

Research Article

Multiple Positive Solutions of Fourth-Order Impulsive Differential Equations with Integral Boundary Conditions and One-Dimensional p -Laplacian

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By using the fixed point theory for completely continuous operator, this paper investigates the existence of positive solutions for a class of fourth-order impulsive boundary value problems with integral boundary conditions and one-dimensional p -Laplacian. Moreover, we offer some interesting discussion of the associated boundary value problems. Upper and lower bounds for these positive solutions also are given, so our work is new.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. For an introduction of the basic theory of impulsive differential equations, see Lakshmikantham et al. [1]; for an overview of existing results and of recent research areas of impulsive differential equations, see Benchohra et al. [2]. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of differential equations (see, e.g., [3–18] and references cited therein).

Moreover, the theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For boundary-value problems with integral boundary conditions and

comments on their importance, we refer the reader to the papers by Gallardo [19], Karakostas and Tsamatos [20], Lomtatidze and Malaguti [21], and the references therein. For more information about the general theory of integral equations and their relation with boundary-value problems, we refer to the book of Corduneanu [22] and Agarwal and O'Regan [23].

On the other hand, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary-value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention in the literature. To identify a few, we refer the reader to [24–46] and references therein. In particular, we would like to mention some results of Zhang et al. [34], Kang et al. [44], and Webb et al. [45]. In [34], Zhang et al. studied the following fourth-order boundary value problem with integral boundary conditions

$$\begin{aligned}x^{(4)}(t) - \lambda f(t, x(t)) &= \theta, \quad 0 < t < 1, \\x(0) = x(1) &= \int_0^1 g(s)x(s)ds, \\x''(0) = x''(1) &= \int_0^1 h(s)x(s)ds,\end{aligned}\tag{1.1}$$

where λ is a positive parameter, $f \in C([0, 1] \times P, P)$, θ is the zero element of E , and $g, h \in L^1[0, 1]$. The authors investigated the multiplicity of positive solutions to problem (1.1) by using the fixed point index theory in cone for strict set contraction operator.

In [44], Kang et al. have improved and generalized the work of [34] by applying the fixed point theory in cone for a strict set contraction operator; they proved that there exist various results on the existence of positive solutions to a class of fourth-order singular boundary value problems with integral boundary conditions

$$\begin{aligned}u^{(4)}(t) &= a(t)f(t, u(t), -u''(t)) + b(t)g(t, u(t), -u''(t)), \quad 0 < t < 1, \\a_1u(0) - b_1u'(0) &= \int_0^1 m_1(s)u(s)ds, \quad c_1u(1) + d_1u'(1) = \int_0^1 n_1(s)u(s)ds, \\a_2u''(0) - b_2u'''(0) &= \int_0^1 m_2(s)u''(s)ds, \quad c_2u''(2) + d_2u'''(1) = \int_0^1 n_2(s)u''(s)ds,\end{aligned}\tag{1.2}$$

where $a, b \in C((0, 1), [0, +\infty))$ and may be singular at $t = 0$ or $t = 1$; $f, g : (0, 1) \times P \setminus \{\theta\} \times P \setminus \{\theta\} \rightarrow P$ are continuous and may be singular at $t = 0, 1$, $u = 0$, and $u'' = 0$; a_i, b_i, c_i , and $d_i \in [0, +\infty)$, and $\rho_i = a_i c_i + a_i d_i + b_i c_i > 0$, and $m_i, n_i \in L^1[0, 1]$ are nonnegative, $i = 1, 2$.

More recently, by using a unified approach, Webb et al. [45] considered the widely studied boundary conditions corresponding to clamped and hinged ends and many nonlocal boundary conditions and established excellent existence results for multiple positive solutions of fourth-order nonlinear equations which model deflections of an elastic beam

$$u^4(t) = g(t)f(t, u(t)), \quad \text{for almost every } t \in (0, 1),\tag{1.3}$$

subject to various boundary conditions

$$\begin{aligned}
 u(0) &= 0, & u(1) &= \alpha[u], & u'(0) &= 0, & u'(1) &= 0, \\
 u(0) &= 0, & u(1) &= 0, & u'(0) &= 0, & u'(1) + \alpha[u] &= 0, \\
 u(0) &= 0, & u(1) &= \alpha[u], & u''(0) &= 0, & u''(1) &= 0, \\
 u(0) &= 0, & u(1) &= 0, & u''(0) &= 0, & u''(1) + \alpha[u] &= 0,
 \end{aligned} \tag{1.4}$$

where $\alpha[u]$ denotes a linear functional on $C[0,1]$ given by $\alpha[u] = \int_0^1 u(s)dA(s)$ involving a Stieltjes integral, and A is a function of bounded variation.

At the same time, we notice that there has been a considerable attention on p -Laplacian BVPs [18, 32, 35, 36, 38, 42] as p -Laplacian appears in the study of flow through porous media ($p = 3/2$), nonlinear elasticity ($p \geq 2$), glaciology ($1 \leq p \leq 4/3$), and so forth. Here, it is worth mentioning that Liu et al. [43] considered the following fourth-order four-point boundary value problem:

$$\begin{aligned}
 (\phi_p(x''(t)))'' &= \omega(t)f(t, x(t)), & t &\in [0, 1], \\
 x(0) &= 0, & x(1) &= ax(\xi), \\
 x''(0) &= 0, & x''(1) &= bx''(\eta),
 \end{aligned} \tag{1.5}$$

where $0 < \xi, \eta < 1$, $0 \leq a < b < 1$, and $f \in C((0,1) \times (0, +\infty), [0, +\infty))$. By using upper and lower solution method, fixed-point theorems, and the properties of Green's function $G(t, s)$ and $H(t, s)$, the authors give sufficient conditions for the existence of one positive solution.

Motivated by works mentioned above, in this paper, we consider the existence of positive solutions for a class of boundary value problems with integral boundary conditions of fourth-order impulsive differential equations:

$$\begin{aligned}
 (\phi_p(y''(t)))'' &= f(t, y(t)), & t &\in J, t \neq t_k, k = 1, 2, \dots, m, \\
 \Delta y'|_{t=t_k} &= -I_k(y(t_k)), & k &= 1, 2, \dots, m, \\
 y(0) = y(1) &= \int_0^1 g(t)y(t)dt, \\
 \phi_p(y''(0)) = \phi_p(y''(1)) &= \int_0^1 h(t)\phi_p(y''(t))dt.
 \end{aligned} \tag{1.6}$$

Here $J = [0, 1]$, $\phi_p(s)$ is p -Laplace operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$, $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$, $R^+ = [0, +\infty)$, t_k ($k = 1, 2, \dots, m$) (where m is fixed positive integer) are fixed points with $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots < t_m < t_{m+1} = 1$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, where $x'(t_k^+)$ and $x'(t_k^-)$ represent the right-hand limit and left-hand limit of $x'(t)$ at $t = t_k$, respectively, and $g, h \in L^1[0, 1]$ is nonnegative.

For the case of $I_k = 0$, $k = 1, 2, \dots, m$, problem (1.6) reduces to the problem studied by Zhang et al. in [33]. By using the fixed point theorem in cone, the authors obtained some

sufficient conditions for the existence and multiplicity of symmetric positive solutions for a class of p -Laplacian fourth-order differential equations with integral boundary conditions.

