

Research Article

Existence and Lyapunov Stability of Periodic Solutions for Generalized Higher-Order Neutral Differential Equations

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Existence and Lyapunov stability of periodic solutions for a generalized higher-order neutral differential equation are established.

1. Introduction

In recent years, there is a good amount of work on periodic solutions for neutral differential equations (see [1–11] and the references cited therein). For example, the following neutral differential equations

$$\begin{aligned} \frac{d}{du}(u(t) - ku(t - \tau)) &= g_1(u(t)) + g_2(u(t - \tau_1)) + p(t), \\ (x(t) + cx(t - r))'' + f(x'(t)) + g(x(t - \tau(t))) &= p(t), \\ (x(t) - cx(t - \sigma))^{(n)} + f(x(t))x'(t) + g\left(\int_{-r}^0 x(t + s)d\alpha(s)\right) &= p(t) \end{aligned} \tag{1.1}$$

have been studied in [1, 3, 8], respectively, and existence criteria of periodic solutions were established for these equations. Afterwards, along with intensive research on the p -Laplacian,

some authors [4, 11] start to consider the following p -Laplacian neutral functional differential equations:

$$\begin{aligned} (\phi_p(x(t) - cx(t - \sigma)))' + g(t, x(t - \tau(t))) &= p(t), \\ (\phi_p((x(t) - cx(t - \sigma)))') + f(x'(t)) + g(x(t - \tau(t))) &= e(t), \end{aligned} \quad (1.2)$$

and by using topological degree theory and some analysis skills, existence results of periodic solutions for (1.2) have been presented.

In general, most of the existing results are concentrated on lower-order neutral functional differential equations, while studies on higher-order neutral functional differential equations are rather infrequent, especially on higher-order p -Laplacian neutral functional differential equations. In this paper, we consider the following generalized higher-order neutral functional differential equation:

$$\left(\varphi_p(x(t) - cx(t - \sigma))^{(l)}\right)^{(n-l)} = F\left(t, x(t), x'(t), \dots, x^{(l-1)}(t)\right), \quad (1.3)$$

where $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ with $p \geq 2$ being a constant, F is a continuous function defined on \mathbb{R}^l and is periodic with respect to t with period T , that is, $F(t, \cdot, \dots, \cdot) = F(t + T, \cdot, \dots, \cdot)$, $F(t, a, 0, \dots, 0) \neq 0$ for all $a \in \mathbb{R}$, and c, σ are constants.

Since the neutral operator is divided into two cases $|c| \neq 1$ and $|c| = 1$, it is natural to study the neutral differential equation separately according to these two cases. The case $|c| = 1$ has been studied in [5]. Now we consider (1.3) for the case $|c| \neq 1$. So throughout this paper, we always assume that $|c| \neq 1$, and the paper is organized as follows. We first transform (1.3) into a system of first-order differential equations, and then by applying Mawhin's continuation theory and some new inequalities, we obtain sufficient conditions for the existence of periodic solutions for (1.3). The Lyapunov stability of periodic solutions for the equation will then be established. Finally, an example is given to illustrate our results.

2. Preparation

First, we recall two lemmas. Let X and Y be real Banach spaces and let $L : D(L) \subset X \rightarrow Y$ be a Fredholm operator with index zero; here $D(L)$ denotes the domain of L . This means that $\text{Im } L$ is closed in Y and $\dim \text{Ker } L = \dim(Y/\text{Im } L) < +\infty$. Consider supplementary subspaces X_1, Y_1 of X, Y , respectively, such that $X = \text{Ker } L \oplus X_1, Y = \text{Im } L \oplus Y_1$. Let $P : X \rightarrow \text{Ker } L$ and $Q : Y \rightarrow Y_1$ denote the natural projections. Clearly, $\text{Ker } L \cap (D(L) \cap X_1) = \{0\}$ and so the restriction $L_P := L|_{D(L) \cap X_1}$ is invertible. Let K denote the inverse of L_P .

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \rightarrow Y$ is said to be L -compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \rightarrow X$ is compact.

Lemma 2.1 (see [12]). *Suppose that X and Y are two Banach spaces, and suppose that $L : D(L) \subset X \rightarrow Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and let $N : \overline{\Omega} \rightarrow Y$ be L -compact on $\overline{\Omega}$. Assume that the following conditions hold:*

- (1) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$,
- (2) $Nx \notin \text{Im } L$, for all $x \in \partial\Omega \cap \text{Ker } L$,

(3) $\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$, where $J : \text{Im } Q \rightarrow \text{Ker } L$ is an isomorphism.

Then, the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2 (see [13]). If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then

$$\int_0^T |\omega(t)|^p dt \leq \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega'(t)|^p dt, \quad (2.1)$$

where p is a fixed real number with $p > 1$ and

$$\pi_p = 2 \int_0^{(p-1)/p} \frac{ds}{(1 - s^p/(p-1))^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}. \quad (2.2)$$

For the sake of convenience, throughout this paper we denote by T a positive real number, and for any continuous function u , we write

$$|u|_0 := \max_{t \in [0, T]} |u(t)|. \quad (2.3)$$

Let $A : C_T \rightarrow C_T$ be the operator on $C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\}$ given by

$$(Ax)(t) := x(t) - cx(t - \sigma), \quad \forall x \in C_T, t \in \mathbb{R}. \quad (2.4)$$

Lemma 2.3. The operator A has a continuous inverse A^{-1} on C_T satisfying the following:

$$(1) \quad [A^{-1}f](t) = \begin{cases} f(t) + \sum_{j=1}^{\infty} c^j f(t - j\sigma), & \text{for } |c| < 1, \forall f \in C_T, \\ -\frac{f(t + \sigma)}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f(t + (j+1)\sigma), & \text{for } |c| > 1, \forall f \in C_T, \end{cases} \quad (2.5)$$

$$(2) \quad |[A^{-1}f](t)| \leq \frac{|f|_0}{|1 - |c||}, \quad \forall f \in C_T, \quad (2.6)$$

$$(3) \quad \int_0^T |[A^{-1}f](t)| dt \leq \frac{1}{|1 - |c||} \int_0^T |f(t)| dt, \quad \forall f \in C_T. \quad (2.7)$$

Remark 2.4. This lemma is basically proved in [3, 10]. For the convenience of the readers, we present a detailed proof here as follows.

