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Research Article

A Quasilinear Parabolic System with Nonlocal Boundary Condition

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We investigate the blow-up properties of the positive solutions to a quasilinear parabolic system with nonlocal boundary condition. We first give the criteria for finite time blowup or global existence, which shows the important influence of nonlocal boundary. And then we establish the precise blow-up rate estimate. These extend the resent results of Wang et al. (2009), which considered the special case $m_1 = m_2 = 1, p_1 = 0, q_2 = 0$, and Wang et al. (2007) , which studied the single equation.

1. Introduction

In this paper, we deal with the following degenerate parabolic system:

$$u_t = \Delta u^{m_1} + u^{p_1} v^{q_1}, \qquad v_t = \Delta v^{m_2} + v^{p_2} u^{q_2}, \quad x \in \Omega, \ \ t > 0 \tag{1.1}$$

with nonlocal boundary condition

$$u(x,t) = \int_\Omega f(x,y) u(y,t) dy, \qquad v(x,t) = \int_\Omega g(x,y) v(y,t) dy, \quad x \in \partial\Omega, \ \ t > 0, \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \overline{\Omega}, \tag{1.3}$$

where $m_i, p_i, q_i > 1, i = 1,2$, and $\Omega \subset R^N$ is a bounded connected domain with smooth boundary. $f(x,y) \not\equiv 0$ and $g(x,y) \not\equiv 0$ for the sake of the meaning of nonlocal boundary are nonnegative continuous functions defined for $x \in \partial\Omega$ and $y \in \overline{\Omega}$, while the initial data v_0, u_0 are positive continuous functions and satisfy the compatibility conditions $u_0(x) = \int_\Omega f(x,y)u_0(y)dy$ and $v_0(x) = \int_\Omega g(x,y)v_0(y)dy$ for $x \in \partial\Omega$, respectively.

Problem (1.1)–(1.3) models a variety of physical phenomena such as the absorption and "downward infiltration" of a fluid (e.g., water) by the porous medium with an internal localized source or in the study of population dynamics (see [1]). The solution $(u(x,t), v(x,t))$ of the problem (1.1)–(1.3) is said to blow up in finite time if there exists $T \in (0,\infty)$ called the blow-up time such that

$$\lim_{t\to T^-}\left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}\right) = +\infty, \tag{1.4}$$

while we say that $(u(x,t), v(x,t))$ exists globally if

$$\sup_{t\in(0,T)}\left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}\right) < +\infty \quad \text{for any } \ T \in (0,\infty). \tag{1.5}$$

Over the past few years, a considerable effort has been devoted to the study of the blow-up properties of solutions to parabolic equations with local boundary conditions, say Dirichlet, Neumann, or Robin boundary condition, which can be used to describe heat propagation on the boundary of container (see the survey papers [2, 3] and references therein). The semilinear case ($m_1 = m_2 = 1, f \equiv 0, g \equiv 0$) of (1.1)–(1.3) has been deeply investigated by many authors (see, e.g., [2–11]). The system turns out to be degenerate if $m_i > 1 (i = 1,2)$; for example, in [12, 13], Galaktionov et al. studied the following degenerate parabolic equations:

$$\begin{gathered} u_t = \Delta u^{m_1} + v^{q_1}, \qquad v_t = \Delta v^{m_2} + u^{p_2}, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \overline{\Omega} \end{gathered} \tag{1.6}$$

with $m_1 > 1$, $m_2 > 1$, $p_2 > 1$, and $q_1 > 1$. They obtained that solutions of (1.6) are global if $p_2 q_1 < m_1 m_2$, and may blow up in finite time if $p_2 q_1 > m_1 m_2$. For the critical case of $p_2 q_1 = m_1 m_2$, there should be some additional assumptions on the geometry of Ω.

Song et al. [14] considered the following nonlinear diffusion system with $m_1 \geq 1, m_2 \geq 1$ coupled via more general sources:

$$\begin{gathered} u_t = \Delta u^{m_1} + u^{p_1}v^{q_1}, \qquad v_t = \Delta v^{m_2} + u^{p_2}v^{q_2}, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = \varepsilon_0 > 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \overline{\Omega}. \end{gathered} \tag{1.7}$$

Recently, the genuine degenerate situation with zero boundary values for (1.7) has been discussed by Lei and Zheng [15]. Clearly, problem (1.6) is just the special case by taking $p_1 = q_2 = 0$ in (1.7) with zero boundary condition.

For the more parabolic problems related to the local boundary, we refer to the recent works [16–20] and references therein.

On the other hand, there are a number of important phenomena modeled by parabolic equations coupled with nonlocal boundary condition of form (1.2). In this case, the solution could be used to describe the entropy per volume of the material (see [21–23]). Over the past decades, some basic results such as the global existence and decay property have been obtained for the nonlocal boundary problem (1.1)–(1.3) in the case of scalar equation (see [24–28]). In particular, in [28], Wang et al. studied the following problem:

$$
\begin{aligned}
u_t &= \Delta u^m + u^p, \quad (x,t) \in \Omega \times (0,t),\\
u(x,t) &= \int_\Omega f(x,y)u(y,t)dy, \quad (x,t) \in \partial\Omega \times (0,t),\\
u(x,0) &= u_0(x), \quad x \in \overline{\Omega},
\end{aligned}
\tag{1.8}
$$

with $m > 1, p > 1$. They obtained the blow-up condition and its blow-up rate estimate. For the special case $m = 1$ in the system (1.8), under the assumption that $\int_\Omega f(x,y)dy = 1$, Seo [26] established the following blow-up rate estimate:

$$
(p-1)^{-(1/(p-1))}(T-t)^{-(1/(p-1))} \le \max_{x\in\overline{\Omega}} u(x,t) \le C_1(T-t)^{-(1/(\gamma-1))},
\tag{1.9}
$$

for any $\gamma \in (1,p)$. For the more nonlocal boundary problems, we also mention the recent works [29–34]. In particular, Kong and Wang in [29], by using some ideas of Souplet [35], obtained the blow-up conditions and blow-up profile of the following system:

$$
u_t = \Delta u + \int_\Omega u^m(x,t)v^n(x,t)dx, \qquad v_t = \Delta v + \int_\Omega u^p(x,t)v^q(x,t)dx, \quad x \in \Omega,\ t > 0
\tag{1.10}
$$

subject to nonlocal boundary (1.2), and Zheng and Kong in [34] gave the condition for global existence or nonexistence of solutions to the following similar system:

$$
u_t = \Delta u + u^m \int_\Omega v^n(x,t)dx, \qquad v_t = \Delta v + v^q \int_\Omega u^p(x,t)dx, \quad x \in \Omega,\ t > 0
\tag{1.11}
$$

with nonlocal boundary condition (1.2). The typical characterization of systems (1.10) and (1.11) is the complete couple of the nonlocal sources, which leads to the analysis of simultaneous blowup.

