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Research Article

Green's Function for Discrete Second-Order Problems with Nonlocal Boundary Conditions

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We investigate a second-order discrete problem with two additional conditions which are described by a pair of linearly independent linear functionals. We have found the solution to this problem and presented a formula and the existence condition of Green's function if the general solution of a homogeneous equation is known. We have obtained the relation between two Green's functions of two nonhomogeneous problems. It allows us to find Green's function for the same equation but with different additional conditions. The obtained results are applied to problems with nonlocal boundary conditions.

1. Introduction

The study of boundary-value problems for linear differential equations was initiated by many authors. The formulae of Green's functions for many problems with classical boundary conditions are presented in [1]. In this book, Green's functions are constructed for regular and singular boundary-value problems for ODEs, the Helmholtz equation, and linear nonstationary equations. The investigation of semilinear problems with Nonlocal Boundary Conditions (NBCs) and the existence of their positive solutions are well founded on the investigation of Green's function for linear problems with NBCs [2–7]. In [8], Green's function for a differential second-order problem with additional conditions, for example, NBCs, has been investigated.

In this paper, we consider a discrete difference equation

$$a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \tag{1.1}$$

where $a^2, a^0 \neq 0$. This equation is analogous to the linear differential equation

$$b_2(x)u''(x) + b_1(x)u'(x) + b_0(x)u(x) = f(x). \tag{1.2}$$

In order to estimate a solution of a boundary value problem for a difference equation, it is possible to use the representation of this solution by Green's function [9].

In [10], Bahvalov et al. established the analogy between the finite difference equations of one discrete variable and the ordinary differential equations. Also, they constructed a Green's function for a grid boundary-value problem in the simplest case (Dirichlet BVP).

The direct method for solving difference equations and an iterative method for solving the grid equations of a general form and their application to difference equations are considered in [11, 12]. Various variants of Thomas' algorithm (monotone, nonmonotone, cyclic, etc.) for one-dimensional three-pointwise equations are described. Also, modern economic direct methods for solving Poisson difference equations in a rectangle with boundary conditions of various types are stated.

Chung and Yau [13] study discrete Green's functions and their relationship with discrete Laplace equations. They discuss several methods for deriving Green's functions. Liu et al. [14] give an application of the estimate to discrete Green's function with a high accuracy analysis of the three-dimensional block finite element approximation.

In this paper, expressions of Green's functions for (1.1) have been obtained using the method of variation of parameters [12]. The advantage of this method is that it is possible to construct the Green's function for a nonhomogeneous equation (1.1) with the variable coefficients a^2, a^1, a^0 and various additional conditions (e.g., NBCs). The main result of this paper is formulated in Theorem 4.1, Lemma 5.3, and Theorem 5.4. Theorem 4.1 can be used to get the solution of an equation with a difference operator with any two linearly independent additional conditions if the general solution of a homogeneous equation is known. Theorem 5.4 gives an expression for Green's function and allows us to find Green's function for an equation with two additional conditions if we know Green's function for the same equation but with different additional conditions. Lemma 5.3 is a partial case of this theorem if we know the special Green's function for the problem with discrete (initial) conditions. We apply these results to BVPs with NBCs: first, we construct the Green's function for classical BCs, then we can construct Green's function for a problem with NBCs directly (Lemma 5.3) or via Green's function for a classical problem (Theorem 5.4). Conditions for the existence of Green's function were found. The results of this paper can be used for the investigation of quasilinear problems, conditions for positiveness of Green's functions, and solutions with various BCs, for example, NBCs.

The structure of the paper is as follows. In Section 2, we review the properties of functional determinants and linear functionals. We construct a special basis of the solutions in Section 3 and introduce some functions that are independent of this basis. The expression of the solution to the second-order linear difference equation with two additional conditions is obtained in Section 4. In Section 5, discrete Green's function definitions of this problem are considered. Then a Green's function is constructed for the second-order linear difference equation. Applications to problems with NBCs are presented in Section 6.

2. Notation

We begin this section with simple properties of determinants. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and $1 < n \in \mathbb{N}$.

For all $a_j^i, b_j^i \in \mathbb{K}$, $i, j = 1, 2$, the equality

$$\begin{vmatrix} \begin{vmatrix} b_1^1 & a_1^1 \\ b_1^2 & a_1^2 \end{vmatrix} & \begin{vmatrix} b_2^1 & a_1^1 \\ b_2^2 & a_1^2 \end{vmatrix} \\ \begin{vmatrix} b_1^1 & a_2^1 \\ b_1^2 & a_2^2 \end{vmatrix} & \begin{vmatrix} b_2^1 & a_2^1 \\ b_2^2 & a_2^2 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{vmatrix} \cdot \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} \tag{2.1}$$

is valid. The proof follows from the Laplace expansion theorem [8].

Let $X = \{0, 1, \ldots, n\}$, $\widetilde{X} = \{0, 1, \ldots, n-2\}$. $F(X) := \{u \mid u : X \to \mathbb{K}\}$ be a linear space of real (complex) functions. Note that $F(X) \cong \mathbb{K}^{n+1}$ and functions δ^i, $i = 0, 1, \ldots, n$, such that $\delta^i(j) = \delta_i^j$ for $j \in X$ (δ_m^n is a Kronecker symbol: $\delta_m^n = 1$ if $m = n$, and $\delta_m^n = 0$ if $m \neq n$), form a basis of this linear space. So, for all $u \in F(X)$, there exists a unique choice of $(u_1, \ldots, u_n) \in \mathbb{K}^n$, such that $u = \sum_{k=0}^n u_k \delta^k$. If we have the vector-function $\mathbf{u} = [u^1, u^2] \in F^2(X)$, then we consider the matrix function $[\mathbf{u}] : X^2 \to M_{2\times 2}(\mathbb{K}) \cong \mathbb{K}^4$ and its functional determinant $D[\mathbf{u}]_{ij} : X^2 \to \mathbb{K}$

$$[\mathbf{u}]_{ij} = \left[u^1, u^2\right]_{ij} := \begin{pmatrix} u_i^1 & u_j^1 \\ u_i^2 & u_j^2 \end{pmatrix},$$
$$D[\mathbf{u}]_{ij} = \det[\mathbf{u}]_{ij} = \det\left[u^1, u^2\right]_{ij} := \begin{vmatrix} u_i^1 & u_j^1 \\ u_i^2 & u_j^2 \end{vmatrix}. \tag{2.2}$$

The Wronskian determinant $W[\mathbf{u}]_i$ in the theory of difference equations is denoted as follows:

$$W[\mathbf{u}]_j := \begin{vmatrix} u_{j-1}^1 & u_{j-1}^2 \\ u_j^1 & u_j^2 \end{vmatrix} = \begin{vmatrix} u_{j-1}^1 & u_j^1 \\ u_{j-1}^2 & u_j^2 \end{vmatrix} = D[\mathbf{u}]_{j-1,j}, \quad j = 1, \ldots, n. \tag{2.3}$$

Let (if $W[\mathbf{u}]_{j+2} \neq 0$)

$$H[\mathbf{u}]_{ij} := \frac{D[\mathbf{u}]_{j+1,i}}{W[\mathbf{u}]_{j+2}} = \frac{D[\mathbf{u}]_{j+1,i}}{D[\mathbf{u}]_{j+1,j+2}}, \quad i \in X, \; j = -1, 0, 1, \ldots, n-2. \tag{2.4}$$

We define $H_{i,n-1}[\mathbf{u}] = H_{in}[\mathbf{u}] = 0$, $i \in X$. Note that $H_{j+1,j} = 0$, $H_{j+2,j} = 1$ for $j \in \widetilde{X}$.