Proof. We split it into the following two cases.

Case 1 ($|c| < 1$). Define an operator $B : C_T \rightarrow C_T$ by

$$(Bx)(t) := cx(t - \sigma), \quad \forall x \in C_T, t \in \mathbb{R}. \quad (2.8)$$

Clearly, $B^j x(t) = c^j x(t - j\sigma)$ and $A = I - B$. Note also that $\|B\| < |c| < 1$. Therefore, A has a continuous inverse $A^{-1} : C_T \rightarrow C_T$ with $A^{-1} = (I - B)^{-1} = \sum_{j=0}^{\infty} B^j$; here $B^0 x(t) := x(t)$. Hence,

$$\left[A^{-1} f \right](t) = \sum_{j=0}^{\infty} \left[B^j f \right](t) = f(t) + \sum_{j=1}^{\infty} c^j f(t - j\sigma), \quad (2.9)$$

and so

$$\begin{aligned} \left| \left[A^{-1} f(t) \right] \right| &= \left| \sum_{j=0}^{\infty} \left[B^j f \right](t) \right| = \left| \sum_{j=0}^{\infty} c^j f(t - j\sigma) \right| \leq \frac{|f|_0}{1 - |c|}, \\ \int_0^T \left| \left[A^{-1} f \right](t) \right| dt &\leq \sum_{j=0}^{\infty} \int_0^T \left| (B^j f)(t) \right| dt = \sum_{j=0}^{\infty} \int_0^T |c^j f(t - j\sigma)| dt \leq \frac{1}{1 - |c|} \int_0^T |f(t)| dt. \end{aligned} \quad (2.10)$$

Case 2 ($|c| > 1$). Define operators

$$\begin{aligned} E : C_T &\longrightarrow C_T, & (Ex)(t) &:= x(t) - \frac{1}{c}x(t + \sigma), \\ B_1 : C_T &\longrightarrow C_T, & (B_1 x)(t) &:= \frac{1}{c}x(t + \sigma). \end{aligned} \quad (2.11)$$

From the definition of the linear operator B_1 , we have

$$\begin{aligned} (B_1^j f)(t) &= \frac{1}{c^j} f(t + j\sigma), \\ \sum_{j=0}^{\infty} (B_1^j f)(t) &= f(t) + \sum_{j=1}^{\infty} \frac{1}{c^j} f(t + j\sigma). \end{aligned} \quad (2.12)$$

Since $\|B_1\| < 1$, the operator E has a bounded inverse $E^{-1} : C_T \rightarrow C_T$ with

$$E^{-1} = (I - B_1)^{-1} = I + \sum_{j=1}^{\infty} B_1^j, \quad (2.13)$$

and so, for any $f \in C_T$,

$$(E^{-1} f)(t) = f(t) + \sum_{j=1}^{\infty} (B_1^j f)(t). \quad (2.14)$$

On the other hand, from $[Ax](t) = x(t) - cx(t - \sigma)$, we have

$$[Ax](t) = x(t) - cx(t - \sigma) = -c \left[x(t - \sigma) - \frac{1}{c}x(t) \right]. \quad (2.15)$$

That is,

$$[Ax](t) = -c(Ex)(t - \sigma). \quad (2.16)$$

Now, for any $f \in C_T$, if $x(t)$ satisfies

$$[Ax](t) = f(t), \quad (2.17)$$

then we have

$$-c(Ex)(t - \sigma) = f(t), \quad (2.18)$$

or

$$(Ex)(t) = -\frac{f(t + \sigma)}{c} = f_1(t). \quad (2.19)$$

So, we have

$$x(t) = (E^{-1}f_1)(t) = f_1(t) + \sum_{j=1}^{\infty} B_1^j f_1(t) = -\frac{f(t + \sigma)}{c} - \sum_{j=1}^{\infty} B_1^j \frac{f(t + \sigma)}{c}. \quad (2.20)$$

So, A^{-1} exists and satisfies

$$\begin{aligned} [A^{-1}f](t) &= -\frac{f(t + \sigma)}{c} - \sum_{j=1}^{\infty} B_1^j \frac{f(t + \sigma)}{c} = -\frac{f(t + \sigma)}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f(t + (j + 1)\sigma), \\ \left| [A^{-1}f](t) \right| &= \left| -\frac{f(t + \sigma)}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f(t + (j + 1)\sigma) \right| \leq \frac{|f|_0}{|c| - 1}. \end{aligned} \quad (2.21)$$

This proves (1) and (2) of Lemma 2.3. Finally, (3) is easily verified. \square

By Hale's terminology [14], a solution $x(t)$ of (1.3) is that $x(t) \in C^1(\mathbb{R}, \mathbb{R})$ such that $Ax \in C^1(\mathbb{R}, \mathbb{R})$ and (1.3) is satisfied on \mathbb{R} . In general, $x(t)$ does not belong to $C^1(\mathbb{R}, \mathbb{R})$. But we can see easily from $(Ax)'(t) = Ax'(t)$ that a solution $x(t)$ of (1.3) must belong to $C^1(\mathbb{R}, \mathbb{R})$. Equation (1.3) is transformed into

$$\left(\varphi_p \left((Ax^{(l)})(t) \right) \right)^{(n-l)} = F(t, x(t), x'(t), \dots, x^{(l-1)}(t)). \quad (2.22)$$

Lemma 2.5 (see [4]). *If $p > 1$, then*

$$\int_0^T \left| (A^{-1}f)(t) \right|^p dt \leq \frac{1}{|1 - |c||^p} \int_0^T |f(t)|^p dt, \quad \forall f \in C_T. \quad (2.23)$$

Now we consider (2.22). Define the conjugate index $q \in (1, 2]$ by $1/p + 1/q = 1$. Introducing new variables

$$\begin{aligned} y_1(t) &= x(t), & y_2(t) &= x'(t), & y_3(t) &= x''(t), \dots, & y_l(t) &= x^{(l-1)}(t), \\ y_{l+1}(t) &= \varphi_p\left(Ax^{(l)}(t)\right), & y_{l+2}(t) &= \left(\varphi_p\left(Ax^{(l)}(t)\right)\right)', \dots, & y_n(t) &= \left(\varphi_p\left(Ax^{(l)}(t)\right)\right)^{(n-l-1)}. \end{aligned} \quad (2.24)$$