For the case of $I_k = 0$, $k = 1, 2, \dots, m$, $g = 0$, $h = 0$, and $p = 2$, problem (1.6) is related to fourth-order two-points boundary value problem of ODE. Under this case, problem (1.6) has received considerable attention (see, e.g., [40–42] and references cited therein). Aftabizadeh [40] showed the existence of a solution to problem (1.6) under the restriction that f is a bounded function. Bai and Wang [41] have applied the fixed point theorem and degree theory to establish existence, uniqueness, and multiplicity of positive solutions to problem (1.6). Ma and Wang [42] have proved that there exist at least two positive solutions by applying the existence of positive solutions under the fact that $f(t, u)$ is either superlinear or sublinear on u by employing the fixed point theorem of cone extension or compression.

Being directly inspired by [18, 34, 43], in the present paper, we consider some existence results for problem (1.6) in a specially constructed cone by using the fixed point theorem. The main features of this paper are as follows. Firstly, comparing with [39–43], we discuss the impulsive boundary value problem with integral boundary conditions, that is, problem (1.6) includes fourth-order two-, three-, multipoint, and nonlocal boundary value problems as special cases. Secondly, the conditions are weaker than those of [33, 34, 46], and we consider the case $I_k \neq 0$. Finally, comparing with [33, 34, 39–43, 46], upper and lower bounds for these positive solutions also are given. Hence, we improve and generalize the results of [33, 34, 39–43, 46] to some degree, and so, it is interesting and important to study the existence of positive solutions of problem (1.6).

The organization of this paper is as follows. We shall introduce some lemmas in the rest of this section. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with problem (1.6). In Section 3, the main results will be stated and proved. Finally, in Section 4, we offer some interesting discussion of the associated problem (1.6).

To obtain positive solutions of problem (1.6), the following fixed point theorem in cones is fundamental, which can be found in [47, page 94].

Lemma 1.1. *Let Ω_1 and Ω_2 be two bounded open sets in Banach space E , such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let P be a cone in E and let operator $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

- (a) $\|Ax\| \geq \|x\|$, for all $x \in P \cap \partial\Omega_1$, and $\|Ax\| \leq \|x\|$, for all $x \in P \cap \partial\Omega_2$;
- (b) $\|Ax\| \leq \|x\|$, for all $x \in P \cap \partial\Omega_1$, and $\|Ax\| \geq \|x\|$, for all $x \in P \cap \partial\Omega_2$.

Then, A has at least one fixed point in $P \cap (\Omega_2 \setminus \overline{\Omega_1})$.

2. Preliminaries

In order to define the solution of problem (1.6), we shall consider the following space.

Let $J' = J \setminus \{t_1, t_2, \dots, t_n\}$, and

$$PC^1[0, 1] = \left\{ x \in C[0, 1] : x'|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), x'(t_k^-) = x'(t_k), \exists x'(t_k^+), k = 1, 2, \dots, m \right\}. \quad (2.1)$$

Then $PC^1[0, 1]$ is a real Banach space with norm

$$\|x\|_{PC^1} = \max\{\|x\|_\infty, \|x'\|_\infty\}, \quad (2.2)$$

where $\|x\|_\infty = \sup_{t \in J} |x(t)|$, $\|x'\|_\infty = \sup_{t \in J} |x'(t)|$.

A function $x \in PC^1[0, 1] \cap C^4(J')$ is called a solution of problem (1.6) if it satisfies (1.6).

To establish the existence of multiple positive solutions in $PC^1[0, 1] \cap C^4(J')$ of problem (1.6), let us list the following assumptions:

(H₁) $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $k = 1, 2, \dots, m$;

(H₂) $g, h \in L^1[0, 1]$ with

$$\begin{aligned} 0 \leq g(t) < (\tau + 1)t^\tau, \quad \tau = 0, 1, 2, \dots, n, \\ 0 \leq h(t) < t^\tau + \frac{\tau}{\tau + 1}, \quad \tau = 0, 1, 2, \dots, n. \end{aligned} \quad (2.3)$$

Write

$$\mu = \int_0^1 g(s) ds, \quad \nu = \int_0^1 h(s) ds. \quad (2.4)$$

From (H₂), it is clear that $\mu \in [0, 1)$, $\nu \in [0, 1)$.

We shall reduce problem (1.6) to an integral equation. To this goal, firstly by means of the transformation

$$\phi_p(y''(t)) = -x(t), \quad (2.5)$$

we convert problem (1.6) into

$$\begin{aligned} x''(t) + f(t, y(t)) &= 0, \quad t \in J, \\ x(0) = x(1) &= \int_0^1 h(t)x(t) dt, \end{aligned} \quad (2.6)$$

$$\begin{aligned} y''(t) &= -\phi_q(x(t)), \quad t \in J, \quad t \neq t_k, \\ \Delta y'|_{t=t_k} &= -I_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ y(0) = y(1) &= \int_0^1 g(t)y(t) dt. \end{aligned} \quad (2.7)$$

Lemma 2.1. *Assume that (H₁) and (H₂) hold. Then problem (2.6) has a unique solution x given by*

$$x(t) = \int_0^1 H(t, s) f(s, y(s)) ds, \quad (2.8)$$

where

$$H(t, s) = G(t, s) + \frac{1}{1-\nu} \int_0^1 G(s, \tau) h(\tau) d\tau, \quad (2.9)$$

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \quad (2.10)$$

Proof. The proof follows by routine calculations. \square

Write $e(t) = t(1-t)$. Then from (2.9) and (2.10), we can prove that $H(t, s)$ and $G(t, s)$ have the following properties.

Proposition 2.2. *If (H_2) holds, then we have*

$$\begin{aligned} H(t, s) &> 0, \quad G(t, s) > 0, \quad \text{for } t, s \in (0, 1), \\ H(t, s) &\geq 0, \quad G(t, s) \geq 0, \quad \text{for } t, s \in J. \end{aligned} \quad (2.11)$$

Proposition 2.3. *For $t, s \in [0, 1]$, we have*

$$e(t)e(s) \leq G(t, s) \leq G(t, t) = t(1-t) = e(t) \leq \bar{e} = \max_{t \in [0, 1]} e(t) = \frac{1}{4}. \quad (2.12)$$

Proposition 2.4. *If (H_2) holds, then for $t, s \in [0, 1]$, we have*

$$\rho e(s) \leq H(t, s) \leq \gamma s(1-s) = \gamma e(s) \leq \frac{1}{4} \gamma, \quad (2.13)$$

where

$$\gamma = \frac{1}{1-\nu}, \quad \rho = \frac{\int_0^1 e(\tau) h(\tau) d\tau}{1-\nu}. \quad (2.14)$$

Proof. By (2.6) and (2.12), we have

$$\begin{aligned} H(t, s) &= G(t, s) + \frac{1}{1-\nu} \int_0^1 G(s, \tau) h(\tau) d\tau \\ &\geq \frac{1}{1-\nu} \int_0^1 G(s, \tau) h(\tau) d\tau \geq \frac{\int_0^1 e(\tau) h(\tau) d\tau}{1-\nu} s(1-s) \\ &= \rho e(s), \quad t \in [0, 1]. \end{aligned} \quad (2.15)$$