If $[\overline{\mathbf{u}}]_{ij} = \mathbf{P} \cdot [\mathbf{u}]_{ij}$, where $\mathbf{P} = (p_n^m) \in M_{2\times 2}(\mathbb{K})$, then

$$\det[\overline{\mathbf{u}}]_{ij} = \det[\mathbf{u}]_{ij} \cdot \det \mathbf{P}, \qquad W[\overline{\mathbf{u}}]_i = W[\mathbf{u}]_i \cdot \det \mathbf{P}. \tag{2.5}$$

If $W[\mathbf{u}] \neq 0$ and $\mathbf{P} \in \mathrm{GL}_2(\mathbb{K}) := \{\mathbf{P} \in M_{2\times 2}(\mathbb{K}) : \det \mathbf{P} \neq 0\}$, then we get $H[\overline{\mathbf{u}}] = H[\mathbf{u}]$. So, the function $H[\mathbf{u}]_{ij}$ is invariant with respect to the basis $\{u^1, u^2\}$ and we write H_{ij}.

Lemma 2.1. *If $\mathbf{w} = [w^1, w^2] \in F^2(X)$, then the equality*

$$\begin{vmatrix} D[\mathbf{w}]_{ik} & D[\mathbf{w}]_{jk} \\ D[\mathbf{w}]_{il} & D[\mathbf{w}]_{jl} \end{vmatrix} = D[\mathbf{w}]_{ij} \cdot D[\mathbf{w}]_{kl}, \quad i, j, k, l \in X, \tag{2.6}$$

is valid.

Proof. If we take $b_1^m = w_i^m$, $b_2^m = w_j^m$, $a_1^m = w_k^m$, $a_2^m = w_l^m$, $m = 1, 2$, in (2.1), then we get equality (2.6). □

Corollary 2.2. *If $\mathbf{w} = [w^1, w^2] \in F(X^2)$, then the equality*

$$W[D[\mathbf{w}]_{\cdot k}, D[\mathbf{w}]_{\cdot l}]_i := \begin{vmatrix} D[\mathbf{w}]_{i-1,k} & D[\mathbf{w}]_{ik} \\ D[\mathbf{w}]_{i-1,l} & D[\mathbf{w}]_{il} \end{vmatrix} = W[\mathbf{w}]_i \cdot D[\mathbf{w}]_{kl}, \tag{2.7}$$

$k, l \in X$, $i = 1, \ldots, n$ is valid.

We consider the space $F^*(X)$ of linear functionals in the space $F(X)$, and we use the notation $\langle f, u\rangle$, $\langle f^k, u_k\rangle$ for the functional f value of the function u. Functionals δ_j, $j = 0, 1, \ldots, n$ form a dual basis for basis $\{\delta^i\}_{i=0}^n$. Thus, $\langle \delta_j, u\rangle = u_j$. If $f \in F^*(X)$, $g \in F^*(Y)$, where $X = \{0, 1, \ldots, n\}$ and $Y = \{0, 1, \ldots, m\}$, then we can define the linear functional (direct product) $f \cdot g \in F^*(X \times Y)$

$$\left\langle f^k \cdot g^l, w_{kl} \right\rangle := \left\langle f^k, \left\langle g^l, w_{kl} \right\rangle \right\rangle, \qquad w_{kl} \in F(X \times Y). \tag{2.8}$$

We define the matrix

$$M(\mathbf{f})[\mathbf{w}] := \begin{pmatrix} \langle f, w^1\rangle & \langle g, w^1\rangle \\ \langle f, w^2\rangle & \langle g, w^2\rangle \end{pmatrix} \tag{2.9}$$

for $\mathbf{f} = (f, g)$, $\mathbf{w} = [w^1, w^2]$, and the determinant

$$D(\mathbf{f})[\mathbf{w}] := \left\langle f^k \cdot g^l, D[\mathbf{w}]_{kl} \right\rangle = \begin{vmatrix} \langle f, w^1\rangle & \langle g, w^1\rangle \\ \langle f, w^2\rangle & \langle g, w^2\rangle \end{vmatrix} = \det M(\mathbf{f})[\mathbf{w}]. \tag{2.10}$$

For example,

$$D(f,\delta_j)[\mathbf{w}] = \left\langle f^k \cdot \delta_j^l, D[\mathbf{w}]_{kl} \right\rangle = \begin{vmatrix} \langle f, w^1\rangle & w_j^1 \\ \langle f, w^2\rangle & w_j^2 \end{vmatrix},$$

$$D(\delta_i,\delta_j)[\mathbf{w}] = \left\langle \delta_i^k \cdot \delta_j^l, D[\mathbf{w}]_{kl} \right\rangle = D[\mathbf{w}]_{ij},$$

$$D(\mathbf{f})\left[\mathbf{w}, w^0\right]_i := D(\mathbf{f},\delta_i)\left[\mathbf{w}, w^0\right] = \left\langle f \cdot g \cdot \delta_i, D\left[\mathbf{w}, w^0\right]\right\rangle = \begin{vmatrix} \langle f, w^1\rangle & \langle g, w^1\rangle & w_i^1 \\ \langle f, w^2\rangle & \langle g, w^2\rangle & w_i^2 \\ \langle f, w^0\rangle & \langle g, w^0\rangle & w_i^0 \end{vmatrix}. \tag{2.11}$$

Let the functions $w^1, w^2 \in F(X)$ be linearly independent.

Lemma 2.3. *Functionals f, g are linearly independent on* $\mathrm{span}\{w^1, w^2\} \subset F(X)$ *if and only if* $D(\mathbf{f})[\mathbf{w}] \neq 0$*.*

Proof. We can investigate the case where $F(X) = \mathrm{span}\{w^1, w^2\}$. The functionals f, g are linearly independent if the equality $\alpha_1 f + \alpha_2 g = 0$ is valid only for $\alpha_1 = \alpha_2 = 0$. We can rewrite this equality as $\langle \alpha_1 f + \alpha_2 g, w\rangle = 0$ for all $w \in \mathrm{span}\{w^1, w^2\}$. A system of functions $\{w^1, w^2\}$ is the basis of the $\mathrm{span}\{w^1, w^2\}$, and the above-mentioned equality is equivalent to the condition below

$$\alpha_1 \begin{pmatrix} \langle f, w^1\rangle \\ \langle f, w^2\rangle \end{pmatrix} + \alpha_2 \begin{pmatrix} \langle g, w^1\rangle \\ \langle g, w^2\rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_1 f + \alpha_2 g, w^1\rangle \\ \langle \alpha_1 f + \alpha_2 g, w^2\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.12}$$

Thus, the functionals f, g are linearly independent if and only if the vectors

$$\begin{pmatrix} \langle f, w^1\rangle \\ \langle f, w^2\rangle \end{pmatrix}, \qquad \begin{pmatrix} \langle g, w^1\rangle \\ \langle g, w^2\rangle \end{pmatrix} \tag{2.13}$$

are linearly independent. But these vectors are linearly independent if and only if

$$\begin{vmatrix} \langle f, w^1\rangle & \langle g, w^1\rangle \\ \langle f, w^2\rangle & \langle g, w^2\rangle \end{vmatrix} \neq 0. \tag{2.14}$$