Using the fact that $\varphi_q \circ \varphi_p \equiv \text{id}$ and by Lemma 2.3, (1.3) can be rewritten as

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= y_3(t), \\ &\vdots \\ y_{l-1}'(t) &= y_l(t), \\ y_l'(t) &= A^{-1}\varphi_q(y_{l+1}(t)), \\ y_{l+1}'(t) &= y_{l+2}(t), \\ &\vdots \\ y_{n-1}'(t) &= y_n(t), \\ y_n'(t) &= F(t, y_1(t), y_2(t), \dots, y_l(t)). \end{aligned} \quad (2.25)$$

It is clear that, if $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ is a T -periodic solution to (2.25), then $y_1(t)$ must be a T -periodic solution to (1.3). Thus, the problem of finding a T -periodic solution for (1.3) reduces to finding one for (2.25).

Define the linear spaces

$$X = Y = \left\{ y = (y_1(\cdot), y_2(\cdot), \dots, y_n(\cdot))^T \in C^0(\mathbb{R}, \mathbb{R}^n) : y(t+T) \equiv y(t) \right\} \quad (2.26)$$

with norm $\|y\| = \max\{\|y_1\|, \|y_2\|, \dots, \|y_n\|\}$. Obviously, X and Y are Banach spaces. Define

$$L : D(L) = \left\{ y \in C^1(\mathbb{R}, \mathbb{R}^n) : y(t+T) = y(t) \right\} \subset X \longrightarrow Y \quad (2.27)$$

by

$$Ly = y' = \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_i \\ \vdots \\ y'_n \end{pmatrix}. \quad (2.28)$$

Moreover, define

$$N : X \longrightarrow Y \quad (2.29)$$

by

$$Ny = \begin{pmatrix} y_2(t) \\ y_3(t) \\ \vdots \\ A^{-1}\varphi_q(y_{l+1}(t)) \\ y_{l+2}(t) \\ \vdots \\ F(t, y_1(t), y_2(t), \dots, y_l(t)) \end{pmatrix}. \quad (2.30)$$

Then, (2.25) can be rewritten as the abstract equation $Ly = Ny$. From the definition of L , one can easily see that $\text{Ker } L = \{y \in C^1(\mathbb{R}, \mathbb{R}^n) : y \text{ is constant}\} \simeq \mathbb{R}^n$ and $\text{Im } L = \{y : y \in X, \int_0^T y(s)ds = 0\}$. So, L is a Fredholm operator with index zero. Let $P : X \rightarrow \text{Ker } L$ and $Q : Y \rightarrow \text{Im } Q$ be defined by

$$Py = \frac{1}{T} \int_0^T y(s)ds, \quad Qy = \frac{1}{T} \int_0^T y(s)ds. \quad (2.31)$$

It is easy to see that $\text{Ker } L = \text{Im } Q = \mathbb{R}^n$. Moreover, for all $y \in Y$, if we write $y^* = y - Q(y)$, we have $\int_0^T y^*(s)ds = 0$ and so $y^* \in \text{Im } L$. This is to say $Y = \text{Im } Q \oplus \text{Im } L$ and $\dim(Y/\text{Im } L) = \dim \text{Im } Q = \dim \text{Ker } L$. So, L is a Fredholm operator with index zero. Let K denote the inverse of $L|_{\text{Ker } P \cap D(L)}$, then we have

$$[Ky](t) = \left(\int_0^T G_1(t,s)y_1(s)ds, \int_0^T G_2(t,s)y_2(s)ds, \dots, \int_0^T G_n(t,s)y_n(s)ds \right)^T, \quad (2.32)$$

where

$$G_i(t, s) = \begin{cases} \frac{s}{T}, & 0 \leq s < t \leq T, \\ \frac{s-T}{T}, & 0 \leq t \leq s \leq T, \end{cases} \quad i = 1, 2, \dots, n. \quad (2.33)$$

From (2.30) and (2.33), it is clear that QN and $K(I - Q)N$ are continuous, and $QN(\overline{\Omega})$ is bounded, and so $K(I - Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$. Hence, N is L -compact on $\overline{\Omega}$. For the function $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ defined as (2.24), we have the following.

Lemma 2.6. *If $y(t) \in C^1(\mathbb{R}, \mathbb{R}^n)$ and $y(t + T) = y(t)$, then*

$$\int_0^T |y'_i(t)|^p dt \leq \frac{1}{|1-c|^p} \left(\frac{T}{\pi_p}\right)^{p(l-i)} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y'_n(t)|^q dt, \quad (2.34)$$

where $1/p + 1/q = 1$, $p \geq 2$, $i = 1, 2, \dots, l-1$.

Proof. From $y_1(0) = y_1(T)$, there is a point $t_1 \in [0, T]$ such that $y'_1(t_1) = 0$. Let $\omega_1(t) = y'_1(t + t_1)$. Then, $\omega_1(0) = \omega_1(T) = 0$. From $y_2(0) = y_2(T)$, there is a point $t_2 \in [0, T]$ such that $y'_2(t_2) = 0$. Let $\omega_2(t) = y'_2(t + t_2)$. Then, $\omega_2(0) = \omega_2(T) = 0$. Continuing this way, we get from $y_{l-1}(0) = y_{l-1}(T)$ a point $t_{l-1} \in [0, T]$ such that $y'_{l-1}(t_{l-1}) = 0$. Let $\omega_{l-1}(t) = y'_{l-1}(t + t_{l-1})$. Then, $\omega_{l-1}(0) = \omega_{l-1}(T) = 0$. From $y_l(t) = y_l(t + T)$, we have $\int_0^T (Ay'_l)(t) dt = \int_0^T (Ay_l)'(t) dt = (Ay_l)(t)|_0^T = 0$, so there is a point $t_l \in [0, T]$ such that $Ay'_l(t_l) = 0$; hence, we have $\varphi_p((Ay'_l)(t_l)) = 0$. Let $\omega_l(t) = \varphi_p((Ay'_l)(t + t_l)) = y'_{l+1}(t + t_l)$. Then, $\omega_l(0) = \omega_l(T) = 0$. Continuing this way, we get from $y_{n-1}(0) = y_{n-1}(T)$ that there is a point $t_{n-1} \in [0, T]$ such that $y'_{n-1}(t_{n-1}) = 0$. Let $\omega_{n-1}(t) = y'_{n-1}(t + t_{n-1})$. Then, $\omega_{n-1}(0) = \omega_{n-1}(T) = 0$. By Lemma 2.2, we have