Recently, Wang and Xiang [30] studied the following semilinear parabolic system with nonlocal boundary condition:

$$u_t - \Delta u = v^p, \qquad v_t - \Delta v = u^q, \ x \in \Omega, \ \ t > 0,$$
$$au(x,t) = \int_\Omega f(x,y)u(y,t)dy, v(x,t) = \int_\Omega g(x,y)v(y,t)dy, \quad x \in \partial\Omega, \ t > 0, \tag{1.12}$$
$$u(x,0) = u_0, \quad v(x,0) = v_0, \quad x \in \Omega,$$

where p and q are positive parameters. They gave the criteria for finite time blowup or global existence, and established blow-up rate estimate.

To our knowledge, there is no work dealing with the parabolic system (1.1) with nonlocal boundary condition (1.2) except for the single equation case, although this is a very classical model. Therefore, the main purpose of this paper is to understand how the reaction terms, the weight functions and the nonlinear diffusion affect the blow-up properties for the problem (1.1)–(1.3). We will show that the weight functions $f(x,y), g(x,y)$ play substantial roles in determining blowup or not of solutions. Firstly, we establish the global existence and finite time blow-up of the solution. Secondly, we establish the precise blowup rate estimates for all solutions which blow up.

Our main results could be stated as follows.

Theorem 1.1. *Suppose that* $\int_\Omega f(x,y)dy \geq 1, \int_\Omega g(x,y)dy \geq 1$ *for any* $x \in \partial\Omega$*. If* $q_2 > p_1 - 1$ *and* $q_1 > p_2 - 1$ *hold, then any solution to* (1.1)–(1.3) *with positive initial data blows up in finite time.*

Theorem 1.2. *Suppose that* $\int_\Omega f(x,y)dy < 1, \int_\Omega g(x,y)dy < 1$ *for any* $x \in \partial\Omega$*.*

(1) *If* $m_1 > p_1, m_2 > p_2$*, and* $q_1 q_2 < (m_1 - p_1)(m_2 - p_2)$*, then every nonnegative solution of* (1.1)–(1.3) *is global.*

(2) *If* $m_1 < p_1$*,* $m_2 < p_2$ *or* $q_1 q_2 > (m_1 - p_1)(m_2 - p_2)$*, then the nonnegative solution of* (1.1)–(1.3) *exists globally for sufficiently small initial values and blows up in finite time for sufficiently large initial values.*

To establish blow-up rate of the blow-up solution, we need the following assumptions on the initial data $u_0(x), v_0(x)$

($H1$) $u_0(x), v_0(x) \in C^{2+\mu}(\Omega) \bigcap (\overline{\Omega})$ for some $0 < \mu < 1$;

($H2$) There exists a constant $\delta \geq \delta_0 > 0$, such tha

$$\Delta u_0^{m_1} + u_0^{p_1} v_0^{q_1} - \delta u_0^{m_1 k_1 + 1}(x) \geq 0, \Delta v_0^{m_2} + v_0^{p_2} u_0^{q_2} - \delta v_0^{m_2 k_2 + 1}(x) \geq 0, \tag{1.13}$$

where δ_0, k_1, and k_2 will be given in Section 4.

Theorem 1.3. *Suppose that* $\int_\Omega f(x,y)dy \leq 1, \int_\Omega g(x,y)dy \leq 1$ *for any* $x \in \partial\Omega$*;* $q_1 > m_2, q_2 > m_1$ *and satisfy* $q_2 > p_1 - 1$ *and* $q_1 > p_2 - 1$*; assumptions (H1)-(H2) hold. If the solution* (u, v)

of (1.1)–(1.3) *with positive initial data* u_0, v_0 *blows up in finite time* $T_\star$, *then there exist constants* $C_i > 0 (i = 1,2,3,4)$ *such that*

$$\begin{aligned}
&C_1(T_\star - t)^{-(q_1-p_2+1)/(q_2 q_1-(1-p_1)(1-p_2))} \\
&\le \max_{x\in\overline{\Omega}} u(x,t) \le C_2(T_\star - t)^{-(q_1-p_2+1)/(q_2q_1-(1-p_1)(1-p_2))}, \quad \textit{for} \quad 0 < t < T_\star, \\
&C_3(T_\star - t)^{-(q_2-p_1+1)/(q_2q_1-(1-p_1)(1-p_2))} \\
&\le \max_{x\in\overline{\Omega}} v(x,t) \le C_4(T_\star - t)^{-(q_2-p_1+1)/(q_2 q_1-(1-p_1)(1-p_2))}, \quad \textit{for}\ 0 < t < T_\star.
\end{aligned} \tag{1.14}$$

This paper is organized as follows. In the next section, we give the comparison principle of the solution of problem (1.1)–(1.3) and some important lemmas. In Section 3, we concern the global existence and nonexistence of solution of problem (1.1)–(1.3) and show the proofs of Theorems 1.1 and 1.2. In Section 4, we will give the estimate of the blow-up rate.

2. Preliminaries

In this section, we give some basic preliminaries. For convenience, we denote that $Q_T = Q \times (0,T), S_T = \partial\Omega \times (0,T)$ for $0 < T < +\infty$. As it is now well known that degenerate equations need not posses classical solutions, we begin by giving a precise definition of a weak solution for problem (1.1)–(1.3).

Definition 2.1. A vector function $(u(x,t), v(x,t))$ *defined on* $\overline{\Omega}_T$, *for some* $T > 0$, *is called a sub (or super) solution of* (1.1)–(1.3), *if all the following hold:*

(1) $u(x,t), v(x,t) \in L^\infty(\Omega_T)$;

(2) $(u(x,t), v(x,t)) \le (\ge)(\int_\Omega f(x,t)u(y,t)dy, \int_\Omega g(x,y)v(y,t)dy)$ for $(x,t) \in S_T$, and $u(x,0) \le (\ge) u_0(x), v(x,0) \le (\ge) v_0(x)$ for almost all $x \in \Omega$;

(3)

$$\begin{aligned}
\int_\Omega u(x,t)\phi(x,t)dx \le (\ge) \int_\Omega u(x,0)\phi(x,0)dx &+ \int_0^t \int_{\Omega_T} (u\phi_\tau + u^{m_1}\Delta\phi + u^{p_1}v^{q_1}\phi)dx\,d\tau \\
&- \int_0^t \int_{\partial\Omega} \frac{\partial\phi}{\partial n}\left(\int_\Omega f(x,y)u(y,\tau)dy\right)^{m_1} dS\,d\tau, \\
\int_\Omega v(x,t)\phi(x,t)dx \le (\ge) \int_\Omega v(x,0)\phi(x,0)dx &+ \int_0^t \int_{\Omega_T} (v\phi_\tau + v^{m_2}\Delta\phi + v^{p_2}u^{q_2}\phi)dx\,d\tau \\
&- \int_0^t \int_{\partial\Omega} \frac{\partial\phi}{\partial n}\left(\int_\Omega g(x,y)u(y,\tau)dy\right)^{m_2} dS\,d\tau,
\end{aligned} \tag{2.1}$$

where n is the unit outward normal to the lateral boundary of Ω_T. For every $t \in [0,T]$ and any ϕ belong to the class of test functions,

$$\Phi \equiv \left\{\phi \in C\left(\overline{\Omega}_T\right); \phi_t, \Delta\phi \in C(\Omega_T) \cap L^2(\Omega_T); \phi \geq 0, \phi(x,t)|_{\partial\Omega\times(0,T)} = 0\right\}. \tag{2.2}$$

A weak solution of (1.1) is a vector function which is both a subsolution and a supersolution of (1.1)-(1.3).