□

If $\overline{\mathbf{f}} = \mathbf{f}\mathbf{P}_f$, $\overline{\mathbf{w}} = \mathbf{P}_w \mathbf{w}$, where $\mathbf{P}_f, \mathbf{P}_w \in M_{2\times 2}(\mathbb{K})$, then

$$D\left(\overline{\mathbf{f}}\right)[\overline{\mathbf{w}}] = \det \mathbf{P}_w \cdot D(\mathbf{f})[\mathbf{w}] \cdot \det \mathbf{P}_f, \tag{2.15}$$

$$D\left(\overline{\mathbf{f}}, h\right)\left[\overline{\mathbf{w}}, w^0\right] = \det \mathbf{P}_w \cdot D(\mathbf{f}, h)\left[\mathbf{w}, w^0\right] \cdot \det \mathbf{P}_f. \tag{2.16}$$

3. Special Basis in a Two-Dimensional Space of Solutions

Let us consider a homogeneous linear difference equation

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = 0, \quad i \in \widetilde{X}, \tag{3.1}$$

where $a^2, a^0 \neq 0$. Let $S \subset F(X)$ a be two-dimensional linear space of solutions, and let $\{u^1, u^2\}$ be a fixed basis of this linear space. We investigate additional equations

$$\langle L_1, u \rangle = 0, \quad \langle L_2, u \rangle = 0, \quad u \in S, \tag{3.2}$$

where $L_1, L_2 \in S^*$ are linearly independent linear functionals, and we use the notation $\mathbf{L} = (L_1, L_2)$. We introduce new functions

$$\overline{v}_i^1 := D(\delta_i, L_2)[\mathbf{u}], \qquad \overline{v}_i^2 := D(L_1, \delta_i)[\mathbf{u}]. \tag{3.3}$$

For these functions $\langle L_m, \overline{v}^n \rangle = \delta_m^n D(\mathbf{L})[\mathbf{u}]$, $m, n = 1, 2$, that is, $\overline{v}^n \in \operatorname{Ker} L_m$ for $m \neq n$. So, the function $\overline{v}^1$ satisfies equation $\langle L_2, u \rangle$, and the function $\overline{v}^2$ satisfies equation $\langle L_1, u \rangle$. Components of the functions $\overline{v}^1$ and $\overline{v}^2$ in the basis $\{u^1, u^2\}$ are

$$\begin{pmatrix} \langle L_2, u^2 \rangle \\ -\langle L_2, u^1 \rangle \end{pmatrix}, \qquad \begin{pmatrix} -\langle L_1, u^2 \rangle \\ \langle L_1, u^1 \rangle \end{pmatrix}, \tag{3.4}$$

respectively. It follows that the functions $\overline{v}^1, \overline{v}^2$ are linearly independent if and only if

$$\begin{vmatrix} \langle L_2, u^2 \rangle & -\langle L_1, u^2 \rangle \\ -\langle L_2, u^1 \rangle & \langle L_1, u^1 \rangle \end{vmatrix} = \begin{vmatrix} \langle L_1, u^1 \rangle & \langle L_2, u^1 \rangle \\ \langle L_1, u^2 \rangle & \langle L_2, u^2 \rangle \end{vmatrix} \neq 0. \tag{3.5}$$

But this determinant is zero if and only if $D(\mathbf{L})[\mathbf{u}] = 0$. We combine Lemma 2.3 and these results in the following lemma.

Lemma 3.1. *Let $\{u^1, u^2\}$ be the basis of the linear space S. Then the following propositions are equivalent:*

(1) *the functionals L_1, L_2 are linearly independent;*

(2) *the functions $\overline{v}^1, \overline{v}^2$ are linearly independent;*

(3) $D(\mathbf{L})[\mathbf{u}] \neq 0$.

If we take $b_1^m = u_i^m$, $b_2^m = u_j^m$, $a_n^m = \langle L_n, u^m \rangle$, $m, n = 1, 2$, in formula (2.1), then we get

$$\begin{vmatrix} D(\delta_i, L_1)[\mathbf{u}] & D(\delta_j, L_1)[\mathbf{u}] \\ D(\delta_i, L_2)[\mathbf{u}] & D(\delta_j, L_2)[\mathbf{u}] \end{vmatrix} = D[\mathbf{u}]_{ij} \cdot D(\mathbf{L})[\mathbf{u}]. \tag{3.6}$$

The left-hand side of this equality is equal to

$$\begin{vmatrix} D(\delta_i, L_2)[\mathbf{u}] & D(\delta_j, L_2)[\mathbf{u}] \\ D(L_1, \delta_i)[\mathbf{u}] & D(L_1, \delta_j)[\mathbf{u}] \end{vmatrix} = \begin{vmatrix} \overline{v}_i^1 & \overline{v}_j^1 \\ \overline{v}_i^2 & \overline{v}_j^2 \end{vmatrix}. \tag{3.7}$$

Finally, we have (see (3.3))

$$D[\overline{\mathbf{v}}] = D[\mathbf{u}] \cdot D(\mathbf{L})[\mathbf{u}]. \tag{3.8}$$

Similarly we obtain

$$W[\overline{\mathbf{v}}] = W[\mathbf{u}] \cdot D(\mathbf{L})[\mathbf{u}]. \tag{3.9}$$

Lemma 3.2. *Let* $\{u^1, u^2\}$ *be a fundamental system of homogeneous equation* (3.1). *Then equality* (3.9) *is valid, and*

$$W[\overline{\mathbf{v}}] \neq 0 \Longleftrightarrow D(\mathbf{L})[\mathbf{u}] \neq 0. \tag{3.10}$$

Propositions in Lemma 3.1 are equivalent to the condition $W[\overline{\mathbf{v}}] \neq 0$.

Corollary 3.3. *If functionals* L_1, L_2 *are linearly independent, that is,* $D(\mathbf{L})[\mathbf{u}] \neq 0$, *and*

$$v_i^1 := \frac{D(\delta_i, L_2)[\mathbf{u}]}{D(\mathbf{L})[\mathbf{u}]}, \qquad v_i^2 := \frac{D(L_1, \delta_i)[\mathbf{u}]}{D(\mathbf{L})[\mathbf{u}]}, \tag{3.11}$$

that is, $\mathbf{v} = \overline{\mathbf{v}}/D(\mathbf{L})$, *then the two bases* $\{v^1, v^2\}$ *and* $\{L_1, L_2\}$ *are biorthogonal:*

$$\langle L_m, v^n \rangle = \delta_m^n, \quad m, n = 1, 2, \tag{3.12}$$

$$D[\mathbf{v}] = \frac{D[\mathbf{u}]}{D(\mathbf{L})}, \qquad W[\mathbf{v}] = \frac{W[\mathbf{u}]}{D(\mathbf{L})}, \qquad H[\mathbf{v}] = H[\mathbf{u}]. \tag{3.13}$$

Remark 3.4. Propositions in Lemma 3.1 are valid if we take $\{v^1, v^2\}$ instead of $\{\overline{v}^1, \overline{v}^2\}$.