On the other hand, noticing $G(t, s) \leq s(1 - s)$, we obtain

$$\begin{aligned}
 H(t, s) &= G(t, s) + \frac{1}{1 - \nu} \int_0^1 G(s, \tau) h(\tau) d\tau \\
 &\leq s(1 - s) + \frac{1}{1 - \nu} \int_0^1 s(1 - s) h(\tau) d\tau \\
 &\leq s(1 - s) \left[1 + \frac{1}{1 - \nu} \int_0^1 h(\tau) d\tau \right] \leq s(1 - s) \frac{1}{1 - \nu} \\
 &= \gamma e(s), \quad t \in [0, 1].
 \end{aligned} \tag{2.16}$$

The proof of Proposition 2.4 is complete. \square

Remark 2.5. From (2.9) and (2.13), we can obtain that

$$\rho e(s) \leq H(s, s) \leq \gamma s(1 - s) = \gamma e(s) \leq \frac{1}{4} \gamma, \quad s \in J. \tag{2.17}$$

Lemma 2.6. *If (H_1) and (H_2) hold, then problem (2.7) has a unique solution y and y can be expressed in the following form:*

$$y(t) = \int_0^1 H_1(t, s) \phi_q(x(s)) ds + \sum_{k=1}^m H_1(t, t_k) I_k(y(t_k)), \tag{2.18}$$

where

$$H_1(t, s) = G(t, s) + \frac{1}{1 - \mu} \int_0^1 G(s, \tau) g(\tau) d\tau, \tag{2.19}$$

and $G(t, s)$ is defined in (2.10).

Proof. First suppose that $y \in PC^1[0, 1] \cap C^2(J')$ is a solution of problem (2.7).

If $t \in [0, t_1)$, it is easy to see by integration of problem (2.7) that

$$y'(t) = y'(0) - \int_0^t \phi_q(x(s)) ds. \tag{2.20}$$

If $t \in [t_1, t_2)$, then integrate from t_1 to t ,

$$\begin{aligned} y'(t) &= y'(t_1^+) - \int_{t_1}^t \phi_q(x(s))ds = y'(t_1^-) + \Delta y'(t_1) - \int_{t_1}^t \phi_q(x(s))ds \\ &= y'(0) - \int_0^{t_1} \phi_q(x(s))ds + \Delta y'(t_1) - \int_{t_1}^t \phi_q(x(s))ds \\ &= y'(0) - \int_0^t \phi_q(x(s))ds - I_1(y(t_1)). \end{aligned} \quad (2.21)$$

Similarly, if $t \in [t_k, t_{k+1})$, we have

$$y'(t) = y'(0) - \int_0^t \phi_q(x(s))ds - \sum_{t_k < t} I_k(y(t_k)). \quad (2.22)$$

Integrating again, we can get

$$y(t) = y(0) + y'(0)t - \int_0^t (t-s)\phi_q(x(s))ds - \sum_{t_k < t} I_k(y(t_k))(t-t_k). \quad (2.23)$$

Letting $t = 1$ in (2.23), we find

$$y'(0) = \int_0^1 (1-s)\phi_q(x(s))ds + \sum_{t_k < 1} I_k(y(t_k))(1-t_k). \quad (2.24)$$

Substituting $y(0) = \int_0^1 g(t)y(t)dt$ and (2.24) into (2.23), we obtain

$$\begin{aligned} y(t) &= y(0) + \int_0^1 t(1-s)\phi_q(x(s))ds + t \sum_{t_k < 1} I_k(y(t_k))(1-t_k) \\ &\quad - \int_0^t (t-s)\phi_q(x(s))ds - \sum_{t_k < t} I_k(y(t_k))(t-t_k) \\ &= \int_0^1 G(t,s)\phi_q(x(s))ds + \int_0^1 g(t)y(t)dt + \sum_{k=1}^m G(t,t_k)I_k(y(t_k)), \end{aligned} \quad (2.25)$$

where

$$\begin{aligned} \int_0^1 g(t)y(t)dt &= \int_0^1 g(t) \left[\int_0^1 g(t)y(t)dt + \int_0^1 G(t,s)\phi_q(x(s))ds + \sum_{k=1}^m G(t,t_k)I_k(y(t_k)) \right] dt \\ &= \int_0^1 g(t)dt \times \int_0^1 g(t)y(t)dt + \iint_0^1 G(t,s)g(t)\phi_q(x(s))ds dt \\ &\quad + \int_0^1 g(t) \left(\sum_{k=1}^m G(t,t_k)I_k(y(t_k)) \right) dt. \end{aligned} \quad (2.26)$$

Therefore, we have

$$\begin{aligned} \int_0^1 g(s)y(s)ds &= \frac{1}{1 - \int_0^1 g(s)ds} \left[\int_0^1 \left(\int_0^1 G(s,r)g(r)dr \right) \phi_q(x(s))ds \right. \\ &\quad \left. + \int_0^1 g(s) \left(\sum_{k=1}^m G(s,t_k)I_k(y(t_k)) \right) ds \right], \\ y(t) &= \int_0^1 G(t,s)\phi_q(x(s))ds + \sum_{k=1}^m G(t,t_k)I_k(y(t_k)) \\ &\quad + \frac{1}{1-\mu} \left[\int_0^1 \left(\int_0^1 G(s,r)g(r)dr \right) \phi_q(x(s))ds \right. \\ &\quad \left. + \int_0^1 g(s) \left(\sum_{k=1}^m G(s,t_k)I_k(y(t_k)) \right) ds \right]. \end{aligned} \quad (2.27)$$

Let

$$H_1(t,s) = G(t,s) + \frac{1}{1-\mu} \int_0^1 G(s,r)g(r)dr. \quad (2.28)$$

Then,

$$y(t) = \int_0^1 H(t,s)\phi_q(x(s))ds + \sum_{k=1}^m H(t,t_k)I_k(y(t_k)), \quad (2.29)$$

and the proof of sufficient is complete.

Conversely, if y is a solution of (2.18).

Direct differentiation of (2.18) implies, for $t \neq t_k$,

$$y'(t) = \int_0^1 (1-s)\phi_q(x(s))ds + \sum_{k=1}^m I_k(y(t_k))(1-t_k) - \int_0^t \phi_q(x(s))ds - \sum_{t_k < t} I_k(y(t_k)). \quad (2.30)$$

Evidently,

$$\begin{aligned} y''(t) &= -\phi_q(x(t)) \\ \Delta y'|_{t=t_k} &= -I_k(y(t_k)), \quad (k = 1, 2, \dots, m), \quad y(0) = y(1) = \int_0^1 g(t)y(t)dt. \end{aligned} \quad (2.31)$$

The Lemma is proved. \square

Remark 2.7. From (2.19), we can prove that the properties of $H_1(t, s)$ are similar to that of $H(t, s)$.

Suppose that y is a solution of problem (1.6). Then from Lemmas 2.6 and 2.1, we have

$$y(t) = \int_0^1 H_1(t, s)\phi_q\left(\int_0^1 H(s, \tau)f(\tau, y(\tau))d\tau\right)ds + \sum_{k=1}^m H_1(t, t_k)I_k(y(t_k)). \quad (2.32)$$

For the sake of applying Lemma 1.1, we construct a cone in $PC^1[0, 1]$ via

$$K = \left\{ x \in PC^1[0, 1] : x \geq 0, \quad x(t) \geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} x(s), \quad t, s \in J \right\}, \quad (2.33)$$

where

$$\rho_1 = \frac{\int_0^1 e(\tau)g(\tau)d\tau}{1 - \mu}, \quad \gamma_1 = \frac{1}{1 - \mu}. \quad (2.34)$$

It is easy to see that K is a closed convex cone of $PC^1[0, 1]$.