Lemma 2.2 (Comparison principle). *Let* $(\underline{u},\underline{v})$ *and* $(\overline{u},\overline{v})$ *be a subsolution and supersolution of* (1.1)–(1.3) *in* Q_T*, respectively. Then* $(\underline{u},\underline{v}) \leq (\overline{u},\overline{v})$ *in* $\overline{\Omega_T}$*, if* $(\underline{u}(x,0),\underline{v}(x,0)) \leq (\overline{u}(x,0),\overline{v}(x,0))$.

Proof. Let $\phi(x,t) \in \Phi$, the subsolution $(\underline{u},\underline{v})$ satisfies

$$\begin{aligned}\int_\Omega \underline{u}(x,t)\phi(x,t)dx \leq{}& \int_\Omega \underline{u}(x,0)\phi(x,0)dx + \int_0^t\int_{\Omega_T}\left(\underline{u}\phi_\tau + \underline{u}^{m_1}\Delta\phi + \underline{u}^{p_1}\underline{v}^{q_1}\phi\right)dx\,d\tau \\ &- \int_0^t\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\left(\int_\Omega f(x,y)\underline{u}(y,\tau)dy\right)^{m_1} dS\,d\tau.\end{aligned} \tag{2.3}$$

On the other hand, the supersolution $(\overline{u},\overline{v})$ satisfies the reversed inequality

$$\begin{aligned}\int_\Omega \overline{u}(x,t)\phi(x,t)dx \geq{}& \int_\Omega \overline{u}(x,0)\phi(x,0)dx + \int_0^t\int_{\Omega_T}\left(\overline{u}\phi_\tau + \overline{u}^{m_1}\Delta\phi + \overline{u}^{p_1}\overline{v}^{q_1}\phi\right)dx\,d\tau \\ &- \int_0^t\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\left(\int_\Omega f(x,y)\overline{u}(y,\tau)dy\right)^{m_1} dS\,d\tau.\end{aligned} \tag{2.4}$$

Set $\omega(x,t) = \underline{u}(x,t) - \overline{u}(x,t)$, we have

$$\begin{aligned}\int_\Omega \omega(x,t)\phi(x,t)dx \leq{}& \int_\Omega \omega(x,0)\phi(x,0)dx + \int_0^t\int_{Q_T}\left(\phi_\tau + \Theta_1(x,s)\Delta\phi + \Theta_2(x,s)\phi\overline{v}^{q_1}\right)\omega\,dx\,d\tau \\ &+ \int_0^t\int_\Omega \phi\underline{u}^{p_1}\left(\Theta_3\left(\underline{v}-\overline{v}\right)\right)dx\,d\tau \\ &- \int_0^t\int_{\partial\Omega}\frac{\partial\phi}{\partial n}m\xi^{m-1}\left(\int_\Omega f(x,y)\omega(y,\tau)dy\right)dS\,d\tau, \quad t\in(0,T),\end{aligned} \tag{2.5}$$

where

$$\begin{gathered}\mathrm{c}\Theta_1(x,t) \equiv \int_0^1 m_1\left(\theta\underline{u} + (1-\theta)\overline{u}\right)^{m_1-1}d\theta, \qquad \Theta_2(x,t) \equiv \int_0^1 p_1\left(\theta\underline{v} + (1-\theta)\overline{v}\right)^{p_1-1}d\theta, \\ \Theta_3(x,t) \equiv \int_0^1 q_1\left(\theta\underline{v} + (1-\theta)\overline{v}\right)^{q_1-1}d\theta.\end{gathered} \tag{2.6}$$

Since $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$are bounded in Ω_T, it follows from $m_1 > 1$, q_1, $p_1 \geq 1$ that $\Theta_i (i = 1,2,3)$ are bounded nonnegative functions. ξ is a function between $\int_\Omega f(x,y)\underline{u}(x,\tau)dy$ and $\int_\Omega f(x,y)\overline{u}(x,\tau)dy$. Noticing that $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are nonnegative bounded function and $\partial\phi/\partial n \leq 0$ on $\partial\Omega$, we choose appropriate function ϕ as in [36] to obtain that

$$\begin{aligned}\int_\Omega \omega(x,t)_+ dx \leq C_1 \int_\Omega \omega(x,0)_+ dx + C_2 \int_0^t \int_\Omega \omega(y,\tau)_+ dy\, d\tau \\ + C_3 \int_0^t \int_\Omega \left[\underline{v} - \overline{v}\right]_+ dx\, d\tau \quad \left(\text{using } \omega(x,0) = \underline{u}(x,0) - \overline{u}(x,0) \leq 0\right).\end{aligned} \tag{2.7}$$

By Gronwall's inequality, we know that $\omega(x,t) = \underline{u}(x,t) - \overline{u}(x,t) \leq 0$, $\underline{v}(x,t) \leq \overline{v}(x,t)$ can be obtained in similar way, then $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$.

Local in time existence of positive classical solutions of the problem (1.1)–(1.3) can be obtained using fixed point theorem (see [37]), the representation formula and the contraction mapping principle as in [38]. By the above comparison principle, we get the uniqueness of the solution to the problem. The proof is more or less standard, so is omitted here. □

Remark 2.3. From Lemma 2.2, it is easy to see that the solution of (1.1)–(1.3) is unique if $p_1, p_2, q_1, q_2 > 1$.

The following comparison lemma plays a crucial role in our proof which can be obtained by similar arguments as in [24, 38–40]

Lemma 2.4. *Suppose that* $w_1(x,t), w_1(x,t) \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ *and satisfy*

$$\begin{aligned}w_{1t} - d_1(x,t)\Delta w_1 \geq c_{11}(x,t)w_1 + c_{21}(x,t)w_2(x,t), \quad (x,t) \in \Omega \times (0,T), \\ w_{2t} - d_2(x,t)\Delta w_2 \geq c_{12}(x,t)w_2 + c_{22}(x,t)w_1(x,t), \quad (x,t) \in \Omega \times (0,T), \\ w_1(x,t) \geq \int_\Omega c_{13}(x,y)w_1(y,t)dy, \quad (x,t) \in \partial\Omega \times (0,T), \\ w_2(x,t) \geq \int_\Omega c_{23}(x,y)w_2(y,t)dy, \quad (x,t) \in \partial\Omega \times (0,T), \\ w_1(x,0) \geq 0, \qquad w_2(x,0) \geq 0, \quad x \in \Omega,\end{aligned} \tag{2.8}$$

where $c_{ij}(x,t)(i = 1,2; j = 1,2,3)$ *are bounded functions and* $d_i(x,t) > 0 (i = 1,2), c_{2j}(x,t) \geq 0, (x,t) \in \Omega \times (0,T)$, *and* $c_{i3}(x,y) \geq 0 (i = 1,2), (x,y) \in \partial\Omega \times \Omega$ *and is not identically zero. Then* $w_i(x,0) > 0 (i = 1,2)$ *for* $x \in \overline{\Omega}$ *imply that* $w_i(x,t) > 0 (i = 1,2)$ *in* Ω_T. *Moreover, if* $c_{i3}(x,y) \equiv 0 (i = 1,2)$ *or if* $\int_\Omega c_{i3}(x,y)dy \leq 1, x \in \partial\Omega$, *then* $w_i(x,0) \geq 0 (i = 1,2)$ *for* $x \in \overline{\Omega}$ *imply that* $w_i(x,t) \geq 0$ *in* Ω_T.