Remark 3.5. If $\{\overline{u}^1, \overline{u}^2\}$ is another fundamental system and $\overline{\mathbf{u}} = \mathbf{P}\mathbf{u}$, where $\mathbf{P} \in \mathrm{GL}_2(\mathbb{K})$, then

$$\frac{D(\delta_i, L_2)[\overline{\mathbf{u}}]}{D(\mathbf{L})[\overline{\mathbf{u}}]} = \frac{D(\delta_i, L_2)[\mathbf{u}]}{D(\mathbf{L})[\mathbf{u}]}, \qquad \frac{D(L_1, \delta_i)[\overline{\mathbf{u}}]}{D(\mathbf{L})[\overline{\mathbf{u}}]} = \frac{D(L_1, \delta_i)[\mathbf{u}]}{D(\mathbf{L})[\mathbf{u}]} \tag{3.14}$$

(see (2.15)). So, the definition of $\mathbf{v} := [v^1, v^2]$ is invariant with respect to the basis $\{u^1, u^2\}$: $v_i^1 = D(\delta_i, L_2)/D(\mathbf{L})$, $v_i^2 = D(L_1, \delta_i)/D(\mathbf{L})$.

4. Discrete Difference Equation with Two Additional Conditions

Let $\{u^1, u^2\}$ be the solutions of a homogeneous equation

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = 0, \quad a_i^2, a_i^0 \neq 0, \ i \in \widetilde{X}. \tag{4.1}$$

Then $D[\mathbf{u}]_{i\cdot}$ is the solution of (4.1), that is,

$$a_i^2 D[\mathbf{u}]_{i+2,j} + a_i^1 D[\mathbf{u}]_{i+1,j} + a_i^0 D[\mathbf{u}]_{ij} = 0, \quad i \in \widetilde{X}, \ j \in X. \tag{4.2}$$

For $j = i+1$, this equality shows that $-a_i^2 W[\mathbf{u}]_{i+2} + a_i^0 W[\mathbf{u}]_{i+1} = 0$, and we arrive at the conclusion that $W[\mathbf{u}]_i \equiv 0$ (the case where $\{u^1, u^2\}$ are linearly dependent solutions) or $W[\mathbf{u}]_i \neq 0$ for all $i = 1, \ldots, n$ (the case of the fundamental system).

In this section, we consider a nonhomogeneous difference equation

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in \widetilde{X}, \tag{4.3}$$

with two additional conditions

$$\langle L_1, u \rangle = g_1 \in \mathbb{K}, \qquad \langle L_2, u \rangle = g_2 \in \mathbb{K}, \tag{4.4}$$

where L_1, L_2 are linearly independent functionals.

4.1. The Solution to a Nonhomogeneous Problem with Additional Homogeneous Conditions

A general solution of (4.1) is $u = C_1 u^1 + C_2 u^2$, where C_1, C_2 are arbitrary constants and $\{u^1, u^2\}$ is the fundamental system of this homogeneous equation. We replace the constants C_1, C_2 by the functions $c_1, c_2 \in F(X)$ (Method of Variation of Parameters [12]), respectively. Then, by substituting

$$u_{f;i} = c_{1;i} u_i^1 + c_{2;i} u_i^2, \quad i \in X, \tag{4.5}$$

into (4.3) and denoting $d_{ki} = \sum_{l=1}^{2} [c_{l;i+k} - c_{l;i}] u_{i+k}^l$, $k = -1, 0, 1, 2$, $i = \max(0, -k), \ldots, \min(n-k, n)$ [12], we obtain

$$\begin{aligned} f_i &= \sum_{k=0}^{2} a_i^k u_{f;i+k} = \sum_{k=0}^{2} a_i^k \sum_{l=1}^{2} c_{l;i+k} u_{i+k}^l = \sum_{k=0}^{2} a_i^k d_{ki} + \sum_{k=0}^{2} a_i^k \sum_{l=1}^{2} c_{l;i} u_{i+k}^l \\ &= \sum_{k=0}^{2} a_i^k d_{ki} + \sum_{l=1}^{2} c_{l;i} \left[\sum_{k=0}^{2} a_i^k u_{i+k}^l \right]. \end{aligned} \tag{4.6}$$

The functions u^1 and u^2 are solutions of the homogeneous equation (4.1). Consequently,

$$f_i = \sum_{k=0}^{2} a_i^k d_{ki}, \quad \text{for } i \in \widetilde{X}. \tag{4.7}$$

Denote $b_{li} = c_{l;i+1} - c_{l;i}$, $l = 1, 2$. We derive $(k = 0, 1, 2)$

$$\begin{gathered} d_{ki} - d_{k-1,i+1} = \sum_{l=1}^{2} (c_{l;i+k} - c_{l;i}) u_{i+k}^l - \sum_{l=1}^{2} (c_{l;i+k} - c_{l;i+1}) u_{i+k}^l = \sum_{l=1}^{2} b_{li} u_{i+k}^l, \\ \sum_{k=0}^{2} a_i^k (d_{ki} - d_{k-1,i+1}) = \sum_{l=1}^{2} b_{li} \sum_{k=0}^{2} a_i^k u_{i+k}^l = 0. \end{gathered} \tag{4.8}$$

Then we rewrite equality (4.7) as ($d_{0i} = 0$ by definition)

$$f_i = \sum_{k=0}^{2} a_i^k d_{ki} = \sum_{k=0}^{2} a_i^k d_{k-1,i+1} = a_i^2 d_{1,i+1} + a_i^0 d_{-1,i+1}. \tag{4.9}$$

We can take $d_{-1,i+1} = 0$, $i = 0, \ldots, n-1$. Then $d_{1,i+1} = f_i / a_i^2$ for all $i \in \widetilde{X}$, and we obtain the following systems:

$$\begin{aligned} b_{1,i+1} u_{i+1}^1 + b_{2,i+1} u_{i+1}^2 &= 0, \\ b_{1,i+1} u_{i+2}^1 + b_{2,i+1} u_{i+2}^2 &= \frac{f_i}{a_i^2}, \end{aligned} \quad i \in \widetilde{X}. \tag{4.10}$$

Since u^1, u^2 are linearly independent, the determinant $W[\mathbf{u}]$ is not equal to zero and system (4.10) has a unique solution

$$b_{1,i+1} = c_{1;i+2} - c_{1;i+1} = -\frac{u_{i+1}^2 f_i}{a_i^2 W[\mathbf{u}]_{i+2}}, \qquad b_{2,i+1} = c_{2;i+2} - c_{2;i+1} = \frac{u_{i+1}^1 f_i}{a_i^2 W[\mathbf{u}]_{i+2}}. \tag{4.11}$$