Define an operator $T : K \rightarrow K$ by

$$(Ty)(t) = \int_0^1 H_1(t, s)\phi_q\left[\int_0^1 H(s, \tau)f(\tau, y(\tau))d\tau\right]ds + \sum_{k=1}^m H_1(t, t_k)I_k(y(t_k)). \quad (2.35)$$

From (2.35), we know that $y \in PC^1[0, 1]$ is a solution of problem (1.6) if and only if y is a fixed point of operator T .

Definition 2.8 (see [1]). The set $S \subset PC^1[0, 1]$ is said to be quasi-equicontinuous in $PC^1[0, 1]$ if for any $\varepsilon > 0$ there exist $\delta > 0$ such that if $u \in S$, $s, t \in J_k$ ($k = 1, 2, \dots, m$), $|s - t| < \delta$, then

$$|u(s) - u(t)| < \varepsilon, \quad |u'(s) - u'(t)| < \varepsilon. \quad (2.36)$$

We present the following result about relatively compact sets in $PC^1(J)$ which is a consequence of the Arzela-Ascoli Theorem. The reader can find its proof partially in [1].

Lemma 2.9. $S \subset PC^1[0, 1]$ is relatively compact if and only if S is bounded and quasi-equicontinuous on $PC^1[0, 1]$.

Write

$$K_{r,R} = \{x \in K : r < \|x\| < R\}, \quad (2.37)$$

where $0 < r < R$.

Lemma 2.10. Suppose that (H_1) and (H_2) hold. Then $T(K) \subset K$ and $T : K_{r,R} \rightarrow K$ is completely continuous.

Proof. For $y \in K$, it is clear that $Ty \geq 0$, $Ty \in PC^1[0, 1]$, and

$$(Ty)(t) \leq \gamma^{q-1} \gamma_1 \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)). \quad (2.38)$$

From (2.35) and Remark 2.5, we obtain the following cases.

Case 1. if $\mu, \nu \in [0, 4/5]$, noticing $0 \leq \rho \leq 1$, then we have

$$\begin{aligned} (Ty)(t) &= \int_0^1 H_1(t, s) \phi_q \left[\int_0^1 H(s, \tau) f(\tau, y(\tau)) d\tau \right] ds + \sum_{k=1}^m H_1(t, t_k) I_k(y(t_k)) \\ &\geq \rho_1 \rho^{q-1} \phi_q \int_0^1 e(s) \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \rho_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\ &\geq \rho_1 \rho^{q-1} \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \rho_1 \rho^{q-1} \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\ &= \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} \gamma^{q-1} \gamma_1 \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \frac{\rho_1 \rho^{q-1}}{\gamma_1} \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\ &\geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} \left[\gamma^{q-1} \gamma_1 \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \right] \\ &\geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} (Ty)(s), \quad t, s \in J. \end{aligned} \quad (2.39)$$

Case 2. if $\mu, \nu \in [4/5, 1)$, noticing $\rho \geq 1$ and $\gamma^{q-1} > \rho^{q-1}$, then we have

$$\begin{aligned}
 (Ty)(t) &= \int_0^1 H_1(t, s) \phi_q \left[\int_0^1 H(s, \tau) f(\tau, y(\tau)) d\tau \right] ds + \sum_{k=1}^m H_1(t, t_k) I_k(y(t_k)) \\
 &\geq \rho_1 \rho^{q-1} \phi_q \int_0^1 e(s) \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \rho_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\
 &= \rho_1 \rho^{q-1} \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \frac{\rho_1 \rho^{q-1}}{\rho^{q-1}} \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\
 &= \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} \gamma^{q-1} \gamma_1 \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \frac{\rho_1 \rho^{q-1}}{\gamma_1 \rho^{q-1}} \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\
 &\geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} \left[\gamma^{q-1} \gamma_1 \int_0^1 e(s) \phi_q \left[\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right] ds + \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \right] \\
 &\geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} (Ty)(s), \quad t, s \in J.
 \end{aligned} \tag{2.40}$$

Therefore, $T(y) \in K$, that is, $T(K) \subset K$. Also, we have $T(K_{r,R}) \subset K$ since $K_{r,R} \subset K$. Hence we have $T : K_{r,R} \rightarrow K$.

Next, we prove that $T : K_{r,R} \rightarrow K$ is completely continuous.

It is obvious that $T : K_{r,R} \rightarrow K$ is continuous. Now we prove T is relatively compact.

Let $B_r = \{x \in PC^1[0, 1] \mid \|x\|_{PC^1} \leq r\}$ be a bounded set. Then, for all $x \in B_r$, we have

$$\begin{aligned}
 \|Tx\|_\infty &\leq \frac{1}{4} \gamma_1 \phi_q \left(\frac{1}{4} \gamma M \right) + \frac{1}{4} \gamma_1 mA = r_0, \\
 \|Tx'\|_\infty &\leq \phi_q \left(\frac{1}{4} \gamma M \right) + mA = r_1.
 \end{aligned} \tag{2.41}$$

Therefore $T(B_r)$ is uniformly bounded.

On the other hand, for all $t, s \in J_k$ ($k = 0, 1, \dots, m$) with $t < s$, we have

$$|(Tx)(t) - (Tx)(s)| = \left| \int_s^t (Tx)'(\tau) d\tau \right| \leq r_1 |t - s| \rightarrow 0, \quad (s \rightarrow t), \tag{2.42}$$

and by the continuity of $H'_1(t, s)$, we have

$$\begin{aligned} |(Tx)'(t) - (Tx)'(s)| &= \left| \int_0^1 [H'_1(t, r) - H'_1(s, r)] \phi_q \left[\int_0^1 H(r, \tau) f(\tau, y(\tau)) d\tau \right] dr \right. \\ &\quad \left. + \sum_{k=1}^m [H'_1(t, t_k) - H'_1(s, t_k)] I_k(y(t_k)) \right| \longrightarrow 0, \quad (s \longrightarrow t), \end{aligned} \quad (2.43)$$

and then $T(B_r)$ is quasi-equicontinuous. It follows that $T(B_r)$ is relatively compact on $PC^1[0, 1]$ by Lemma 2.9. So T is completely continuous. \square

3. Main Results

In this section, we apply Lemma 1.1 to establish the existence of positive solutions of problem (1.6). We begin by introducing the following conditions on $f(t, y)$ and $I_k(y)$.