Denote that

$$A = \begin{pmatrix} m_1 - p_1 & -q_1 \\ -q_2 & m_2 - p_2 \end{pmatrix}, \quad l = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}. \tag{2.9}$$

We give some lemmas that will be used in the following section. Please see [41] for their proofs.

Lemma 2.5. *If* $m_1 > p_1, m_2 > p_2$, *and* $q_1q_2 < (m_1 - p_1)(m_2 - p_2)$, *then there exist two positive constants* l_1, l_2, *such that* $Al = (1,1)^T$. *Moreover,* $A(cl) > (0,0)^T$ *for any* $c > 0$.

Lemma 2.6. *If* $m_1 < p_1, m_2 < p_2$ *or* $q_1q_2 > (m_1-p_1)(m_2-p_2)$, *then there exist two positive constants* l_1, l_2, *such that* $Al < (0,0)^T$. *Moreover,* $A(cl) < (0,0)^T$ *for any* $c > 0$.

3. Global Existence and Blowup in Finite Time

Compared with usual homogeneous Dirichlet boundary data, the weight functions $f(x,y)$ and $g(x,y)$ play an important role in the global existence or global nonexistence results for problem (1.1)–(1.3).

Proof of Theorem 1.1. We consider the ODE system

$$\begin{gathered} F'(t) = F^{p_1}H^{q_1}(t), \qquad H'(t) = H^{p_2}F^{q_2}(t), \quad t > 0, \\ F(0) = a > 0, \qquad H(0) = b > 0, \end{gathered} \tag{3.1}$$

where $a = (1/2)\min_{\overline{\Omega}} u_0(x), b = (1/2)\min_{\overline{\Omega}} v_0(x)$, and we use the assumption $u_0, v_0 > 0$.

Set

$$\begin{aligned} F_0 &= \left(\frac{(q_2 - p_1 + 1)^{q_1}(q_1 - p_2 + 1)^{1-p_2}}{(q_1q_2 - (p_1 - 1)(p_2 - 1))^{q_1 - p_2 + 1}} \right)^{1/(q_1q_2 - (p_1-1)(p_2-1))} \\ &\quad \times (T_1 - t)^{-(q_1 - p_2 + 1)/(q_1q_2 - (p_1-1)(p_2-1))}, \\ H_0 &= \left(\frac{(q_1 - p_2 + 1)^{q_2}(q_2 - p_1 + 1)^{1-p_1}}{(q_1q_2 - (p_1 - 1)(p_2 - 1))^{q_2 - p_1 + 1}} \right)^{(1/q_1q_2 - (p_1-1)(p_2-1))} \\ &\quad \times (T_2 - t)^{-(q_2 - p_1 + 1)/(q_1q_2 - (p_1-1)(p_2-1))}, \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} T_1 &= a^{-(q_1 q_2 - (p_1-1)(p_2-1))/(q_1 - p_2 + 1)} \left(\frac{(q_2 - p_1 + 1)^{q_1}(q_1 - p_2 + 1)^{1-p_2}}{(q_1q_2 - (p_1 - 1)(p_2 - 1))^{q_1 - p_2 + 1}} \right)^{1/(q_1 - p_2 + 1)}, \\ T_2 &= b^{-(q_1 q_2 - (p_1-1)(p_2-1))/(q_2 - p_1 + 1)} \left(\frac{(q_1 - p_2 + 1)^{q_2}(q_2 - p_1 + 1)^{1-p_1}}{(q_1q_2 - (p_1 - 1)(p_2 - 1))^{q_2 - p_1 + 1}} \right)^{1/(q_2 - p_1 + 1)}. \end{aligned} \tag{3.3}$$

It is easy to check that (F_0, H_0) is the unique solution of the ODE problem (3.1), then $q_2 > p_1 - 1$ and $q_1 > p_2 - 1$ imply that (F_0, H_0) blows up in finite time. Under the assumption that $\int_\Omega f(x,y)dy \geq 1, \int_\Omega g(x,y)dy \geq 1$ for any $x \in \partial\Omega$, (F_0, H_0) is a subsolution of problem

(1.1)–(1.3). Therefore, by Lemma 2.2, we see that the solution (u,v) of problem (1.1)–(1.3) satisfies $(u,v) \geq (F_0,H_0)$ and then (u,v) blows up in finite time. □

Proof of Theorem 1.2. (1) Let $\Psi_1(x)$ be the positive solution of the linear elliptic problem

$$-\Delta\Psi_1(x) = \epsilon_1, \quad x \in \Omega, \qquad \Psi_1(x) = \int_\Omega f(x,y)dy, \quad x \in \partial\Omega, \tag{3.4}$$

and $\Psi_2(x)$ be the positive solution of the linear elliptic problem

$$-\Delta\Psi_2(x) = \epsilon_2, \quad x \in \Omega, \qquad \Psi_2(x) = \int_\Omega g(x,y)dy, \quad x \in \partial\Omega, \tag{3.5}$$

where ϵ_1,ϵ_2 are positive constant such that $0 \leq \Psi_1(x) \leq 1, 0 \leq \Psi_2(x) \leq 1$. We remark that $\int_\Omega f(x,y)dy < 1$ and $\int_\Omega g(x,y)dy < 1$ ensure the existence of such ϵ_1,ϵ_2.