Then

$$c_{1;i} = -\sum_{j=0}^{i-2} \frac{u_{j+1}^2 f_j}{a_j^2 W[\mathbf{u}]_{j+2}} + c_{1;1}, \quad c_{2;i} = \sum_{j=0}^{i-2} \frac{u_{j+1}^1 f_j}{a_j^2 W[\mathbf{u}]_{j+2}} + c_{2;1}, \quad i = 2, \ldots, n, \tag{4.12}$$

and the formula for solution of nonhomogeneous equation (with the conditions $u_0 = u_1 = 0$) is

$$u_i = \sum_{j=0}^{i-2} \frac{f_j}{a_j^2 W[\mathbf{u}]_{j+2}} \begin{vmatrix} u_{j+1}^1 & u_i^1 \\ u_{j+1}^2 & u_i^2 \end{vmatrix} = \sum_{j=0}^{i-2} \frac{D[\mathbf{u}]_{j+1,i}}{W[\mathbf{u}]_{j+2}} \frac{f_j}{a_j^2} = \sum_{j=0}^{i-2} \frac{H_{ij}}{a_j^2} f_j \tag{4.13}$$

for $i = 2, \dots, n$. We introduce a function $H^{\theta} \in F(X \times \widetilde{X})$:

$$H_{ij}^{\theta} := \frac{\theta_{i-j} H_{ij}}{a_j^2}, \qquad \theta_i := \begin{cases} 1 & i > 0, \\ 0 & i \leq 0. \end{cases} \tag{4.14}$$

Then we rewrite (4.13) and the conditions $u_0 = 0$, $u_1 = 0$ as follows:

$$u_i = \sum_{j=0}^{n-2} H_{ij}^{\theta} f_j = \left(H_{ij}^{\theta}, f_j \right)_X = \left(H_{i,\cdot}^{\theta}, f \right)_X, \quad i \in X, \tag{4.15}$$

where $(w, g)_X = (w_l, g_l)_X := \sum_{l=0}^{n-2} w_l g_l$, $w, g \in F(\widetilde{X})$. So, we derive a formula for the general solution $u_i = (H_{i,\cdot}^{\theta}, f)_X + C_1 u_i^1 + C_2 u_i^2$. We use this formula for the special basis $\{v^1, v^2\}$ (see (3.11)). In this case, we have

$$u_i = \left(H_{i,\cdot}^{\theta}, f \right)_X + C_1 v_i^1 + C_2 v_i^2, \quad i \in X. \tag{4.16}$$

Let there be homogeneous conditions

$$\langle L_1, u \rangle = 0, \qquad \langle L_2, u \rangle = 0. \tag{4.17}$$

So, by substituting general solution (4.16) into homogeneous additional conditions, we find (see (3.12))

$$\begin{aligned} C_1 &= -\left\langle L_1^k, \left(H_{k,\cdot}^{\theta}, f \right)_X \right\rangle = -\left(\left\langle L_1^k, H_{k,\cdot}^{\theta} \right\rangle, f \right)_X, \\ C_2 &= -\left\langle L_2^k, \left(H_{k,\cdot}^{\theta}, f \right)_X \right\rangle = -\left(\left\langle L_2^k, H_{k,\cdot}^{\theta} \right\rangle, f \right)_X. \end{aligned} \tag{4.18}$$

Next we obtain a formula for solution in the case of difference equation with two additional homogeneous conditions

$$\begin{aligned} u_{f;i} &= \left(H_{i,\cdot}^{\theta}, f \right)_X - v_i^1 \left(\left\langle L_1^k, H_{k,\cdot}^{\theta} \right\rangle, f \right)_X - v_i^2 \left(\left\langle L_2^k, H_{k,\cdot}^{\theta} \right\rangle, f \right)_X \\ &= \left(\left\langle \delta_i^k - \mathbf{L}^k \mathbf{v}_i, H_{k,\cdot}^{\theta} \right\rangle, f \right)_X, \end{aligned} \tag{4.19}$$

where $v_i^1 = D(\delta_i, L_2)/D(\mathbf{L})$, $v_i^2 = D(L_1, \delta_i)/D(\mathbf{L})$, $\mathbf{v}_i = [v_i^1, v_i^2]$, $\mathbf{L}^k = (L_1^k, L_2^k)$, $i, k \in X$, $\mathbf{L}^k \mathbf{v}_i := L_1^k v_i^1 + L_2^k v_i^2$.

4.2. A Homogeneous Equation with Additional Conditions

Let us consider the homogeneous equation (4.1) with the additional conditions (4.4)

$$\mathcal{L} u = 0, \qquad \langle L_1, u \rangle = g_1, \qquad \langle L_2, u \rangle = g_2. \tag{4.20}$$

We can find the solution

$$u_{0;i} = g_1 \cdot v_i^1 + g_2 \cdot v_i^2, \quad i \in X, \tag{4.21}$$

to this problem if the general solution is inserted into the additional conditions.

The solution of nonhomogeneous problems is of the form $u_i = u_{f;i} + u_{0;i}$ (see (4.19) and (4.21)). Thus, we get a simple formula for solving problem (4.3)-(4.4).

Theorem 4.1. *The solution of problem* (4.3)-(4.4) *can be expressed by the formula*

$$u_i = \left(\left\langle \delta_i^k - \mathbf{L}^k \mathbf{v}_i, H_{k,\cdot}^{\theta} \right\rangle, f \right)_X + g_1 \cdot v_i^1 + g_2 \cdot v_i^2, \quad i \in X. \tag{4.22}$$

Formula (4.22) can be effectively employed to get the solutions to the linear difference equation, with various a^0, a^1, a^2, any right-hand side function f, and any functionals L_1, L_2 and any g_1, g_2, provided that the general solution of the homogeneous equation is known. In this paper, we also use (4.22) to get formulae for Green's function.

4.3. Relation between Two Solutions

Next, let us consider two problems with the same nonhomogeneous difference equation with a difference operator as in the previous subsection

$$\begin{gathered} \mathcal{L}u = f, \qquad \mathcal{L}v = f, \\ \langle l_m, u \rangle = f_m, \quad m = 1,2, \qquad \langle L_m, v \rangle = F_m, \quad m = 1,2, \end{gathered} \tag{4.23}$$

and $D(\mathbf{L}) \neq 0$. The difference $w = v - u$ satisfies the problem

$$\begin{gathered} \mathcal{L}w = 0, \\ \langle L_m, w \rangle = F_m - \langle L_m, u \rangle, \quad m = 1,2. \end{gathered} \tag{4.24}$$

Thus, it follows from formula (4.21) that

$$w_i = (F_1 - \langle L_1, u \rangle) v_i^1 + (F_2 - \langle L_2, u \rangle) v_i^2, \quad i \in X, \tag{4.25}$$

or

$$v_i = u_i + (F_1 - \langle L_1, u \rangle) \frac{D(\delta_i, L_2)}{D(\mathbf{L})} + (F_2 - \langle L_2, u \rangle) \frac{D(L_1, \delta_i)}{D(\mathbf{L})}, \quad i \in X, \tag{4.26}$$

and we can express the solution of the second problem (4.23) via the solution of the first problem.

Corollary 4.2. *The relation*

$$v_i = \frac{1}{D(\mathbf{L})[\mathbf{u}]}\begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & u_i^1 \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle & u_i^2 \\ \langle L_1, u\rangle - F_1 & \langle L_2, u\rangle - F_2 & u_i \end{vmatrix}, \quad i \in X, \tag{4.27}$$

between the two solutions of problems (4.23) *is valid.*

Proof. If we expand the determinant in (4.27) according to the last row, then we get formula (4.26). □

Remark 4.3. The determinant in formula (4.27) is equal to

$$\begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & u_i^1 \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle & u_i^2 \\ \langle L_1, u\rangle & \langle L_2, u\rangle & u_i \end{vmatrix} - \begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & u_i^1 \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle & u_i^2 \\ F_1 & F_2 & 0 \end{vmatrix}. \tag{4.28}$$

In this way, we can rewrite (4.27) as

$$v_i = \frac{D(\mathbf{L}, \delta_i)[\mathbf{u}, u]}{D(\mathbf{L})[\mathbf{u}]} + \frac{F_1 D(\delta_i, L_2)[\mathbf{u}] + F_2 D(L_1, \delta_i)[\mathbf{u}]}{D(\mathbf{L})[\mathbf{u}]}, \quad i \in X. \tag{4.29}$$

Note that in this formula the function u is in the first term only and v_i is invariant with regard to the basis $\{u^1, u^2\}$.