$$\begin{aligned} \int_0^T |y'_1(t)|^p dt &= \int_0^T |\omega_1(t)|^p dt \\ &\leq \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega'_1(t)|^p dt \\ &= \left(\frac{T}{\pi_p}\right)^p \int_0^T |y'_2(t)|^p dt \\ &= \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega_2(t)|^p dt \\ &\leq \left(\frac{T}{\pi_p}\right)^{2p} \int_0^T |\omega'_2(t)|^p dt \\ &\vdots \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{T}{\pi_p}\right)^{p(l-1)} \int_0^T |\omega'_{l-1}(t)|^p dt \\
&= \left(\frac{T}{\pi_p}\right)^{p(l-1)} \int_0^T |y'_l(t)|^p dt.
\end{aligned}
\tag{2.35}$$

By Lemma 2.5 and Lemma 2.2, we have

$$\begin{aligned}
\int_0^T |y'_l(t)|^p dt &= \int_0^T \left| A^{-1} \varphi_q(y_{l+1}(t)) \right|^p dt \\
&\leq \frac{1}{|1-|c||^p} \int_0^T |\varphi_q(y_{l+1}(t))|^p dt \\
&= \frac{1}{|1-|c||^p} \int_0^T |y_{l+1}(t)|^{pq-p} dt \\
&= \frac{1}{|1-|c||^p} \int_0^T |y_{l+1}(t)|^q dt \\
&= \frac{1}{|1-|c||^p} \int_0^T |\omega_l(t)|^q dt \\
&\leq \frac{1}{|1-|c||^p} \left(\frac{T}{\pi_q}\right)^q \int_0^T |\omega'_l(t)|^q dt \\
&= \frac{1}{|1-|c||^p} \left(\frac{T}{\pi_q}\right)^q \int_0^T |y_{l+2}(t)|^q dt \\
&\vdots \\
&\leq \frac{1}{|1-|c||^p} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |\omega'_{n-1}(t)|^q dt \\
&= \frac{1}{|1-|c||^p} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y'_n(t)|^q dt.
\end{aligned}
\tag{2.36}$$

Combining (2.35) and (2.36), we get

$$\int_0^T |y'_1(t)|^p dt \leq \frac{1}{|1-|c||^p} \left(\frac{T}{\pi_p}\right)^{p(l-1)} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y'_n(t)|^q dt.
\tag{2.37}$$

Similarly, we get

$$\int_0^T |y'_i(t)|^p dt \leq \frac{1}{|1-|c||^p} \left(\frac{T}{\pi_p}\right)^{p(l-i)} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y'_n(t)|^q dt. \quad (2.38)$$

This completes the proof of Lemma 2.6. \square

Remark 2.7. In particular, if we take $p = 2$, then $q = 2$ and

$$\pi_p = \pi_q = \pi_2 = 2 \int_0^{(2-1)/2} \frac{ds}{(1-s^2/(2-1))^{1/2}} = \frac{2\pi(2-1)^{1/2}}{2 \sin(\pi/2)} = \pi. \quad (2.39)$$

In this case, (2.34) is transformed into

$$\int_0^T |y_i(t)|^2 dt \leq \frac{1}{|1-|c||^2} \left(\frac{T}{\pi}\right)^{2(n-i)} \int_0^T |y'_n(t)|^2 dt. \quad (2.40)$$

3. Main Results

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel.

(H_1) There exists a constant $D > 0$ such that

$$z_1 F(t, z_1, z_2, \dots, z_l) > 0, \quad \forall (t, z_1, z_2, \dots, z_l) \in [0, T] \times \mathbb{R}^l, \quad \text{with } |z_1| > D. \quad (3.1)$$

(H_2) There exists a constant $M > 0$ such that

$$|F(t, z_1, z_2, \dots, z_l)| \leq M, \quad \forall (t, z_1, z_2, \dots, z_l) \in [0, T] \times \mathbb{R}^l. \quad (3.2)$$

(H_3) There exist nonnegative constants $\alpha_1, \alpha_2, \dots, \alpha_l, m$ such that

$$|F(t, z_1, z_2, \dots, z_l)| \leq \alpha_1 |z_1| + \alpha_2 |z_2| + \dots + \alpha_l |z_l| + m, \quad \forall (t, z_1, z_2, \dots, z_l) \in [0, T] \times \mathbb{R}^l. \quad (3.3)$$

(H_4) There exist nonnegative constants $\gamma_1, \gamma_2, \dots, \gamma_n$ such that

$$|F(t, u_1, u_2, \dots, u_n) - F(t, v_1, v_2, \dots, v_n)| \leq \gamma_1 |u_1 - v_1| + \gamma_2 |u_2 - v_2| + \dots + \gamma_n |u_n - v_n| \quad (3.4)$$

for all $(t, u_1, u_2, \dots, u_n), (t, v_1, v_2, \dots, v_n) \in [0, T] \times \mathbb{R}^n$.