(H₃) There exist numbers $0 < r < R < +\infty$ such that

$$f(t, y) \leq \phi_p \left(\frac{1}{2\gamma^{q-1}} (1 - \mu)r \right), \quad t \in J, \quad 0 \leq y \leq r, \quad (3.1)$$

$$I_k(y) \leq \frac{7}{8m} (1 - \mu)r, \quad 0 \leq y \leq r,$$

$$f(t, y) \geq \phi_p \left(\frac{\gamma_1 \gamma^{q-1}}{\rho_1^2 \rho^{2(q-1)} t_1^q (1 - t_m)^q (t_m - t_1)^q} R \right), \quad t \in J, \quad R \leq y < \infty, \quad (3.2)$$

where μ is defined in (2.4), ρ, γ are defined in (2.14), respectively, and write

$$\begin{aligned} f^\beta &= \limsup_{y \rightarrow \beta} \max_{t \in J} \frac{f(t, y)}{\phi_p(y)}, & f_\beta &= \liminf_{y \rightarrow \beta} \min_{t \in J} \frac{f(t, y)}{\phi_p(y)}, \\ I^\beta(k) &= \limsup_{y \rightarrow \beta} \frac{I_k(y)}{y}, \end{aligned} \quad (3.3)$$

where β denotes 0^+ or $+\infty$.

Theorem 3.1. *Assume that (H₁)–(H₃) hold. Then problem (1.6) has at least one positive solution $y(t)$, $t \in J$ with*

$$\frac{\rho^{q-1} \rho_1}{\gamma_1 \gamma^{q-1}} r \leq y(t) \leq \frac{\gamma_1 \gamma^{q-1}}{\rho^{q-1} \rho_1} R, \quad t \in J. \quad (3.4)$$

Proof. Let T be the cone preserving, completely continuous operator that was defined by (2.35). For $y \in K$ with $\|y\|_{PC^1} = r$, (2.13), (2.19), and (3.1) imply

$$\begin{aligned}
 (Ty)(t) &\leq \gamma_1 \gamma^{q-1} \int_0^1 e(s) \phi_q \left(\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau \right) ds + \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\
 &\leq \gamma_1 \gamma^{q-1} \frac{1}{4} \left(\frac{1}{4} \right)^{q-1} \phi_q \left(\int_0^1 f(\tau, y(\tau)) d\tau \right) ds + \frac{1}{4} \gamma_1 \sum_{k=1}^m I_k(y(t_k)) \\
 &\leq \gamma_1 \gamma^{q-1} \frac{1}{4} \left(\frac{1}{4} \right)^{q-1} \phi_q \left(\int_0^1 \phi_p \left(\frac{1}{2\gamma^{q-1}} (1-\mu)r \right) d\tau \right) ds + \frac{1}{4} \gamma_1 m \times \frac{7}{8m} (1-\mu)r \\
 &\leq \left(\frac{1}{4} \right)^q \frac{1}{2} r + \frac{7}{32} r \leq \frac{1}{8} r + \frac{7}{32} r < r = \|y\|_{PC^1}, \\
 |(Ty)'(t)| &\leq \int_0^1 |H'_t(t, s)| \phi_q \left(\int_0^1 H(s, \tau) f(\tau, y(\tau)) d\tau \right) ds + \sum_{k=1}^m |H'_t(t, t_k)| I_k(y(t_k)) \\
 &\leq \gamma^{q-1} \left(\frac{1}{4} \right)^{q-1} \phi_q \left(\int_0^1 f(\tau, y(\tau)) d\tau \right) + \sum_{k=1}^m I_k(y(t_k)) \\
 &\leq \gamma^{q-1} \left(\frac{1}{4} \right)^{q-1} \phi_q \left(\int_0^1 \left(\frac{1}{2\gamma^{q-1}} (1-\mu)r \right) d\tau \right) + m \times \frac{7}{8m} (1-\mu)r \\
 &\leq \frac{1}{8} (1-\mu)r + \frac{7}{8} r \leq r = \|y\|_{PC^1},
 \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
 H'_t(t, s) = G'_t(t, s) &= \begin{cases} -s & \text{if } 0 \leq s \leq t \leq 1, \\ 1-s & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \\
 \max_{t, s \in J, t \neq s} |H'_t(t, s)| &= \max_{t, s \in J, t \neq s} |G'_t(t, s)| = 1.
 \end{aligned} \tag{3.6}$$

Now if we let $\Omega_1 = \{y \in K : \|y\|_{PC^1} < r\}$, then (3.5) shows that

$$\|Ty\|_{PC^1} \leq \|y\|_{PC^1}, \quad y \in \partial\Omega_1. \tag{3.7}$$

Further, let

$$R_1 = \frac{\gamma_1 \gamma^{q-1}}{\rho^{q-1} \rho_1} R, \tag{3.8}$$

$$\Omega_2 = \{y \in K : \|y\|_{PC^1} < R_1\}.$$

Then, $y \in K$ and $\|y\|_{PC^1} = R_1$ implies

$$y(t) \geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} y(s), \quad t, s \in J, \quad (3.9)$$

that is,

$$y(t) \geq \frac{\rho_1 \rho^{q-1}}{\gamma^{q-1} \gamma_1} R_1 = R, \quad t \in J. \quad (3.10)$$

Hence, $y(t) \geq R$ for all $t \in J$. Therefore, for all $t \in J$, (3.2) implies

$$\begin{aligned} (Ty)(t) &\geq \int_0^1 H_1(t,s) \phi_q \left[\int_0^1 H(s,\tau) f(\tau, y(\tau)) d\tau \right] ds \\ &\geq \rho_1 \rho^{q-1} \int_{t_1}^{t_m} e(s) \phi_q \left[\int_{t_1}^{t_m} e(\tau) \phi_p \left(\frac{\gamma_1 \gamma^{q-1}}{\rho_1^2 \rho^{2(q-1)} t_1^q (1-t_m)^q (t_m-t_1)^q} R \right) d\tau \right] ds \\ &\geq \rho_1 \rho^{q-1} t_1 (1-t_m) t_1^{q-1} (1-t_m)^{q-1} (t_m-t_1)^q \times \frac{\gamma_1 \gamma^{q-1}}{\rho_1^2 \rho^{2(q-1)} t_1^q (1-t_m)^q (t_m-t_1)^q} R \\ &= \frac{\gamma_1 \gamma^{q-1}}{\rho_1 \rho^{q-1}} R = R_1 = \|y\|_{PC^1}, \end{aligned} \quad (3.11)$$

that is, $y \in \partial\Omega_2$ implies

$$\|Ty\|_{PC^1} \geq \|y\|_{PC^1}. \quad (3.12)$$

Applying (b) of Lemma 1.1 to (3.7) and (3.12) yields that T has a fixed point $y \in \overline{\Omega_2} \setminus \Omega_1$ with $r \leq \|y\|_{PC^1} \leq R_1$. Hence, since for $y \in K$ we have $y(t) \geq (\rho_1 \rho^{q-1} / \gamma_1 \gamma^{q-1}) y(s)$, $t, s \in J$, it follows that (3.4) holds. This and Lemma 2.9 complete the proof. \square

As a special case of Theorem 3.1, we can prove the following results.

Corollary 3.2. *Assume that (H_1) and (H_2) hold. If $f^0 = 0$, $I^0(k) = 0$, and $f_\infty = \infty$, then, for $r > 0$ being sufficiently small and $R > 0$ being sufficiently large, BVP (1.6) has at least one positive solution $y(t)$, $t \in J$ with property (3.4).*

Proof. The proof is similar to that of Theorem 3.1 of [6]. \square

In Theorem 3.3, we assume the following condition on $f(t, y)$ and $I_k(y)$.