5. Green's Functions

5.1. Definitions of Discrete Green's Functions

We propose a definition of Green's function (see [9, 12]). In this section, we suppose that $\mathbb{K} = \mathbb{R}$ and $X_n := X = \{0, 1, \ldots, n\}$. Let $A : F(X_n) \to F(X_{n-m}) = \operatorname{Im} A$ be a linear operator, $0 \le m \le n$. Consider an operator equation $Au = f$, where $u \in F(X_n)$ is unknown and $f \in F(X_{n-m})$ is given. This operator equation, in a discrete case, is equivalent to the system of linear equations

$$\sum_{i=0}^{n} a_{ji} u_i = f_j, \quad j = 0, 1, \ldots, n-m, \tag{5.1}$$

that is, $\mathbf{Au} = \mathbf{f}$, where $\mathbf{u} \in \mathbb{R}^{n+1}$, $\mathbf{f} \in \mathbb{R}^{n-m+1}$, $\mathbf{A} = (a_{ji}) \in M_{(n+1)\times(n-m+1)}(\mathbb{R})$, rank $\mathbf{A} = n - m + 1$. We have dim Ker $A = m$. In the case $m > 0$, we must add additional conditions if we want to get a unique solution. Let us add $M - n + m$ homogeneous linear equations

$$\sum_{i=0}^{n} b_{ji} u_i = 0, \quad j = 1, \ldots, M - n + m, \tag{5.2}$$

where $\mathbf{B} = (b_{ji}) \in M_{(n+1)\times(M-n+m)}(\mathbb{R})$, rank $\mathbf{B} = M - n + m$, and denote

$$\tilde{a}_{ji} := \begin{cases} a_{ji}, & j = 0, 1, \dots, n-m, \\ b_{j-n+m,i}, & j = n-m+1, \dots, M, \end{cases} \quad i \in X_n,$$
$$\tilde{f}_j := \begin{cases} f_j, & j = 0, 1, \dots, n-m, \\ 0, & j = n-m+1, \dots, M. \end{cases} \tag{5.3}$$

We have a system of linear equations $\widetilde{\mathbf{A}}\mathbf{u} = \tilde{\mathbf{f}}$, where $\tilde{\mathbf{f}} = (\tilde{f}_j) \in M_{(n+1)\times 1}(\mathbb{R})$, $\widetilde{\mathbf{A}} = (\tilde{a}_{ji}) \in M_{(n+1)\times(M+1)}(\mathbb{R})$. The necessary condition for a unique solution is $M \geq n$. Additional equations (5.2) define the linear operator $B : F(X_n) \to F(X_{M-n+m})$ and the additional operator equation $Bu = 0$, and we have the following problem:

$$Au = f, \qquad Bu = 0. \tag{5.4}$$

If solution of (5.4) allows the following representation:

$$u_i = \sum_{j=0}^{n-m} G_{ij} f_j, \quad i \in X_n, \tag{5.5}$$

then $G \in F(X_n \times X_{n-m})$ is called *Green's function* of operator A with the additional condition $Bu = 0$. Green's function exists if $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$. This condition is equivalent to $\det \widetilde{\mathbf{A}} \neq 0$ for $M = n$. In this case, we can easily get an expression for Green's function in representation (5.5) from the Kramer formula or from the formula for $\mathbf{u} = \widetilde{\mathbf{A}}^{-1}\tilde{\mathbf{f}}$. If $\widetilde{\mathbf{A}}^{-1} = (g_{ij})$, then $G_{ij} = g_{ij}$ for $i \in X_n$, $j \in X_{n-m}$ and $\mathbf{AG} = \mathbf{E}$, $\mathbf{BG} = \mathbf{O}$, where $\mathbf{G} = (G_{ij}) \in M_{(n+1)\times(n-m+1)}(\mathbb{R})$ (or $\sum_{k=0}^{n} a_{ik} G_{kj} = \delta_j^i$, $i \in X_{n-m}$, $\sum_{k=0}^{n} b_{ik} G_{kj} = 0$, $i \in X_m$, $j \in X_{n-m}$). So, $[G_{0j}, \dots, G_{nj}]$ is a unique solution of problem (5.4) with $f_j = [\delta_j^0, \dots, \delta_j^n]$, $j \in X_{n-m}$.

Example 5.1. In the case $m = 2$, formula (5.5) can be written as

$$u_i = \sum_{j=0}^{n-2} G_{ij} f_j = \left(G_{i,\cdot}, f\right)_{\widetilde{X}}, \quad i \in X_n. \tag{5.6}$$

The function $H^\theta \in F(X \times \widetilde{X})$ is an example of Green's function for (4.3) with discrete (initial) conditions $u_0 = u_1 = 0$. In the case $m = 2$, formula (5.6) is the same as (4.15), $\widetilde{X} = X_{n-2}$.

Remark 5.2. Let us consider the case $m = 2$. If $f_i = \overline{f}_{i+1}$, where the function $\overline{f}$ is defined on $\overline{X} := \{1, 2, \dots, n-1\}$, then we use the shifted Green's function $\overline{G} \in F(X \times \overline{X})$

$$u_i = \sum_{j=1}^{n-1} \overline{G}_{ij} \overline{f}_j, \quad \overline{G}_{ij} := G_{i,j-1}, \ i \in X_n. \tag{5.7}$$

For finite-difference schemes, discrete functions are defined in points $x_i \in [0, L]$ and $f_i = f(x_i)$. In this paper, we introduce meshes

$$\overline{\omega}^h = \{0 = x_0 < x_1 < \cdots < x_n = L\},$$
$$\omega^h = \overline{\omega}^h \setminus \{x_0, x_n\}, \qquad \widetilde{\omega}^h = \overline{\omega}^h \setminus \{x_{n-1}, x_n\} \tag{5.8}$$

with the step sizes $h_i = x_i - x_{i-1}$, $1 \leq i \leq n$, $h_0 = h_{n+1} = 0$, and a semi-integer mesh

$$\omega^h_{1/2} = \left\{x_{i+1/2} \mid x_{i+1/2} = \frac{x_i + x_{i+1}}{2},\ 0 \leq i \leq n-1\right\} \tag{5.9}$$

with the step sizes $h_{i+1/2} = (h_i + h_{i+1})/2$, $0 \leq i \leq n$. We define the inner product

$$(U, V)_{\overline{\omega}^h} := \sum_{i=0}^{n} U_i V_i h_{i+1/2}, \tag{5.10}$$

where $U, V \in F(\overline{\omega}^h)$, and the following mesh operators:

$$(\delta Z)_{i+1/2} = \frac{Z_{i+1} - Z_i}{h_{i+1}}, \quad Z \in F\left(\overline{\omega}^h\right), \qquad (\delta Z)_i = \frac{Z_{i+1/2} - Z_{i-1/2}}{h_{i+1/2}}, \quad Z \in F\left(\omega^h_{1/2}\right). \tag{5.11}$$