Denote that

$$\max_{\overline{\Omega}}\Psi_1 = \overline{K}_1, \quad \min_{\overline{\Omega}}\Psi_1 = \underline{K}_1; \quad \max_{\overline{\Omega}}\Psi_2 = \overline{K}_2, \quad \min_{\overline{\Omega}}\Psi_2 = \underline{K}_2. \tag{3.6}$$

We define the functions $\overline{u},\overline{v}$ as following:

$$\overline{u}(x,t) = \overline{u}(x) = M^{l_1}\Psi_1^{1/m_1}, \qquad \overline{v}(x,t) = \overline{v}(x) = M^{l_2}\Psi_2^{1/m_2}, \tag{3.7}$$

where M is a constant to be determined later. Then, we have

$$\begin{aligned}\overline{u}(x,t)\mid_{x\in\partial\Omega} &= M^{l_1}\Psi_1^{1/m_1} = M^{l_1}\left(\int_\Omega f(x,y)dy\right)^{1/m_1} \\ &> M^{l_1}\int_\Omega f(x,y)dy \geq M^{l_1}\int_\Omega f(x,t)\Psi_1^{1/m_1}(y)dy = \int_\Omega f(x,y)\overline{u}(y)dy.\end{aligned} \tag{3.8}$$

In a similar way, we can obtain that

$$|\overline{v}(x,t)|_{x\in\partial\Omega} > \int_\Omega g(x,y)\overline{v}(y)dy, \tag{3.9}$$

here, we used $0 \leq \Psi_1(x) \leq 1, 0 \leq \Psi_2(x) \leq 1, \int_\Omega f(x,y)dy < 1$, and $\int_\Omega g(x,y)dy < 1$.

On the other hand, we have

$$\begin{aligned}\overline{u}_t - \Delta\overline{u}^{m_1} - \overline{u}^{p_1}\overline{v}^{q_1} &= M^{l_1m_1}\varepsilon_1 - M^{p_1l_1+l_2q_1}\Psi_1^{p_1/m_1}\Psi_2^{q_1/m_2} \\ &\geq M^{l_1m_1}\varepsilon_1 - M^{p_1l_1+l_2q_1}\overline{K}_1^{p_1/m_1}\overline{K}_2^{q_1/m_2},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\overline{v}_t - \Delta\overline{v}^{m_2} - \overline{v}^{p_2}\overline{u}^{q_2} &= M^{l_2m_2}\varepsilon_2 - M^{p_2l_2+l_1q_2}\Psi_2^{p_2/m_2}\Psi_1^{q_2/m_1} \\ &\geq M^{l_2m_2}\varepsilon_2 - M^{p_2l_2+l_1q_2}\overline{K}_2^{p_2/m_2}\overline{K}_1^{q_2/m_1}.\end{aligned} \tag{3.11}$$

Let

$$M_1 = \left(\frac{\overline{K}_1^{p_1/m_1} \overline{K}_2^{q_1/m_2}}{\varepsilon_1} \right)^{1/(l_1 m_1 - p_1\, l_1 - l_2 q_1)},$$
$$M_2 = \left(\frac{\overline{K}_2^{p_2/m_2} \overline{K}_1^{q_2/m_1}}{\varepsilon_2} \right)^{1/(l_2\, m_2 - p_2\, l_2 - l_1\, q_2)}. \tag{3.12}$$

If $m_1 > p_1, m_2 > p_2$, and $q_1 p_2 < (m_1 - \mathrm{p}_1)(m_2 - p_2)$, by Lemma 2.5, there exist positive constants l_1, l_2 such that

$$p_1 l_1 + q_1 l_2 < m_1 l_1, \qquad q_2 l_2 + p_2 l_2 < n_2 l_2. \tag{3.13}$$

Therefore, we can choose M sufficiently large, such that

$$M > \max\{M_1, M_2\}, \tag{3.14}$$

$$M^{l_1} \Psi_1^{(1/m\_1)} \geq u_0(x), \qquad M^{l_2} \Psi_2^{(1/m\_2)} \geq v_0(x). \tag{3.15}$$

Now, it follows from (3.8)–(3.15) that $(\overline{u}, \overline{v})$ defined by (3.7) is a positive supersolution of (1.1)–(1.3).

By comparison principle, we conclude that $(u, v) \leq (\overline{u}, \overline{v})$, which implies (u, v) exists globally.

(2) If $m_1 < p_1$, $m_2 < p_2$ or $(m_1 - p_1)(m_2 - p_2) < q_1 q_2$, by Lemma 2.6, there exist positive constants l_1, l_2 such that

$$p_1 l_1 + q_1 l_2 > m_1 l_1, \qquad q_2 l_2 + p_2 l_2 > n_2 l_2. \tag{3.16}$$

So we can choose $M = \min\{M_1, M_2\}$. Furthermore, assume that $u_0(x), v_0(x)$ are small enough to satisfy (3.15). It follows that $(\overline{u}, \overline{v})$ defined by (3.7) is a positive supersolution of (1.1)–(1.3). Hence, (u, v) exists globally.

Due to the requirement of the comparison principle we will construct blow-up subsolutions in some subdomain of Ω in which $u, v > 0$. We use an idea from Souplet [42] and apply it to degenerate equations. Let $\varphi(x)$ be a nontrivial nonnegative continuous function and vanished on $\partial\Omega$. Without loss of generality, we may assume that $0 \in \Omega$ and $\varphi(0) > 0$. We will construct a blow-up positive subsolution to complete the proof.

Set

$$\underline{u}(x,t) = \frac{1}{(T-t)^{l_1}} \omega^{(1/m_1)} \left(\frac{|x|}{(T-t)^{\sigma}} \right), \qquad \underline{u}(x,t) = \frac{1}{(T-t)^{l_2}} \omega^{(1/m_2)} \left(\frac{|x|}{(T-t)^{\sigma}} \right), \tag{3.17}$$

with

$$\omega(r) = \frac{R^3}{12} - \frac{R}{4} r^2 + \frac{1}{6} r^3, \qquad r = \frac{|x|}{(T-t)}, \quad 0 \leq r \leq R, \tag{3.18}$$

where $l_1, l_2, \sigma > 0$ and $0 < T < 1$ are to be determined later. Clearly, $0 \leq \omega(r) \leq R^3/12$ and $\omega(r)$ is nonincreasing since $\omega'(r) = r(r-R)/2 \leq 0$. Note that

$$\operatorname{supp} \underline{u}(\cdot,t) = \operatorname{supp} \underline{v}(\cdot,t) = \overline{B\left(0, R(T-t)^{\sigma}\right)} \subset \overline{B(0, RT^{\sigma})} \subset \Omega, \tag{3.19}$$

for sufficiently small $T > 0$. Obviously, $(\underline{u}, \underline{v})$ becomes unbounded as $t \to T^-$, at the point $x = 0$. Calculating directly, we obtain that

$$\underline{u}_t - \Delta \underline{u}^{m_1}(x,t) = \frac{m_1 l_1 \omega^{1/m_1}(r) + \sigma r \omega'(r) \omega^{(1-m_1)/m_1}}{m_1 (T-t)^{l_1+1}} + \frac{R-2r}{2(T-t)^{m_1+2\sigma}} + \frac{(N-1)(R-r)}{2(T-t)^{m_1 l_1+\sigma}} \tag{3.20}$$

$$\leq \frac{l_1 \left(R^3/12\right)^{1/m_1}}{(T-t)^{l_1+1}} + \frac{NR-(N+1)r}{2(T-t)^{m_1 l_1+2\sigma}}, \tag{3.21}$$

notice that $T < 1$ is sufficiently small.