(H₄) There exist numbers $0 < r < R < +\infty$ such that

$$f(t, y) \geq \phi_p \left(\frac{\gamma_1 \gamma^{q-1}}{\rho^2 \rho_1^{2(q-1)} t_1^q (1-t_m)^q (t_m-t_1)^q} y \right), \quad t \in J, \quad 0 \leq y \leq r, \tag{3.13}$$

$$f(t, y) \leq \phi_p \left(\frac{1-\mu}{4} y \right), \quad t \in J, \quad R \leq y < \infty, \tag{3.14}$$

$$I_k(y) \leq \frac{1-\mu}{4m} y, \quad R \leq y < \infty, \tag{3.15}$$

where ρ and γ are defined in (2.14), and write

$$\begin{aligned} M &= \max_{t \in J, y \in K, \|y\|_{PC^1} = R} f(t, y), & N &= \max_{y \in K, \|y\|_{PC^1} = R} I_k(y), \\ \eta &= \max \left\{ M, \phi_p \left(\frac{1-\mu}{4} \|y\|_{PC^1} \right) \right\}. \end{aligned} \tag{3.16}$$

Theorem 3.3. Assume that (H₁), (H₂), and (H₄) hold. Then problem (1.6) has at least one positive solution $y(t)$, $t \in J$ with

$$\frac{\rho^{q-1} \rho_1}{\gamma_1 \gamma^{q-1}} r \leq y(t) \leq \max \left\{ 2R, \frac{4}{3(1-\mu)} \left(\gamma^{q-1} \left(\frac{1}{4} \right)^{q-1} (2\eta)^{q-1} + mN \right) \right\}, \quad \forall t \in J. \tag{3.17}$$

Proof. For $y \in K$ with $\|y\|_{PC^1} = r$, (3.13) implies

$$\begin{aligned} (Ty)(t) &\geq \int_0^1 H_1(t, s) \phi_q \left[\int_0^1 H(s, \tau) f(\tau, y(\tau)) d\tau \right] ds \\ &\geq \rho_1 \rho^{q-1} \int_{t_1}^{t_m} e(s) \phi_q \left[\int_{t_1}^{t_m} e(\tau) \phi_p \left(\frac{\gamma_1 \gamma^{q-1}}{\rho_1^2 \rho^{2(q-1)} t_1^q (1-t_m)^q (t_m-t_1)^q} y(\tau) \right) d\tau \right] ds \\ &\geq \rho_1 \rho^{q-1} t_1 (1-t_m) t_1^{q-1} (1-t_m)^{q-1} (t_m-t_1)^q \\ &\quad \times \frac{\gamma_1 \gamma^{q-1}}{\rho_1^2 \rho^{2(q-1)} t_1^q (1-t_m)^q (t_m-t_1)^q} \frac{\rho_1 \rho^{q-1}}{\gamma_1 \gamma^{q-1}} \|y\|_{PC^1} = \|y\|_{PC^1}, \end{aligned} \tag{3.18}$$

that is, $y \in \partial\Omega_1$ implies

$$\|Ty\|_{PC^1} \geq \|y\|_{PC^1}. \quad (3.19)$$

Next, we turn to (3.14) and (3.15). From (3.14), (3.15), and (3.16), we have

$$f(t, y) \leq M + \phi_p\left(\frac{1-\mu}{4}y\right), \quad I_k(y) \leq \frac{1-\mu}{4m}y + N. \quad (3.20)$$

Let

$$R_2 = \max\left\{2R, \frac{4}{3(1-\mu)}\left(\gamma^{q-1}\left(\frac{1}{4}\right)^{q-1}(2\eta)^{q-1} + mN\right)\right\}, \quad (3.21)$$

$$\Omega_3 = \{y \in K : \|y\|_{PC^1} < R_2\}.$$

Thus, for $y \in \partial\Omega_3$, we have

$$\begin{aligned} (Ty)(t) &\leq \gamma_1 \gamma^{q-1} \int_0^1 e(s) \phi_q\left(\int_0^1 e(\tau) f(\tau, y(\tau)) d\tau\right) ds + \gamma_1 \sum_{k=1}^m e(t_k) I_k(y(t_k)) \\ &\leq \gamma_1 \gamma^{q-1} \frac{1}{4} \left(\frac{1}{4}\right)^{q-1} \phi_q\left(\int_0^1 f(\tau, y(\tau)) d\tau\right) + \frac{1}{4} \gamma_1 \sum_{k=1}^m I_k(y(t_k)) \\ &\leq \gamma_1 \gamma^{q-1} \frac{1}{4} \left(\frac{1}{4}\right)^{q-1} \phi_q\left(\int_0^1 \left(M + \phi_p\left(\frac{1}{4}(1-\mu)y(\tau)\right)\right) d\tau\right) \\ &\quad + \frac{1}{4} \gamma_1 m \times \left(N + \frac{1}{4m}(1-\mu)r\right) \quad (3.22) \\ &\leq \gamma_1 \gamma^{q-1} \frac{1}{4} \left(\frac{1}{4}\right)^{q-1} \phi_q\left(\int_0^1 (2\eta) d\tau\right) + \frac{1}{4} \gamma_1 m \times \left(N + \frac{1}{4m}(1-\mu)\|y\|_{PC^1}\right) \\ &\leq \gamma_1 \gamma^{q-1} \left(\frac{1}{4}\right)^q (2\eta)^{q-1} + \frac{1}{16} \|y\|_{PC^1} + \frac{1}{4} \gamma_1 m \times N \\ &\leq \frac{3}{4} \|y\|_{PC^1} + \frac{7}{32} \|y\|_{PC^1} < \|y\|_{PC^1}, \end{aligned}$$

$$\begin{aligned}
|(Ty)'(t)| &\leq \int_0^1 |H'_i(t,s)| \phi_q \left(\int_0^1 H(s,\tau) f(\tau, y(\tau)) d\tau \right) ds + \sum_{k=1}^m |H'_i(t,t_k)| I_k(y(t_k)) \\
&\leq \gamma^{q-1} \left(\frac{1}{4}\right)^{q-1} \phi_q \left(\int_0^1 f(\tau, y(\tau)) d\tau \right) + \sum_{k=1}^m I_k(y(t_k)) \\
&\leq \gamma^{q-1} \left(\frac{1}{4}\right)^{q-1} \phi_q \left(\int_0^1 \left(M + \phi_p \left(\frac{1}{4} (1-\mu) y(\tau) \right) \right) d\tau \right) \\
&\quad + m \times \left(N + \frac{1}{4m} (1-\mu) \|y\|_{PC^1} \right) \\
&\leq \gamma^{q-1} \left(\frac{1}{4}\right)^{q-1} \phi_q \left(\int_0^1 2\eta d\tau \right) + m \times \left(N + \frac{1}{4m} (1-\mu) \|y\|_{PC^1} \right) \\
&\leq \gamma^{q-1} \left(\frac{1}{4}\right)^{q-1} \phi_q \left(\int_0^1 2\eta d\tau \right) + m \times \frac{1}{4m} (1-\mu) \|y\|_{PC^1} + mN \\
&\leq \gamma^{q-1} \left(\frac{1}{4}\right)^{q-1} (2\eta)^{q-1} + m \times \frac{1}{4m} (1-\mu) \|y\|_{PC^1} + mN \\
&\leq \frac{3}{4} (1-\mu) \|y\|_{PC^1} + \frac{4}{4} \|y\|_{PC^1} \leq (1-\mu) \|y\|_{PC^1} \leq \|y\|_{PC^1}.
\end{aligned} \tag{3.23}$$

Then, (3.22) and (3.23) imply

$$\|Ty\|_{PC^1} \leq \|y\|_{PC^1}, \quad y \in \partial\Omega_3. \tag{3.24}$$

Applying (a) of Lemma 1.1 to (3.19) and (3.24) yields that T has a fixed point $y \in \overline{\Omega_3} \setminus \Omega_1$ with $r \leq \|y\|_{PC^1} \leq R_2$. Hence, since for $y \in K$ we have $y(t) \geq (\rho\rho^{q-1}/\gamma_1\gamma^{q-1})y(s)$, $t, s \in J$, it follows that (3.17) holds. This and Lemma 2.9 complete the proof. \square

As a special case of Theorem 3.3, we can prove the following results.