Theorem 3.1. *If (H_1) and (H_2) hold, then (1.3) has at least one nonconstant T -periodic solution.*

Proof. Consider the equation

$$Ly = \lambda Ny, \quad \lambda \in (0, 1). \quad (3.5)$$

Let $\Omega_1 = \{y \in D(L) : Ly = \lambda Ny, \lambda \in (0, 1)\}$. If $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \Omega_1$, then

$$\begin{aligned} y_1'(t) &= \lambda y_2(t), \\ y_2'(t) &= \lambda y_3(t), \\ &\vdots \\ y_{l-1}'(t) &= \lambda y_l(t), \\ y_l'(t) &= \lambda \varphi_q(y_{l+1}(t)), \\ y_{l+1}'(t) &= \lambda y_{l+2}(t), \\ &\vdots \\ y_{n-1}'(t) &= \lambda y_n(t), \\ y_n'(t) &= \lambda F(t, y_1(t), y_2(t), \dots, y_l(t)). \end{aligned} \quad (3.6)$$

We first claim that there exists a constant $\xi \in \mathbb{R}$ such that

$$|y_1(\xi)| \leq D. \quad (3.7)$$

Integrating the last equation of (3.6) over $[0, T]$, we have

$$\int_0^T F(t, y_1(t), y_2(t), \dots, y_l(t)) dt = 0. \quad (3.8)$$

By the continuity of F , there exists $\xi \in [0, T]$ such that

$$F(\xi, y_1(\xi), \dots, y_l(\xi)) = 0. \quad (3.9)$$

From assumption (H_1) , we get (3.7). As a consequence, we have

$$|y_1(t)| = \left| y_1(\xi) + \int_{\xi}^t y_1'(s) ds \right| \leq D + \int_0^T |y_1'(s)| ds. \quad (3.10)$$

On the other hand, multiplying both sides of the last equation of (3.6) by $y'_n(t)$ and integrating over $[0, T]$, using assumption (H_2) , we have

$$\begin{aligned}
 \int_0^T |y'_n(t)|^2 dt &= \lambda \int_0^T F(t, y_1(t), y_2(t), \dots, y_l(t)) y'_n(t) dt \\
 &\leq \int_0^T |F(t, y_1(t), y_2(t), \dots, y_l(t))| |y'_n(t)| dt \\
 &\leq M \int_0^T |y'_n(t)| dt \\
 &\leq MT^{1/2} \left(\int_0^T |y'_n(t)|^2 dt \right)^{1/2}.
 \end{aligned} \tag{3.11}$$

It is easy to see that there exists a constant $M'_n > 0$ (independent of λ) such that

$$\int_0^T |y'_n(t)|^2 dt \leq M'_n. \tag{3.12}$$

From $y_{n-1}(0) = y_{n-1}(T)$, there exists a point $t_1 \in [0, T]$ such that $y_n(t_1) = 0$. By Hölder's inequality, we have

$$|y_n(t)| \leq \int_0^T |y'_n(t)| dt \leq T^{1/2} \left(\int_0^T |y'_n(t)|^2 dt \right)^{1/2} \leq T^{1/2} M_n'^{1/2} := M_n. \tag{3.13}$$

From $y_{n-2}(0) = y_{n-2}(T)$, there exists a point $t_2 \in [0, T]$ such that $y_{n-1}(t_2) = 0$, and we have

$$|y_{n-1}(t)| \leq \int_0^T |y'_{n-1}(t)| dt = \int_0^T |\lambda y_n(t)| dt \leq \int_0^T |y_n(t)| dt \leq TM_n := M_{n-1}. \tag{3.14}$$

Continuing this way for y_{n-2}, \dots, y_{l+1} , we get

$$|y_{l+1}(t)| \leq TM_{l+2} := M_{l+1}. \tag{3.15}$$

Hence,

$$\begin{aligned}
 |y_l(t)| &\leq \int_0^T |y'_l(t)| dt \leq \int_0^T \left| \lambda A^{-1} \varphi_q(y_{l+1}(t)) \right| dt \leq \frac{1}{|1-c|} \int_0^T |y_{l+1}(t)|^{q-1} dt \\
 &\leq \frac{1}{|1-c|} T M_{l+1}^{q-1} := M_l, \\
 |y_{l-1}(t)| &\leq T M_l := M_{l-1}, \\
 &\vdots \\
 |y_2(t)| &\leq T M_3 := M_2.
 \end{aligned} \tag{3.16}$$

Meanwhile, from (3.10), we get

$$|y_1(t)| \leq D + \int_0^T |y'_1(t)| dt \leq D + T M_2 := M_1. \tag{3.17}$$

Let $M_0 = \max\{M_1, M_2, \dots, M_n\}$. Then, obviously $\|y_1\| \leq M_0$, $\|y_2\| \leq M_0, \dots$, and $\|y_n\| \leq M_0$.

Let $\Omega_2 = \{y \in \text{Ker } L : Ny \in \text{Im } L\}$. If $y \in \Omega_2$, then $y \in \text{Ker } L$, which means that $y = \text{constant}$ and $QNy = 0$. We see that

$$\begin{aligned}
 y_2 &= 0, \\
 y_3 &= 0, \\
 &\vdots \\
 y_n &= 0, \\
 F(t, y_1, 0, \dots, 0) &= 0.
 \end{aligned} \tag{3.18}$$

So,

$$|y_1| \leq D \leq M_0, \quad y_2 = y_3 = \dots = y_n = 0 \leq M_0. \tag{3.19}$$

Now take $\Omega = \{y = (y_1, y_2, \dots, y_n)^\top \in X : \|y_1\| < M_0 + 1, \|y_2\| < M_0 + 1, \dots, \|y_n\| < M_0 + 1\}$. By the analysis above, it is easy to see that $\bar{\Omega}_1 \subset \Omega$, $\bar{\Omega}_2 \subset \Omega$, and conditions (1) and (2) of Lemma 2.1 are satisfied.