Similarly, we have

$$\underline{v}_t - \Delta \underline{v}^{m_2}(x,t) \leq \frac{l_2 \left(R^3/12\right)^{1/m_2}}{(T-t)^{l_2+1}} + \frac{NR-(N+1)r}{2(T-t)^{m_2 l_2+2\sigma}}. \tag{3.22}$$

Case 1. If $0 \leq r \leq NR/(N+1)$, we have $\omega(r) \geq (3N+1)R^3/12(N+1)^3$, then

$$\begin{aligned} \underline{u}^{p_1}\underline{\mathrm{v}}^{q_1} &= \frac{\omega^{p_1/m_1}\omega^{q_1/m_2}}{(T-t)^{p_1 l_1+q_1 l_1}} \geq \frac{\left(R^3/12\right)^{(q_1/m_2)}}{(T-t)^{p_1 l_1+q_1 l_2}} \left(\frac{R^3(3N+1)}{12(N+1)^3}\right)^{p_1/m_1}, \\ \underline{v}^{p_2}\underline{u}^{q_2} &= \frac{\omega^{p_2/m_2}\omega^{q_2/m_1}}{(T-t)^{p_2 l_2+q_2 l_2}} \geq \frac{\left(R^3/12\right)^{(p_1/m_1)}}{(T-t)^{p_2 l_2+q_2 l_1}} \left(\frac{R^3(3N+1)}{12(N+1)^3}\right)^{q_1/m_2}. \end{aligned} \tag{3.23}$$

Hence,

$$\begin{aligned} \underline{u}_t - \Delta \underline{u}^{m_1}(x,t) - \underline{u}^{p_1}\underline{v}^{q_1} &\leq \frac{l_1\left(R^3/12\right)^{1/m_1}}{(T-t)^{l_1+1}} - \frac{\left(R^3/12\right)^{q_1/m_2}}{(T-t)^{p_1 l_1+q_1 l_2}} \left(\frac{R^3(3N+1)}{12(N+1)^3}\right)^{p_1/m_1}, \\ \underline{v}_t - \Delta \underline{v}^{m_2}(x,t) - \underline{v}^{p_2}\underline{u}^{q_2} &\leq \frac{l_2\left(R^3/12\right)^{1/m_2}}{(T-t)^{l_2+1}} - \frac{\left(R^3/12\right)^{p_1/m_1}}{(T-t)^{p_2 l_2+q_2 l_1}} \left(\frac{R^3(3N+1)}{12(N+1)^3}\right)^{q_1/m_2}. \end{aligned} \tag{3.24}$$

Case 2. If $NR/(N+1) < r \leq R$, then

$$\begin{aligned} \underline{u}_t - \Delta \underline{u}^{m_1}(x,t) - \underline{u}^{p_1}\underline{v}^{q_1} &\leq \frac{l_1\left(R^3/12\right)^{1/m_1}}{(T-t)^{l_1+1}} + \frac{NR-(N+1)r}{2(T-t)^{m_1 l_1+2\sigma}}, \\ \underline{v}_t - \Delta \underline{v}^{m_2}(x,t) - \underline{v}^{p_2}\underline{u}^{q_2} &\leq \frac{l_2\left(R^3/12\right)^{1/m_2}}{(T-t)^{l_2+1}} + \frac{NR-(N+1)r}{2(T-t)^{m_2 l_2+2\sigma}}. \end{aligned} \tag{3.25}$$

By Lemma 2.6, there exist positive constants l_1, l_2 large enough to satisfy

$$p_1 l_1 + q_1 l_2 > m_1 l_1 + 1, \qquad q_2 l_1 + p_2 l_2 > m_2 l_2 + 1, \quad (m_1 - 1) l_1 > 1, \ (m_2 - 1) l_2 > 1, \tag{3.26}$$

and we can choose $\sigma > 0$ be sufficiently small that

$$\sigma < \max\left\{ \frac{p_1 l_1 + q_1 l_2 - m_1 l_1}{2}, \frac{p_2 l_2 + q_2 l_1 - m_2 l_2}{2} \right\}. \tag{3.27}$$

Thus, we have

$$p_1 l_1 + q_1 l_2 > m_1 l_1 + 2\sigma > l_1 + 1, \qquad p_2 l_2 + q_2 l_1 > m_2 l_2 + 2\sigma > l_2 + 1. \tag{3.28}$$

Hence, for sufficiently small $T > 0$, (3.24) and (3.25) imply that

$$\underline{u}_t - \Delta \underline{u}^{m_1}(x,t) - \underline{u}^{p_1} \underline{v}^{q_1} \le 0, \qquad (x,t) \in \Omega \times (0,T), \tag{3.29}$$

$$\underline{v}_t - \Delta \underline{v}^{m_2}(x,t) - \underline{v}^{p_2} \underline{u}^{q_2} \le 0, \qquad (x,t) \in \Omega \times (0,T). \tag{3.30}$$

Since $\varphi(0) > 0$ and $\varphi(x)$ is continuous, there exist two positive constants ρ and ε such that $\varphi(x) \ge \varepsilon$, for all $x \in B(0,\rho) \subset \Omega$. Choose T small enough to insure $B(0, RT^{\sigma}) \subset B(0,\rho)$, hence $\underline{u} \le 0, \underline{v} \le 0$ on $\partial\Omega \times (0,T)$. Under the assumption that $\int_\Omega f(x,y)dy < 1$ and $\int_\Omega g(x,y)dy < 1$ for any $\partial\Omega$, we have $\underline{u}(x,t) \le \int_\Omega f(x,y)\underline{u}(y,t)dy, \underline{v}(x,t) \le \int_\Omega f(x,y)\underline{v}(y,t)dy$ and $x \in \partial\Omega \times (0,T)$. Furthermore, choose $u_0(x), v_0(x)$ so large that $u_0(x) > \underline{u}(x,0), v_0(x) > \underline{v}(x,0)$. By comparison principle, we have $(\underline{u}, \underline{v}) \le (u,v)$. It shows that solution (u,v) to (1.1)–(1.3) blows up in finite time. □

4. Blow-Up Rate Estimates

In this section, we will estimate the blow-up rate of the blow-up solution of (1.1). Throughout this section, we will assume that

$$q_1 > m_2, \quad q_2 > m_1 \quad \text{and satisfy} \quad q_2 > p_1 - 1, \quad q_1 > p_2 - 1. \tag{4.1}$$