Corollary 3.4. *Assume that (H_1) and (H_2) hold. If $f_0 = \infty$ and $f^\infty = 0$, $I^\infty(k) = 0$; then, for $r > 0$ being sufficiently small and $R > 0$ being sufficiently large, BVP (1.6) has at least one positive solution $y(t)$, $t \in J$ with property (3.17).*

Proof. The proof is similar to that of Theorem 3.2 of [6]. \square

Theorem 3.5. *Assume that (H_1) , (H_2) , (3.1) of (H_3) and (3.14) and (3.15) of (H_4) hold. In addition, letting f and I_k satisfy the following condition:*

(H_5) *There is a $\xi > 0$ such that $(\rho_1\rho^{q-1}/\gamma_1\gamma^{q-1})\xi \leq y \leq \xi$ and $t \in J$ implies*

$$f(t, y) > \phi_p \left(\frac{1}{\rho_1\rho^{q-1}t_1^q(1-t_m)^q(t_m-t_1)^q\xi} \right), \tag{3.25}$$

Then, problem (1.6) has at least two positive solutions $y^*(t)$ and $y^{**}(t)$ with

$$\frac{\rho}{\gamma}l \leq y^*(t) < \xi < y^{**}(t) \leq L, \quad t \in J, \quad (3.26)$$

where $l > 0$ and $L > 0$ satisfy

$$0 < l < \xi < L. \quad (3.27)$$

Proof. If (3.1) of (H_3) holds, similar to the proof of (3.7), we can prove that

$$\|Ty\|_{PC^1} \leq \|y\|_{PC^1}, \quad y \in K, \quad \|y\|_{PC^1} = l. \quad (3.28)$$

If (3.14) and (3.15) of (H_4) hold, similar to the proof of (3.23), we have

$$\|Ty\|_{PC^1} \leq \|y\|_{PC^1}, \quad y \in K, \quad \|y\|_{PC^1} = L. \quad (3.29)$$

Finally, we show that

$$\|Ty\|_{PC^1} > \|y\|_{PC^1}, \quad y \in K, \quad \|y\|_{PC^1} = \xi. \quad (3.30)$$

In fact, for $y \in K$ with $\|y\|_{PC^1} = \xi$, then by (2.18), we have

$$y(t) \geq \frac{\rho_1 \rho^{q-1}}{\gamma_1 \gamma^{q-1}} \|y\|_{PC^1} = \frac{\rho_1 \rho^{q-1}}{\gamma_1 \gamma^{q-1}} \xi, \quad (3.31)$$

and it follows from (H_5) that

$$\begin{aligned} (Ty)(t) &\geq \int_0^1 H_1(t,s) \phi_q \left[\int_0^1 H(s,\tau) f(\tau, y(\tau)) d\tau \right] ds \\ &> \rho_1 \rho^{q-1} \int_{t_1}^{t_m} e(s) \phi_q \left[\int_{t_1}^{t_m} e(\tau) \phi_p \left(\frac{1}{\rho_1 \rho^{q-1} t_1^q (1-t_m)^q (t_m-t_1)^q} \xi \right) d\tau \right] ds \\ &\geq \rho_1 \rho^{q-1} t_1 (1-t_m) t_1^{q-1} (1-t_m)^{q-1} (t_m-t_1)^q \\ &\quad \times \frac{1}{\rho_1 \rho^{q-1} t_1^q (1-t_m)^q (t_m-t_1)^q} \xi \\ &= \xi, \end{aligned} \quad (3.32)$$

which implies that (3.30) holds.

Applying Lemma 1.1 to (3.28), (3.29), and (3.30) yields that T has two fixed point y^* and y^{**} with $y^* \in K_{l,\xi} = \{x \in K, l \leq \|x\|_{PC^1} < \xi\}$, and $y^{**} \in K_{\xi,L} = \{x \in K, \xi < \|x\|_{PC^1} \leq L\}$.

Hence, since for $y^* \in K$ we have $y^*(t) \geq (\rho\rho^{q-1}/\gamma_1\gamma^{q-1})y^*(s)$, $t, s \in J$, it follows that (3.26) holds. This and Lemma 2.9 complete the proof. \square

Remark 3.6. Similar to the proof of that of [5], we can prove that problem (1.6) can be generalized to obtain many positive solutions.

4. Discussion

In this section, we offer some interesting discussions associated with problem (1.6).

Discussion 1. Generally, it is difficult to obtain the upper and lower bounds of positive solutions for nonlinear higher-order boundary value problems (see, e.g., [33, 34, 39–43, 46, 48, 49] and their references).

For example, we consider the following problems:

$$\begin{aligned} (\phi_p(y''(t)))'' &= f(t, y(t)), \quad t \in J, \quad t \neq t_k, \quad k = 1, 2, \dots, m, \\ \Delta y'|_{t=t_k} &= -I_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ y(0) &= 0, \quad y(1) = \int_0^1 g(t)y(t)dt, \\ \phi_p(y''(0)) &= 0, \quad \phi_p(y''(1)) = \int_0^1 h(t)\phi_p(y''(t))dt. \end{aligned} \tag{4.1}$$

Here $J = [0, 1]$, $\phi_p(s)$ is p -Laplace operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$, $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$, $R^+ = [0, +\infty)$, t_k ($k = 1, 2, \dots, m$) (where m is fixed positive integer) are fixed points with $0 < t_1 < t_2 < \dots < t_k < \dots < t_m < 1$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, where $x'(t_k^+)$ and $x'(t_k^-)$ represent the right-hand limit and left-hand limit of $x'(t)$ at $t = t_k$, respectively, and $g, h \in L^1[0, 1]$ is nonnegative.

By means of the transformation (2.5), we can convert problem (4.1) into

$$\begin{aligned} x''(t) + f(t, y(t)) &= 0, \quad t \in J, \\ x(0) &= 0, \quad x(1) = \int_0^1 h(t)x(t)dt, \end{aligned} \tag{4.2}$$

$$\begin{aligned} y''(t) &= -\phi_q(x(t)), \quad t \in J, \quad t \neq t_k, \\ \Delta y'|_{t=t_k} &= -I_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ y(0) &= 0, \quad y(1) = \int_0^1 g(t)y(t)dt. \end{aligned} \tag{4.3}$$

Using the similar proof of that of Lemmas 2.1 and 2.6, we can obtain the following results. In addition, if we replace μ, ν by μ^*, ν^* in (H_2) , respectively, then we obtain (H_2^*) , where

$$\mu^* = \int_0^1 sg(s)ds, \quad \nu^* = \int_0^1 sh(s)ds. \quad (4.4)$$

Lemma 4.1. *If (H_1) and (H_2^*) hold, then BVP (4.2) has a unique solution x and x can be expressed in the form*

$$x(t) = \int_0^1 H^*(t, s) f(s, y(s)) ds, \quad (4.5)$$

where

$$H^*(t, s) = G(t, s) + \frac{t}{1 - \int_0^1 sh(s)ds} \int_0^1 G((s, \tau)h\tau) d\tau, \quad (4.6)$$

$G(t, s)$ is defined in (2.10).