Next we show that condition (3) of Lemma 2.1 is also satisfied. Define an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$ as follows:

$$J(y_1, y_2, \dots, y_n)^\top := (y_n, y_1, \dots, y_{n-1})^\top. \tag{3.20}$$

Let $H(\mu, y) = \mu y + (1 - \mu)JQNy$, $(\mu, y) \in [0, 1] \times \Omega$. Then, for all $(\mu, y) \in (0, 1) \times (\partial\Omega \cap \text{Ker } L)$,

$$H(\mu, y) = \begin{pmatrix} \mu y_1 + \frac{1 - \mu}{T} \int_0^T F(t, y_1, 0, \dots, 0) dt \\ y_2 \\ \vdots \\ A^{-1} \varphi_q(y_{l+1}) \\ \vdots \\ y_n \end{pmatrix}. \quad (3.21)$$

From (H_1) , it is obvious that $yH(\mu, y) > 0$ for all $(\mu, y) \in (0, 1) \times (\partial\Omega \cap \text{Ker } L)$. Therefore,

$$\begin{aligned} \deg\{JQN, \Omega \cap \text{Ker } L, 0\} &= \deg\{H(0, y), \Omega \cap \text{Ker } L, 0\} \\ &= \deg\{H(1, y), \Omega \cap \text{Ker } L, 0\} \\ &= \deg\{I, \Omega \cap \text{Ker } L, 0\} \neq 0, \end{aligned} \quad (3.22)$$

which means that condition (3) of Lemma 2.1 is also satisfied. By applying Lemma 2.1, we conclude that equation $Ly = Ny$ has a solution $y(t)^* = (y_1^*(t), y_2^*(t), \dots, y_n^*(t))^T$ on $\bar{\Omega}$; that is, (1.3) has a T -periodic solution $y_1^*(t)$ with $\|y_1^*\| < M_0 + 1$.

Finally, observe that $y_1^*(t)$ is not constant. For, if $y_1^* \equiv a$ (constant), then from (1.3) we have $F(t, a, 0, \dots, 0) \equiv 0$, which contradicts the assumption that $F(t, a, 0, \dots, 0) \neq 0$. The proof is complete. \square

Theorem 3.2. *If (H_1) and (H_3) hold, then (1.3) has at least one nonconstant T -periodic solution if one of the following conditions holds:*

$$(1) \quad p > 2,$$

$$(2) \quad p = 2 \text{ and } 1/|1 - |c||[(\alpha_1 T + \alpha_2)(T/\pi)^{l-1} + \alpha_3(T/\pi)^{l-2} + \dots + \alpha_l(T/\pi)](T/\pi)^{n-1} < 1.$$

Proof. Let Ω_1 be defined as in Theorem 3.1. If $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \Omega_1$, then from the proof of Theorem 3.1 we have

$$y_n'(t) = \lambda F(t, y_1(t), y_2(t), \dots, y_l(t)), \quad (3.23)$$

$$|y_1|_0 \leq D + \int_0^T |y_1'(s)| ds. \quad (3.24)$$

We claim that $|y_n|_0$ is bounded.

Multiplying both sides of (3.23) by $\varphi_q(y'_n(t))$ and integrating over $[0, T]$, by using assumption (H_3) , we have

$$\begin{aligned}
\int_0^T |y'_n(t)|^q dt &= \lambda \int_0^T F(t, y_1(t), y_2(t), \dots, y_l(t)) \varphi_q(y'_n(t)) dt \\
&\leq \int_0^T |F(t, y_1(t), y_2(t), \dots, y_l(t))| |\varphi_q(y'_n(t))| dt \\
&\leq \alpha_1 \int_0^T |y_1(t)| |\varphi_q(y'_n(t))| dt + \alpha_2 \int_0^T |y_2(t)| |\varphi_q(y'_n(t))| dt \\
&\quad + \dots + \alpha_l \int_0^T |y_l(t)| |\varphi_q(y'_n(t))| dt + m \int_0^T |\varphi_q(y'_n(t))| dt \\
&\leq \alpha_1 |y_1|_0 \int_0^T |\varphi_q(y'_n(t))| dt + \alpha_2 \int_0^T |y_2(t)| |\varphi_q(y'_n(t))| dt \\
&\quad + \dots + \alpha_l \int_0^T |y_l(t)| |\varphi_q(y'_n(t))| dt + m \int_0^T |\varphi_q(y'_n(t))| dt \\
&\leq \alpha_1 \left(D + \int_0^T |y'_1(t)| dt \right) \int_0^T |\varphi_q(y'_n(t))| dt + \alpha_2 \int_0^T |y_2(t)| |\varphi_q(y'_n(t))| dt \\
&\quad + \dots + \alpha_l \int_0^T |y_l(t)| |\varphi_q(y'_n(t))| dt + m \int_0^T |\varphi_q(y'_n(t))| dt.
\end{aligned} \tag{3.25}$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\int_0^T |y'_n(t)|^q dt &\leq \alpha_1 \left[D + T^{1/p} \left(\int_0^T |y'_1(t)|^q dt \right)^{1/q} \right] T^{1/q} \left(\int_0^T |\varphi_q(y'_n(t))|^p dt \right)^{1/p} \\
&\quad + \alpha_2 \left(\int_0^T |y_2(t)|^q dt \right)^{1/q} \left(\int_0^T |\varphi_q(y'_n(t))|^p dt \right)^{1/p} \\
&\quad + \dots + \alpha_l \left(\int_0^T |y_l(t)|^q dt \right)^{1/q} \left(\int_0^T |\varphi_q(y'_n(t))|^p dt \right)^{1/p} \\
&\quad + m T^{1/q} \left(\int_0^T |\varphi_q(y'_n(t))|^p dt \right)^{1/p} \\
&\leq \alpha_1 T \cdot T^{(p-2)/q(p-1)} \left(\int_0^T |y'_1(t)|^p dt \right)^{1/q(p-1)} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} \\
&\quad + \alpha_2 T^{(p-2)/q(p-1)} \left(\int_0^T |y_2(t)|^p dt \right)^{1/q(p-1)} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} + \dots
\end{aligned}$$

$$\begin{aligned}
& + \alpha_l T^{(p-2)/q(p-1)} \left(\int_0^T |y_l(t)|^p dt \right)^{1/q(p-1)} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} \\
& + (\alpha_1 D + m) T^{1/q} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} \\
= & \alpha_1 T \cdot T^{(p-2)/p} \left(\int_0^T |y'_1(t)|^p dt \right)^{1/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} \\
& + \alpha_2 T^{(p-2)/p} \left(\int_0^T |y_2(t)|^p dt \right)^{1/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} + \\
& \dots + \alpha_l T^{(p-2)/p} \left(\int_0^T |y_l(t)|^p dt \right)^{1/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} \\
& + (\alpha_1 D + m) T^{1/q} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p}.
\end{aligned} \tag{3.26}$$