To obtain the estimate, we firstly introduce some transformations. Let $U(x,t) = u^{m_1}(x,t), V(x,t) = (m_2/m_1)^{m_2/(m_2-1)} v^{m_2}(x,t)$, then problem (1.1)–(1.3) becomes

$$\begin{gathered} U_t = U^{r_1}(\Delta U + aU^{p_3}V^{q_3}(x,t)), \qquad V_t = V^{r_2}(\Delta V + bV^{p_4}U^{q_4}(x,t)), \quad x \in \Omega, t > 0, \\ U(x,t) = \left(\int_\Omega f(x,y) U^{m_3}(y,t)dy\right)^{m_1}, \qquad V(x,t) = \left(\int_\Omega g(x,y) V^{m_4}(y,t)dy\right)^{m_2}, \\ x \in \partial\Omega, \ t > 0, \\ U(x,0) = U_0(x), \qquad V(x,0) = V_0(x), \quad x \in \Omega, \end{gathered} \tag{4.2}$$

where $U_0(x) = u_0^{m_1}(x)$, $V_0(x) = (m_2/m_1)^{m_2/(m_2-1)} v_0^{m_2}(x)$; $m_3 = 1/m_1 < 1$, $m_4 = 1/m_2 < 1$; $p_3 = p_1/m_1$, $q_3 = q_1/m_2$, $p_4 = p_2/m_2$, $q_4 = q_2/m_1$; $0 < r_1 = (m_1-1)/m_1 < 1$, $0 < r_2 = (m_2-1)/m_2 < 1$; $a = (m_1/m_2)^{q_1/(m_1-1)}$, $b = (m_1/m_2)^{(p_2-m_2)/(m_2-1)}$. By the conditions (4.1), we have $q_3 > 1, q_4 > 1$ and satisfy that $q_4 - p_3 - r_1 + 1 > 0, q_3 - p_4 - r_2 + 1 > 0$. Under this transformation, assumptions $(H1)$-$(H3)$ become

$(H1')$ $U_0(x), V_0(x) \in C^{2+\mu}(\Omega) \cap (\overline{\Omega})$, for some $0 < \mu < 1$;

$(H2')$ there exists a constant $\delta \geq \delta_0 > 0$, such that

$$\Delta U_0 + aU_0^{p_3} V_0^{q_3} - \delta U_0^{k_1+1-r_1}(x) \geq 0, \qquad \Delta V_0 + bV_0^{p_4} U_0^{q_4} - \delta V_0^{k_2+1-r_2}(x) \geq 0, \tag{4.3}$$

where δ_0, k_1, k_2 will be given later.

By the standard method [16, 42], we can show that system (4.2) has a smooth nonnegative solution (U, V), provided that U_0, V_0 satisfy the hypotheses $(H1)'$-$(H2)'$. We thus assume that the solution (U, V) of problem (4.2) blows up in the finite time $T_\star$. Denote $M_1(t) = \max_{\overline{\Omega}} U(x,t), M_2(t) = \max_{\overline{\Omega}} V(x,t)$. We can obtain the blow-up rate from the following lemmas.

Lemma 4.1. *Suppose that* $U_0(x), V_0(x)$ *satisfy* $(H1)'$-$(H2)'$, *then there exists a positive constant* K_1 *such that*

$$\begin{aligned} &M_1(t)^{q_4-p_3-r_1+1} + M_2(t)^{q_3-p_4-r_2+1} \\ &\quad \geq C_1(T_\star - t)^{-((q_3-p_4-r_2+1)(q_4-p_3-r_1+1))/(q_3\ q_4-(1-r_1-p_3)(1-r_2-p_4))}. \end{aligned} \tag{4.4}$$

Proof. By (4.2), we have (see [43])

$$M_1' \leq aM_1^{p_3+r_1} M_2^{q_3}, \quad M_2' \leq bM_1^{q_4} M_2^{p_4+r_2}. \tag{4.5}$$

Noticing that $q_4 - p_3 - r_1 + 1 > 0$ and $q_3 - p_4 - r_2 + 1 > 0$, hence we have

$$\begin{aligned} &\left(M_1^{q_4-p_3-r_1+1}(t) + M_2^{q_3-p_4-r_2+1}(t)\right)' \\ &\leq \left(a(q_4 - p_3 - r_1 + 1) + b(q_3 - p_4 - r_2 + 1)\right) M_1^{q_4}(t) M_2^{q_3}(t) \\ &\leq C_2\left(M_1^{q_4-p_3-r_1+1}(t) + M_2^{q_3-p_4-r_2+1}\right)^{((q_4-p_3-r_1+1)q_3+(q_3-p_4-r_2+1)q_4)/((q_4-p_3-r_1+1)(q_3-p_4-r_2+1))}, \end{aligned} \tag{4.6}$$

by virtue of Young's inequality. Integrating (4.6) from t to $T_\star$, we can obtain (4.4). □

Lemma 4.2. *Suppose that* U_0, V_0 *satisfy* $(H1)'$-$(H2)'$, (U, V) *is a solution of* (4.2). *Then*

$$U_t - \delta U^{k_1+1} \geq 0, \qquad V_t - \delta V^{k_2+1} \geq 0, \quad (x,t) \in \Omega \times (0, T_\star), \tag{4.7}$$

where

$$k_1 = \frac{q_4q_3 - (1-r_1-p_3)(1-r_2-p_4)}{q_3 - r_2 - p_4 + 1}, \qquad k_2 = \frac{q_4q_3 - (1-r_1-p_3)(1-r_2-p_4)}{q_4 - r_1 - p_3 + 1},$$

$$\delta_1 = \frac{ak_1(1+k_1-p_3)}{r_1(2k_1+1-r_1-p_3)}\left(\frac{1+k_1-p_3}{q_3+k_2}\right)^{(q_3\ (2k_1+1-r_1-p_3))/(k_1\ (q_3+k_2))},$$

$$\delta_2 = \frac{bk_2(1+k_2-p_4)}{r_2(2k_2+1-r_2-p_3)}\left(\frac{1+k_2-p_4}{q_4+k_1}\right)^{(q_4(2k_2+1-r_2-p_4))/(k_2\ (q_4+k_1))}, \tag{4.8}$$

$$\delta > \delta_0 = \max\{|\delta_1|, |\delta_2| > 0\}.$$

Proof. Set $J_1(x,t) = U_t - \delta U^{k_1+1}$, $J_2(x,t) = V_t - \delta V^{k_2+1}$, $(x,t) \in \Omega \times (0, T_\star)$, a straightforward computation yields

$$\begin{aligned} J_{1t} - U^{r_1}\Delta J_1 - \left(2\delta r_1 U^{k_1} + ap_3U^{r_1+p_3-1}V^{q_3}\right)J_1 - aq_3U^{r_1+p_3}V^{q_3-1}J_2 \\ = r_1U^{-1}J_1^2 + \delta k_1(k_1+1)U^{k_1+r_1-1}|\nabla U|^2 + r_1\delta^2U^{2k_1+1} \\ + aq_3\delta U^{r_1+p_3}V^{q_3+k_2} - a\delta(1+k_1-p_3)U^{k_1+r_1+p_3}V^{q_3} \\ \geq r_1\delta^2U^{2k_1+1} + aq_3\delta U^{r_1+p_3}V^{q_3+k_2} - a\delta(1+k_1-p_3)U^{k_1+r_1+p_3}V^{q_3}. \end{aligned} \tag{4.9}$$