If $A : F(\overline{\omega}^h) \to F(\omega)$ and $f \in F(\omega)$, where $\omega = \overline{\omega}^h, \omega^h, \widetilde{\omega}^h$, then we define the Green's function $G \in F(\overline{\omega}^h \times \omega)$

$$u_i = \sum_{j : x_j \in \omega} G_{ij} f_j, \quad i \in X_n. \tag{5.12}$$

For many applications another discrete Green's function G^h is used [9, 11]

$$u_i = \sum_{j=0}^{n} G^h_{ij} f_j h_{j+1/2} = \left(G^h_{i,\cdot}, f\right)_{\overline{\omega}^h}, \quad i \in X_n, \tag{5.13}$$

where $f_j = 0$ for $x_j \in \overline{\omega}^h \setminus \omega$. The relations between these functions are

$$G^h_{ij} = \frac{G_{ij}}{h_{j+1/2}} \quad \text{for } j : x_j \in \omega, \qquad G^h_{ij} = 0 \quad \text{for } j : x_j \in \overline{\omega}^h \setminus \omega. \tag{5.14}$$

So, if we know the function G_{ij}, then we can calculate G^h_{ij}, and vice versa. If $h_i \equiv 1$ ($L = n$), then G^h_{ij} coincides with G_{ij}.

Note that the Wronskian determinant can be defined by the following formula (see [10]):

$$W^h[\mathbf{u}]_j = \begin{vmatrix} u^1_{j-1} & u^2_{j-1} \\ \delta u^1_{j-1/2} & \delta u^2_{j-1/2} \end{vmatrix} = \begin{vmatrix} u^1_{j-1} & u^2_{j-1} \\ \dfrac{u^1_j - u^1_{j-1}}{h_j} & \dfrac{u^2_j - u^2_{j-1}}{h_j} \end{vmatrix} = \frac{W[\mathbf{u}]_j}{h_j}, \quad j = 1, \dots, n. \tag{5.15}$$

5.2. Green's Functions for a Linear Difference Equation with Additional Conditions

Let us consider the nonhomogeneous equation (4.3) with the operator: $\mathcal{L} : U \to F(X)$, where additional homogeneous conditions define the subspace $U = \{\mathbf{u} \in F(X) : \langle L_1, u\rangle = 0, \langle L_2, u\rangle = 0\}$.

Lemma 5.3. *Green's function for problem* (4.3) *with the homogeneous additional conditions* $\langle L_1, u\rangle = 0$, $\langle L_2, u\rangle = 0$, *where functionals* L_1 *and* L_2 *are linearly independent, is equal to*

$$G_{ij} = \frac{D(\mathbf{L}, \delta_i)\left[\mathbf{u}, H^\theta_{\cdot,j}\right]}{D(\mathbf{L})[\mathbf{u}]}, \quad i \in \mathrm{X},\ j \in \tilde{\mathrm{X}}. \tag{5.16}$$

Proof. In the previous section, we derived a formula of the solution (see Theorem 4.1 for $g_1, g_2 = 0$)

$$u_i = \left(\left\langle \delta^k_i - \mathbf{L}^k \mathbf{v}_i, H^\theta_{k,\cdot}\right\rangle, f\right)_{X}, \quad i \in X, \tag{5.17}$$

where $v^1_i = D(\delta_i, L_2)/D(\mathbf{L})$, $v^2_i = D(L_1, \delta_i)/D(\mathbf{L})$. So, Green's function is equal to

$$G_{ij} = \left\langle \delta^k_i - \mathbf{L}^k \mathbf{v}_i, H^\theta_{kj}\right\rangle = H^\theta_{ij} - \left\langle L^k_1, H^\theta_{kj}\right\rangle \frac{D(\delta_i, L_2)}{D(\mathbf{L})} - \left\langle L^k_2, H^\theta_{kj}\right\rangle \frac{D(L_1, \delta_i)}{D(\mathbf{L})}. \tag{5.18}$$

We have

$$D(\mathbf{L}, \delta_i)\left[\mathbf{u}, H^\theta_{\cdot,j}\right] = \begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & u^1_i \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle & u^2_i \\ \left\langle L^k_1, H^\theta_{kj}\right\rangle & \left\langle L^k_2, H^\theta_{kj}\right\rangle & H^\theta_{ij} \end{vmatrix}, \tag{5.19}$$

too. If we expand this determinant according to the last row and divide by $D(\mathbf{L})[\mathbf{u}]$, then we get the right-hand side of (5.18). The lemma is proved. □

If $\overline{\mathbf{u}} = \mathbf{P}\mathbf{u}$, where $\mathbf{P} \in \mathrm{GL}_2(\mathbb{R})$, then we get that Green's function $G_{ij} = G[\overline{\mathbf{u}}]_{ij} = G[\mathbf{u}]_{ij}$, that is, it is invariant with respect to the basis $\{u^1, u^2\}$.

For the theoretical investigation of problems with NBCs, the next result about the relations between Green's functions G^u_{ij} and G^v_{ij} of two nonhomogeneous problems

$$\begin{gathered} Lu = f, \qquad Lv = f, \\ \langle l_m, u\rangle = 0, \quad m = 1,2, \qquad \langle L_m, v\rangle = 0, \quad m = 1,2, \end{gathered} \tag{5.20}$$

with the same f, is useful.

Theorem 5.4. *If Green's function G^u exists and the functionals L_1 and L_2 are linearly independent, then*

$$G^v_{ij} = \frac{D(\mathbf{L},\delta_i)\left[\mathbf{u}, G^u_{\cdot,j}\right]}{D(\mathbf{L})[\mathbf{u}]}, \quad i \in X,\ j \in \widetilde{X}. \tag{5.21}$$

Proof. We have equality (4.26) (the case $F_1, F_2 = 0$)

$$v = u - \langle L_1, u\rangle v^1 - \langle L_2, u\rangle v^2. \tag{5.22}$$

If $u_i = (G^u_{i,\cdot}, f)_X$, then

$$v_i = u_i - \sum_{k=0}^{n} u_k L_1^k v_i^1 - \sum_{k=0}^{n} u_k L_2^k v_i^2 = \left(G^u_{i,\cdot} - \sum_{k=0}^{n} G^u_{k,\cdot} L_1^k v_i^1 - \sum_{k=0}^{n} G^u_{k,\cdot} L_2^k v_i^2, f \right)_X. \tag{5.23}$$

So, Green's function G^v is equal to

$$\begin{aligned} G^v_{ij} &= G^u_{ij} - \sum_{k=0}^{n} G^u_{kj} L_1^k v_i^1 - \sum_{k=0}^{n} G^u_{kj} L_2^k v_i^2 \\ &= \left\langle \delta_i^k - L_1^k v_i^1 - L_2^k v_i^2, G^u_{kj} \right\rangle = \left\langle \delta_i^k - \mathbf{L}^k \mathbf{v}_i, G^u_{kj} \right\rangle. \end{aligned} \tag{5.24}$$

A further proof of this theorem repeats the proof of Lemma 5.3 (we have G^u instead of H^θ). □