Applying Lemma 2.6 and (3.26), we have

$$\begin{aligned}
\int_0^T |y'_n(t)|^q dt & \leq \alpha_1 T \cdot T^{(p-2)/p} \frac{1}{|1-|c||} \left(\frac{T}{\pi_p} \right)^{l-1} \left(\frac{T}{\pi_q} \right)^{q(n-1)/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{2/p} \\
& + \alpha_2 T^{(p-2)/p} \frac{1}{|1-|c||} \left(\frac{T}{\pi_p} \right)^{l-1} \left(\frac{T}{\pi_q} \right)^{q(n-1)/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{2/p} + \\
& \dots + \alpha_l T^{(p-2)/p} \frac{1}{|1-|c||} \left(\frac{T}{\pi_p} \right) \left(\frac{T}{\pi_q} \right)^{q(n-1)/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{2/p} \\
& + (\alpha_1 D + m) T^{1/q} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p} \\
& \leq \frac{1}{|1-|c||} \left[(\alpha_1 T + \alpha_2) \left(\frac{T}{\pi_p} \right)^{l-1} + \alpha_3 \left(\frac{T}{\pi_p} \right)^{l-2} + \dots + \alpha_l \left(\frac{T}{\pi_p} \right) \right] \\
& \times T^{(p-2)/p} \left(\frac{T}{\pi_q} \right)^{q(n-1)/p} \cdot \left(\int_0^T |y'_n(t)|^q dt \right)^{2/p} \\
& + (\alpha_1 D + m) T^{1/q} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/p}.
\end{aligned} \tag{3.27}$$

Case 1. If $p = 2$ and $1/|1 - |c||[(\alpha_1 T + \alpha_2)(T/\pi)^{l-1} + \alpha_3(T/\pi)^{l-2} + \dots + \alpha_l(T/\pi)](T/\pi)^{n-l} < 1$, then it is easy to see that there exists a constant $M'_n > 0$ (independent of λ) such that

$$\int_0^T |y'_n(t)|^q dt \leq M'_n. \quad (3.28)$$

Case 2. If $p > 2$, then it is easy to see that there exists a constant $M'_n > 0$ (independent of λ) such that

$$\int_0^T |y'_n(t)|^q dt \leq M'_n. \quad (3.29)$$

From $y_{n-1}(0) = y_{n-1}(T)$, there exists a point $t_1 \in [0, T]$ such that $y_n(t_1) = 0$. By Hölder's inequality, we have

$$|y_n(t)| \leq \int_0^T |y'_n(t)| dt \leq T^{1/p} \left(\int_0^T |y'_n(t)|^q dt \right)^{1/q} \leq T^{1/p} M_n^{1/q} := M_n. \quad (3.30)$$

This proves the claim, and the rest of the proof of the theorem is identical to that of Theorem 3.1. \square

Remark 3.3. If (1.3) takes the form

$$\left(\varphi_p(x(t) - cx(t - \sigma))^{(l)} \right)^{(n-l)} = F(t, x(t), x'(t), \dots, x^{(l-1)}(t)) + e(t), \quad (3.31)$$

where $e(t) \in C(\mathbb{R}, \mathbb{R})$, $e(t + T) = e(t)$ and $\int_0^T e(t) dt = 0$, then the results of Theorems 3.1 and 3.2 still hold.

Remark 3.4. If $p = 2$, then (1.3) is transformed into

$$(x(t) - cx(t - \sigma))^{(n)} = F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad (3.32)$$

and the results of Theorems 3.1 and 3.2 still hold.

Next, we study the Lyapunov stability of the periodic solutions of (3.32).

Theorem 3.5. *Assume that (H_4) holds. Then every T -periodic solution of (3.32) is Lyapunov stable.*

Proof. Let

$$z_1(t) = x(t), \quad z_2(t) = x'(t), \dots, z_n(t) = x^{(n-1)}(t). \quad (3.33)$$

Then, system (3.32) is transformed into

$$\begin{aligned} z_1'(t) &= z_2(t), \\ z_2'(t) &= z_3(t), \\ &\vdots \\ z_n'(t) &= A^{-1}F(t, z_1(t), z_2(t), \dots, z_n(t)). \end{aligned} \tag{3.34}$$

Suppose now that $z^*(t) = (z_1^*(t), z_2^*(t), \dots, z_n^*(t))^T$ is a T -periodic solution of (3.34). Let $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T$ be any arbitrary solution of (3.34). For any $k = 1, \dots, n$, write $w_k(t) = z_k(t) - z_k^*(t)$. Then, it follows from (3.34) that

$$\begin{aligned} w_1'(t) &= w_2(t), \\ w_2'(t) &= w_3(t), \\ &\vdots \\ w_n'(t) &= A^{-1}(F(t, z_1(t), z_2(t), \dots, z_n(t)) - F(t, z_1^*(t), z_2^*(t), \dots, z_n^*(t))), \end{aligned} \tag{3.35}$$

and so

$$\begin{aligned} |w_1'(t)| &= |w_2(t)|, \\ |w_2'(t)| &= |w_3(t)|, \\ &\vdots \\ |w_n'(t)| &= \left| A^{-1}(F(t, z_1(t), z_2(t), \dots, z_n(t)) - F(t, z_1^*(t), z_2^*(t), \dots, z_n^*(t))) \right|. \end{aligned} \tag{3.36}$$

Let $u_k^{(l)}(t) = |w_k^{(l)}(t)|$, $l = 0, 1$, $k = 1, 2, \dots, n$. Then,

$$\begin{aligned} u_1'(t) &= u_2(t), \\ u_2'(t) &= u_3(t), \\ &\vdots \\ u_n'(t) &= \left| A^{-1}(F(t, z_1(t), z_2(t), \dots, z_n(t)) - F(t, z_1^*(t), z_2^*(t), \dots, z_n^*(t))) \right|. \end{aligned} \tag{3.37}$$

