambda_1(a,b), 0) \in C$.

Proof of Theorem 1.1. It is clear that any solution of the form $(1, u)$ yields a solution u of (1.9). We will show that $\mathcal{C}$ crosses the hyperplane $\{1\} \times X$. To do this, it is enough to show that $\mathcal{C}$ joins $(\lambda_1(a, b), 0)$ to $(\lambda_1(c, d), \infty)$. Let $(\mu_n, y_n) \in \mathcal{C}$ satisfy

$$\mu_n + \|y_n\|_X \longrightarrow \infty, \quad n \longrightarrow \infty, \tag{3.14}$$

we note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of (3.8) for $\lambda = 0$ and $\mathcal{C} \cap (\{0\} \times X) = \emptyset$. $\qquad\square$

Case 1 $(\lambda_1(c, d) < 1 < \lambda_1(a, b))$. In this case, we show that

$$(\lambda_1(c, d), \lambda_1(a, b)) \subseteq \{\lambda \in \mathbb{R} \mid \exists (\lambda, u) \in \mathcal{C}\}. \tag{3.15}$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$\mu_n \in (0, M], \tag{3.16}$$

then $\mathcal{C}$ joins $(\lambda_1(a, b), 0)$ to $(\lambda_1(c, d), \infty)$.

From (3.16), we have that $\|y_n\|_X \to \infty$. We divide the equation

$$Ly_n = \mu_n(cy_n - dy_n'') + \mu_n \xi(t, y_n, y_n'') \tag{3.17}$$

by $\|y_n\|_X$ and set $\overline{y_n} = y_n / \|y_n\|_X$. Since $\overline{y_n}$ is bounded in X, choosing a subsequence and relabeling if necessary, we see that $\overline{y_n} \to \overline{y}$ for some $\overline{y} \in X$ with $\|\overline{y}\|_X = 1$. Moreover, we have from (3.7) and Lemma 2.5 that

$$\lim_{n \to \infty} \frac{|\xi(t, y_n(t), y_n''(t))|}{\|y_n\|_X} = 0. \tag{3.18}$$

Since $|\xi(t, y_n(t), y_n''(t))| / \|y_n\|_X \leq \widetilde{\xi}(\sqrt{2}\|y_n\|_X) / \|y_n\|_X$. Thus,

$$\overline{y}(t) = \iint_0^1 H_1(t, x) H_2(x, s) \overline{\mu} [c\overline{y}(s) - d\overline{y}''(s)] \, ds \, dx, \tag{3.19}$$

where $\overline{\mu} := \lim_{n \to \infty} \mu_n$, again choosing a subsequence and relabeling if necessary. Thus,

$$L\overline{y}(t) = \overline{\mu}(c\overline{y}(t) - d\overline{y}''(t)), \quad t \in [0, 1]. \tag{3.20}$$

This together with Theorem 2.6 imply that $\overline{\mu} = \lambda_1(c, d)$. Therefore, $\mathcal{C}$ joins $(\lambda_1(a, b), 0)$ to $(\lambda_1(c, d), \infty)$.

Step 2. We show that there exists a constant M be such that $\mu_n \in (0, M]$ for all n.

By Lemma 1.4, we only need to show that A has a linear minorant V and there exists a $(\mu, y) \in (0, \infty) \times P$ such that $\|y\|_X = 1$ and $\mu V y \geq y$.

By(H3), there exist constants $a_0, b_0 \in [0, +\infty)$ satisfying $a_0^2 + b_0^2 > 0$ and

$$f(t, u, p) \geq a_0 u - b_0 p, (t, u, p) \in [0, 1] \times [0, +\infty) \times (-\infty, 0]. \tag{3.21}$$

For $u \in X$, let

$$V u(t) = \iint_0^1 H_1(t, x) H_2(x, s) [a_0 u(s) - b_0 u''(s)] ds dx, \tag{3.22}$$

then V is a linear minorant of A. Moreover,

$$
\begin{aligned}
V\left(\frac{4}{\pi^2} e(t)\right) &= \iint_0^1 H_1(t, x) H_2(x, s) \left[a_0 \frac{4}{\pi^2} e(s) + b_0 e(s)\right] ds dx \\
&= \left(\frac{4}{\pi^2} a_0 + b_0\right) \iint_0^1 H_1(t, x) H_2(x, s) e(s) ds dx \\
&= \left(\frac{4}{\pi^2} a_0 + b_0\right) \iint_0^1 \left[G(t, x) + \frac{1-t}{1-\mu} \int_0^1 G(x, \tau) g(\tau) d\tau\right] H_2(x, s) e(s) ds dx \\
&\geq \left(\frac{4}{\pi^2} a_0 + b_0\right) \frac{\iint_0^1 \left[\int_0^1 G(x, \tau) g(\tau) d\tau\right] H_2(x, s) e(s) ds dx}{1 - \mu} (1 - t) \\
&:= \Gamma_5 \frac{4}{\pi^2} (1 - t),
\end{aligned}
\tag{3.23}
$$

where

$$\Gamma_5 = \frac{(a_0 + (\pi^2/4) b_0) \iint_0^1 \left[\int_0^1 G(x, \tau) g(\tau) d\tau\right] H_2(x, s) e(s) ds dx}{1 - \mu}. \tag{3.24}$$

Combining this with (2.39), we conclude that $\lambda V((4/\pi^2) e(t)) \geq (4/\pi^2) e(t)$, here, $\lambda := (\Gamma_5 \rho_1)^{-1}$. Therefore, we have that from Lemma 1.4 that $|\mu_n| \leq \lambda$.

Case 2 $(\lambda_1(a, b) < 1 < \lambda_1(c, d))$. In this case, if $(\mu_n, y_n) \in \mathcal{C}$ such that

$$\lim_{n \to \infty} (\mu_n + \|y_n\|_X) = \infty, \tag{3.25}$$

and $\lim_{n \to \infty} \mu_n = \infty$, then

$$(\lambda_1(a, b), \lambda_1(c, d)) \subseteq \{\lambda \in (0, \infty) \mid (\lambda, u) \in \mathcal{C}\} \tag{3.26}$$

and, moreover, $(\{1\} \times X) \cap \mathcal{C} \neq \emptyset$.

Assume that there exists $M > 0$, such that for all $n \in \mathbb{N}$,

$$\mu_n \in (0, M]. \tag{3.27}$$

Applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\mu_n, y_n) \longrightarrow (\lambda_1(c, d), \infty), \quad n \longrightarrow \infty. \tag{3.28}$$

Again $\mathcal{C}$ joins $(\lambda_1(a, b), 0)$ to $(\lambda_1(c, d), \infty)$ and the result follows.

Acknowledgments

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