If $1 + k_1 \leq p_3$, obviously we have

$$J_{1t} - U^{r_1}\Delta J_1 - \left(2\delta r_1U^{k_1} + ap_3U^{r_1+p_3-1}V^{q_3}\right)J_1 - aq_3U^{r_1+p_3}V^{q_3-1}J_2 \geq 0. \tag{4.10}$$

Otherwise, noticing that $k_1/(2k_1+1-r_1-p_3)+q_3/(q_3+k_2) = 1$, by virtue of Young's inequality,

$$U^{k_1}V^{q_3} \leq \frac{k_1}{2k_1+1-r_1-p_3}\left(\theta U^{k_1}\right)^{(2k_1+1-r_1-p_3)/k_1} + \frac{q_3}{q_3+k_2}\left(\frac{V^{q_3}}{\theta}\right)^{(q_3+k_2)/q_3}, \tag{4.11}$$

where $\theta = ((k_1+1-p_3)/(q_3+k_2\ ))^{q_3/(q_3+k_2)}$, we have

$$\begin{aligned} J_{1t} - U^{r_1}\Delta J_1 - \left(2\delta r_1U^{k_1} + ap_3U^{r_1+p_3-1}V^{q_3}\right)J_1 - aq_3U^{r_1+p_3}V^{q_3-1}J_2 \\ \geq r_1\delta^2U^{2k_1+1} + aq_3\delta U^{r_1+p_3}V^{q_3+k_2} - a\delta(1+k-p_3)U^{k_1+r_1+p_3}V^{q_3} \\ \geq r_1\delta(\delta-\delta_1)U^{2k_1} \geq 0. \end{aligned} \tag{4.12}$$

Similarly, we also have

$$J_{2t} - V^{r_2}\Delta J_2 - \left(2\delta r_2V^{k_2} + bp_4V^{r_2+p_4}\right)J_2 - bq_4V^{r_2+p_4}U^{q_4-1}J_1 \geq 0. \tag{4.13}$$

Fix $(x,t) \in \partial\Omega \times (0,T_\star)$, we have

$$\begin{aligned} J_1(x,t) &= U_t - \delta U^{k_1+1} \\ &= \left(\int_\Omega f(x,y)u(y,t)\right)^{m_1-1}\left(\int_\Omega m_1 f(x,y)u_t(y,t)dy - \delta\left(\int_\Omega f(x,y)u(y,t)dy\right)^\lambda\right), \end{aligned} \tag{4.14}$$

where $\lambda = m_1k_1 + 1 > 1$. Since $U_t(x,t) = J_1(x,r) + \delta U^{k_1+1}$, we have

$$\begin{aligned} &\int_\Omega m_1 f(x,y)u_t(y,t)dy - \delta\left(\int_\Omega f(x,t)u(y,t)dy\right)^\lambda \\ &\quad = \int_\Omega f(x,y)U^{(1-m_1)/m_1}\, J_1(y,t)dy \\ &\quad\quad + \delta\left(\int_\Omega f(x,y)U^{\lambda/m_1}\,(y,t) - \left(\int_\Omega f(x,y)U^{1/m_1}(y,t)dy\right)^\lambda\right). \end{aligned} \tag{4.15}$$

Noticing that $0 < \Phi(x) = \int_\Omega f(x,y)dy \le 1, x \in \partial\Omega$, by virtue of Jensen's inequality, we have

$$\begin{aligned} &\int_\Omega f(x,y)U^{\lambda/m_1}(y,t)dy - \left(\int_\Omega f(x,y)U^{1/m_1}(y,t)dy\right)^\lambda \\ &\quad \ge \int_\Omega f(x,y)dy\left(\frac{\int_\Omega f(x,y)U^{1/m_1}(y,t)dy}{\int_\Omega f(x,y)dy}\right)^\lambda - \left(\int_\Omega f(x,t)U^{1/m_1}(y,t)\right)^\lambda \\ &\quad \ge \Phi(x)\left(\int_\Omega f(x,t)U^{1/m_1}(y,t)\frac{dy}{\Phi(x)}\right)^\lambda - \left(\int_\Omega f(x,y)U^{1/m_1}(y,t)dy\right)^\lambda \\ &\quad = \left(\frac{1}{\Phi(x)^{\lambda-1}}\right)\left(\int_\Omega f(x,y)U^{1/m_1}(y,t)dy\right)^\lambda \ge 0, \end{aligned} \tag{4.16}$$

here, we used $\lambda > 1$ and $0 < \Phi(x) \le 1$ in the last inequality. Hence $(x,t) \in \partial\Omega \times (0,T_\star)$,

$$J_1(x,t) \ge \left(\int_\Omega f(x,t)U^{1/m_1}(y,t)dy\right)^{m_1-1}\int_\Omega f(x,y)U^{(1-m_1)/m_1}(y,t)J_1dy. \tag{4.17}$$

Similarly, we also have

$$J_2(x,t) \ge \left(\int_\Omega g(x,t)V^{1/m_2}(y,t)dy\right)^{m_2-1}\int_\Omega g(x,y)V^{(1-m_2)/m_2}(y,t)J_2dy. \tag{4.18}$$

On the other hand, $(H1)'$-$(H2)'$ imply that $J_1(x,0) \geq 0, J_2(x,0) \geq 0, x \in \Omega$. Combined inequalities (4.12)-(4.18) and Lemma 2.4, we obtain $J_1 \geq 0, J_2 \geq 0$, that is, (4.7) holds.Integrating (4.7) from t to $T_\star$, we conclude that

$$M_1(t) \leq C_3(T_\star - t)^{-(q_3-p_4-r_2+1)/(q_4\ q_3-(1-r_1-p_3)(1-r_2-p_4))},$$
$$M_2(t) \leq C_4(T_\star - t)^{-(q_4-p_3-r_1+1)/(q_4\ q_3-(1-r_1-p_3)(1-r_2-p_4))}. \tag{4.19}$$

where C_3, C_4 are positive constants independent of t. It follows from Lemma 4.1 and (4.19), we have the following lemma.

Lemma 4.3. *Suppose that $U_0(x), V_0(x)$ satisfy $(H1)'$-$(H3)'$. If (U, V) is the solution of system* (4.2) *and blows up in finite time $T_\star$, then there exist positive constants $C_i (i = 3, 4, 5, 6)$ such that*

$$C_5 \leq \max_{x\in\overline{\Omega}} U(x,t)(T_\star - t)^{(q_3-p_4-r_2+1)/(q_4\ q_3-(1-r_1-p_3)(1-r_2-p_4))} \leq C_3, \quad \text{for} \quad 0 < t < T_\star,$$
$$C_6 \leq \max_{x\in\overline{\Omega}} V(x,t)(T_\star - t)^{(q_4-p_3-r_1+1)/(q_4\ q_3-(1-r_1-p_3)(1-r_2-p_4))} \leq C_4, \quad \text{for} \quad 0 < t < T_\star. \tag{4.20}$$

According the transform and Lemma 4.3, *we can obtain* Theorem 1.3. □

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