Take $\beta = \max\{\gamma_1/|1 - |c||, \gamma_2/|1 - |c|| + 1, \dots, \gamma_n/|1 - |c|| + 1\} + 1$, and define a function $V(\cdot)$ by

$$V(t, u_1, \dots, u_n) := e^{-\beta t} \sum_{k=1}^n u_k(t). \tag{3.38}$$

Let $U(u_1, \dots, u_n) = \sum_{k=1}^n y_k(t)$. It is obvious that $V(t, u_1, \dots, u_n) > 0$ and $V(t, u_1, \dots, u_n) \geq U(u_1, \dots, u_n) > 0$. From (H_4) and Lemma 2.3, we get

$$\begin{aligned}
 \dot{V}(t, u_1, \dots, u_n) &= -\beta e^{-\beta t} \left(\sum_{k=1}^n u_k(t) \right) + e^{-\beta t} (u_2(t) + \dots + u_n(t)) \\
 &\quad + e^{-\beta t} \left| A^{-1} (F(t, z_1(t), z_2(t), \dots, z_n(t)) - F(t, z_1^*(t), z_2^*(t), \dots, z_n^*(t))) \right| \\
 &\leq -\beta e^{-\beta t} \left(\sum_{k=1}^n u_k(t) \right) + e^{-\beta t} (u_2(t) + \dots + u_n(t)) \\
 &\quad + \frac{e^{-\beta t}}{|1 - |c||} \left| (F(t, z_1(t), z_2(t), \dots, z_n(t)) - F(t, z_1^*(t), z_2^*(t), \dots, z_n^*(t))) \right| \\
 &\leq -\beta e^{-\beta t} \left(\sum_{k=1}^n u_k(t) \right) + e^{-\beta t} (u_2(t) + \dots + u_n(t)) \\
 &\quad + \frac{e^{-\beta t}}{|1 - |c||} (\gamma_1 |z_1(t) - z_1^*(t)| + \dots + \gamma_n |z_n(t) - z_n^*(t)|) \\
 &= -\beta e^{-\beta t} \left(\sum_{k=1}^n u_k(t) \right) + e^{-\beta t} (u_2(t) + \dots + u_n(t)) \\
 &\quad + \frac{e^{-\beta t}}{|1 - |c||} (\gamma_1 |w_1(t)| + \dots + \gamma_n |w_n(t)|) \\
 &= -\beta e^{-\beta t} \left(\sum_{k=1}^n u_k(t) \right) + e^{-\beta t} (u_2(t) + \dots + u_n(t)) \\
 &\quad + \frac{e^{-\beta t}}{|1 - |c||} (\gamma_1 u_1(t) + \dots + \gamma_n u_n(t)) \\
 &= \left(-\beta + \frac{\gamma_1}{|1 - |c||} \right) u_1(t) e^{-\beta t} + \sum_{k=2}^n \left(-\beta + 1 + \frac{\gamma_k}{|1 - |c||} \right) u_k(t) e^{-\beta t} \\
 &< 0.
 \end{aligned} \tag{3.39}$$

Hence, V is a Lyapunov function for nonautonomous (3.32) (see [15, page 50]), and so the T -periodic solution z^* of (3.32) is Lyapunov stable. \square

Finally, we present an example to illustrate our result.

Example 3.6. Consider the n -order delay differential equation

$$\left(\varphi_p(x(t) - 3x(t - \sigma)) \right)^{(n-4)} = \frac{1}{3\pi} x(t) + \frac{1}{8} \sin x'(t) + \frac{1}{8} \cos x''(t) \sin 2t + \frac{1}{8} \sin x'''(t). \tag{3.40}$$

Here p is a constant with $p \geq 2$. Comparing with (1.3), we have $c = 3$ and

$$F(t, z_1, z_2, z_3, z_4) = \frac{1}{3\pi} z_1 + \frac{1}{8} \sin z_2 + \frac{1}{8} \cos z_3 \sin 2t + \frac{1}{8} \sin z_4. \quad (3.41)$$

Observe that F has period $T = \pi$ and satisfies

$$F(t, a, 0, 0, 0) = \frac{1}{3\pi} a + \frac{1}{8} \sin 2t \neq 0, \quad \forall a \in \mathbb{R}. \quad (3.42)$$

Pick $D = 3\pi$. Then,

$$\begin{aligned} |z_1 F(t, z_1, z_2, z_3, z_4)| &= |z_1| \cdot \left| \frac{1}{3\pi} z_1 + \frac{1}{8} \sin z_2 + \frac{1}{8} \cos z_3 \sin 2t + \frac{1}{8} \sin z_4 \right| \\ &\geq |z_1| \cdot \left| \frac{1}{3\pi} |z_1| - \left| \frac{1}{8} \sin z_2 + \frac{1}{8} \cos z_3 \sin 2t + \frac{1}{8} \sin z_4 \right| \right| \\ &\geq 3\pi \cdot \left[1 - \frac{3}{8} \right] \\ &> 0 \end{aligned} \quad (3.43)$$

for all $(t, z_1, z_2, z_3, z_4) \in [0, T] \times \mathbb{R}^4$ with $|z_1| > D = 3\pi$. Hence, (H_1) holds. On the other hand, since

$$\begin{aligned} |F(t, z_1, z_2, z_3, z_4)| &\leq \frac{1}{3\pi} |z_1| + \left| \frac{1}{8} \sin z_2 + \frac{1}{8} \cos z_3 \sin 2t + \frac{1}{8} \sin z_4 \right| \\ &< \frac{1}{3\pi} |z_1| + 1, \end{aligned} \quad (3.44)$$

assumption (H_3) holds with $\alpha_1 = 1/3\pi$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$, and $m = 1$.

Case 1. If $p > 2$, then by (1) of Theorem 3.2, (3.40) has at least one nonconstant π -periodic solution.

Case 2. If $p = 2$, then

$$\begin{aligned} &\frac{1}{|1 - |c||} \left[(\alpha_1 T + \alpha_2) \left(\frac{T}{\pi} \right)^3 + \alpha_3 \left(\frac{T}{\pi} \right)^2 + \alpha_4 \left(\frac{T}{\pi} \right) \right] \left(\frac{T}{\pi} \right)^{n-4} \\ &= \frac{1}{2} \times \left[\left(\frac{1}{3\pi} \times \pi + 0 \right) \times 1 + 0 \times 1 + 0 \times 1 \right] \times 1 \\ &= \frac{1}{2} \times \frac{1}{3} < 1. \end{aligned} \quad (3.45)$$

So by (2) of Theorem 3.2, (3.40) has at least one nonconstant π -periodic solution.

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