Lemma 4.2. *If (H_1) and (H_2^*) hold, then BVP (4.3) has a unique solution y and y can be expressed in the form*

$$y(t) = \int_0^1 H_1^*(t, s) \phi_q(x(s)) ds + \sum_{k=1}^m H_1^*(t, t_k) I_k(y(t_k)), \quad (4.7)$$

where

$$H_1^*(t, s) = G(t, s) + \frac{t}{1 - \int_0^1 sg(s)ds} \int_0^1 G(s, \tau)g(\tau) d\tau. \quad (4.8)$$

It is not difficult to prove that $H^*(t, s)$ and $H_1^*(t, s)$ have the similar properties to that of $H(t, s)$ and $H_1(t, s)$. But for $t \in J$, $H^*(t, s)$ and $H_1^*(t, s)$ have no property (2.13). In fact, if $t \in [t_1, t_m]$, then we can prove that $H^*(t, s)$ and $H_1^*(t, s)$ have the following properties.

Proposition 4.3. *If (H_2^*) holds, then for $t \in [t_1, t_m], s \in [0, 1]$, we have*

$$\rho^* e(s) \leq H^*(t, s) \leq \gamma^* s(1 - s) = \gamma^* e(s) \leq \frac{1}{4} \gamma^*, \quad (4.9)$$

$$\rho_1^* e(s) \leq H_1^*(t, s) \leq \gamma_1^* s(1 - s) = \gamma_1^* e(s) \leq \frac{1}{4} \gamma_1^*, \quad (4.10)$$

where

$$\begin{aligned}\gamma^* &= \frac{1 + \int_0^1 (t_m - \tau)g(\tau)d\tau}{1 - \mu^*}, & \rho^* &= \frac{\int_0^1 e(\tau)g(\tau)d\tau}{1 - \mu^*}t_1(1 - t_m), \\ \gamma_1^* &= \frac{1 + \int_0^1 (t_m - \tau)h(\tau)d\tau}{1 - \nu^*}, & \rho_1^* &= \frac{\int_0^1 e(\tau)h(\tau)d\tau}{1 - \nu^*}t_1(1 - t_m).\end{aligned}\tag{4.11}$$

Proof. We only consider (4.9). By (4.3), we have

$$\begin{aligned}H^*(t, s) &= G(t, s) + \frac{t}{1 - \mu^*} \int_0^1 G(s, \tau)g(\tau)d\tau \\ &\geq \frac{t_1}{1 - \mu^*} \int_0^1 G(s, \tau)g(\tau)d\tau \\ &\geq \frac{\int_0^1 e(\tau)g(\tau)d\tau}{1 - \mu^*}s(1 - s)t_1(1 - t_m) \\ &= \rho^*e(s), \quad t \in [t_1, t_m], \quad s \in [0, 1].\end{aligned}\tag{4.12}$$

On the other hand, noticing $G(t, s) \leq s(1 - s)$, we obtain

$$\begin{aligned}H^*(t, s) &= G(t, s) + \frac{t}{1 - \mu^*} \int_0^1 G(s, \tau)g(\tau)d\tau \\ &\leq s(1 - s) + \frac{t_m}{1 - \mu^*} \int_0^1 s(1 - s)g(\tau)d\tau \\ &\leq s(1 - s) \left[1 + \frac{t_m}{1 - \mu^*} \int_0^1 g(\tau)d\tau \right] \\ &\leq s(1 - s) \frac{1 + \int_0^1 (t_m - \tau)g(\tau)d\tau}{1 - \mu^*} \\ &= \gamma^*e(s), \quad t \in [0, 1].\end{aligned}\tag{4.13}$$

Similarly, we can prove that (4.10) holds, too. \square

Remark 4.4. Since $\mu^* \in [0, 1)$, $\gamma^* > 0$.

From (4.9) and (4.10), we can only define a cone K^* by

$$K^* = \left\{ y \in PC^1[0, 1] : y \geq 0, \quad y(t) \geq \frac{\rho_1^*(\rho^*)^{q-1}}{\gamma_1^*(\gamma^*)^{q-1}}y(s), \quad t \in [t_1, t_m], \quad s \in J \right\},\tag{4.14}$$

which implies that we cannot obtain the lower bounds for the positive solutions of problem (4.1).

5. Example

To illustrate how our main results can be used in practice, we present an example.

Example 5.1. Consider the following boundary value problem:

$$\begin{aligned}
 (\phi_3(y''(t)))'' &= f(t, y(t)), \quad t \in J, \quad t \neq \frac{1}{3}, \quad t \neq \frac{2}{3}, \\
 \Delta y'|_{t=1/3} &= -\frac{1}{32}y\left(\frac{1}{3}\right), \\
 \Delta y'|_{t=2/3} &= -\frac{1}{32}y\left(\frac{2}{3}\right), \\
 y(0) = y(1) &= \int_0^1 \frac{1}{2}y(t)dt, \\
 \phi_3(y''(0)) = \phi_3(y''(1)) &= \int_0^1 t\phi_3(y''(t))dt,
 \end{aligned} \tag{5.1}$$

where $t_1 = 1/3$, $t_2 = 2/3$, $m = 2$, $p = 3$, $q = 3/2$, $g(t) = 1/2$, $h(t) = t$, and

$$f(t, y) = \begin{cases} \frac{1}{256}y^2, & 0 \leq y \leq 1, \\ \left(5668704 - \frac{1}{256}\right)y + \left(\frac{2}{256} - 5668704\right), & 1 \leq y \leq 2, \\ 5668704y^2, & y \geq 2. \end{cases} \tag{5.2}$$

Conclusion. Equation(5.1) has at least one positive solution $y(t)$ for $t \in J$ with

$$\frac{1}{12\sqrt{12}} \leq y(t) \leq 24\sqrt{12}. \tag{5.3}$$

Proof. By simple computation, we have $\mu = 1/2$, $\nu = 1/2$, $\gamma = 2$, $\rho = 1/6$, $\gamma_1 = 2$, and $\rho_1 = 1/6$. Select $r = 1$, and $R = 2$, then for $0 < r < R < +\infty$, we have

$$\begin{aligned}
 f(t, y) &\leq \frac{1}{256}, \quad \forall 0 \leq y \leq 1, \\
 I_k(y) &= \frac{1}{32}y \leq \frac{7}{32}, \quad \forall 0 \leq y \leq 1, \\
 f(t, y) &\geq 5668704, \quad \forall y \geq 2.
 \end{aligned} \tag{5.4}$$

By Theorem 3.1, (5.1) has a positive solution $y(t)$ with $1/12\sqrt{12} \leq y(t) \leq 24\sqrt{12}$. \square

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