Remark 5.5. Instead of formula (5.18), we have

$$G^v_{ij} = G^u_{ij} - \langle L_1^k, G^u_{kj}\rangle \frac{D(\delta_i, L_2)}{D(\mathbf{L})} - \langle L_2^k, G^u_{kj}\rangle \frac{D(L_1,\delta_i)}{D(\mathbf{L})}. \tag{5.25}$$

We can write the determinant in formula (5.21) in the explicit way

$$G^v_{ij} = \frac{D(\mathbf{L},\delta_i)\left[\mathbf{u}, G^u_{\cdot,j}\right]}{D(\mathbf{L})[\mathbf{u}]} = \frac{1}{D(\mathbf{L})[\mathbf{u}]} \begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & u_i^1 \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle & u_i^2 \\ \left\langle L_1^k, G^u_{kj}\right\rangle & \left\langle L_2^k, G^u_{kj}\right\rangle & G^u_{ij} \end{vmatrix}. \tag{5.26}$$

Formulaes (5.25) and (5.26) easily allow us to find Green's function for an equation with two additional conditions if we know Green's function for the same equation, but with other additional conditions. The formula

$$u_i = \left(G_{i,\cdot}, f\right)_X + g_1 v_i^1 + g_2 v_i^2, \quad i \in X \tag{5.27}$$

can be used to get the solutions of the equations with a difference operator with any two linear additional (initial or boundary or nonlocal boundary) conditions if the general solution of a homogeneous equation is known.

6. Applications to Problems with NBC

Let us investigate Green's function for the problem with nonlocal boundary conditions

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in \widetilde{X}, \tag{6.1}$$

$$\langle L_1, u \rangle := \langle \kappa_0, u \rangle - \gamma_0 \langle \varkappa_0, u \rangle = 0, \tag{6.2}$$

$$\langle L_2, u \rangle := \langle \kappa_1, u \rangle - \gamma_1 \langle \varkappa_1, u \rangle = 0. \tag{6.3}$$

We can write many problems with nonlocal boundary conditions (NBC) in this form, where $\langle \kappa_m, u \rangle := \langle \kappa_m^i, u_i \rangle$, $m = 0, 1$, is a classical part and $\langle \varkappa_m, u \rangle := \langle \varkappa_m^i, u_i \rangle$, $m = 0, 1$, is a nonlocal part of boundary conditions.

If $\gamma_0, \gamma_1 = 0$, then problem (6.1)–(6.3) becomes classical. Suppose that there exists Green's function G_{ij}^{cl} for the classical case. Then Green's function exists for problem (6.1)–(6.3) if $\vartheta = D(\mathbf{L})[\mathbf{u}] \neq 0$. For $L_m = \kappa_m - \gamma_m \varkappa_m$, $m = 0, 1$, we derive

$$\vartheta = D(\kappa_0 \cdot \kappa_1)[\mathbf{u}] - \gamma_0 D(\varkappa_0 \cdot \kappa_1)[\mathbf{u}] - \gamma_1 D(\kappa_0 \cdot \varkappa_1)[\mathbf{u}] + \gamma_0 \gamma_1 D(\varkappa_0 \cdot \varkappa_1)[\mathbf{u}]. \tag{6.4}$$

Since $\langle \kappa_m^k, G_{kj}^{\mathrm{cl}} \rangle = 0$, $m = 0, 1$, we can rewrite formula (5.26) as

$$\begin{aligned} G_{ij} &= G_{ij}^{\mathrm{cl}} + \gamma_0 v_i^1 \left\langle \varkappa_1^k, G_{kj}^{\mathrm{cl}} \right\rangle + \gamma_1 v_i^2 \left\langle \varkappa_2^k, G_{kj}^{\mathrm{cl}} \right\rangle \\ &= G_{ij}^{\mathrm{cl}} + \gamma_0 \left\langle \varkappa_1^k, G_{kj}^{\mathrm{cl}} \right\rangle \frac{D(\delta_i, L_2)}{\vartheta} + \gamma_1 \left\langle \varkappa_2^k, G_{kj}^{\mathrm{cl}} \right\rangle \frac{D(L_1, \delta_i)}{\vartheta} \\ &= \frac{1}{\vartheta} \begin{vmatrix} \langle L_1, u^1 \rangle & \langle L_2, u^1 \rangle & u_i^1 \\ \langle L_1, u^2 \rangle & \langle L_2, u^2 \rangle & u_i^2 \\ -\gamma_0 \left\langle \varkappa_1^k, G_{kj}^{\mathrm{cl}} \right\rangle & -\gamma_1 \left\langle \varkappa_2^k, G_{kj}^{\mathrm{cl}} \right\rangle & G_{ij}^{\mathrm{cl}} \end{vmatrix}. \end{aligned} \tag{6.5}$$

Example 6.1. Let us consider the differential equation with two nonlocal boundary conditions

$$\begin{gathered} -u'' = f(x), \quad x \in (0, 1), \\ u(0) = \gamma_0 u(\xi_0), \quad u(1) = \gamma_1 u(\xi_1), \quad 0 < \xi_0, \ \xi_1 < 1. \end{gathered} \tag{6.6}$$

We introduce a mesh $\overline{\omega}^h$ (see (5.8)). Denote $u_i = u(x_i)$, $f_i = f(x_i)$ for $x_i \in \overline{\omega}^h$. Then problem (6.6) can be approximated by a finite-difference problem (scheme)

$$-\delta^2 u_i = f_i, \qquad x_i \in \omega^h, \tag{6.7}$$

$$u_0 = \gamma_0 u_{s_0}, \qquad u_n = \gamma_1 u_{s_1}. \tag{6.8}$$

We suppose that the points ξ_0, ξ_1 are coincident with the grid points, that is, $\xi_0 = x_{s_0}$, $\xi_1 = x_{s_1}$.

We rewrite (6.7) in the following form:

$$a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_{i+1}, \quad i \in \widetilde{X}, \tag{6.9}$$

where

$$a_i^2 = -\frac{1}{h_{i+2}h_{i+3/2}}, \quad a_i^1 = \frac{2}{h_{i+2}h_{i+1}}, \quad a_i^0 = -\frac{1}{h_{i+1}h_{i+3/2}}, \quad i \in \widetilde{X}. \tag{6.10}$$

We can take the following fundamental system: $u_i^1 = 1$, $u_i^2 = x_i$. Then

$$\begin{gathered} D[\mathbf{u}]_{ij} = \begin{vmatrix} u_i^1 & u_j^1 \\ u_i^2 & u_j^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ x_i & x_j \end{vmatrix} = x_j - x_i, \quad i,j \in X, \ \ W_j = h_j, \ \ j = 1,\dots,n, \\ H_{ij} = \frac{x_i - x_{j+1}}{h_{j+2}}, \quad j = -1,0,1,\dots,n-2, \quad H_{i,n-1} = H_{in} = 0, \quad i \in X. \end{gathered} \tag{6.11}$$

As a result, we obtain

$$H_{ij}^{\theta} = \frac{\theta_{i-j} H_{ij}}{a_j^2} = \theta_{i-j}(x_{j+1} - x_i) h_{j+3/2}. \tag{6.12}$$

For a problem with the boundary conditions $u_0 = u_n = 0$ we have $D(L)[\mathbf{u}] = 1$,

$$\begin{aligned} D(L,\delta_i)\left[\mathbf{u}, H_{\cdot,j}^{\theta}\right] &= H_{ij}^{\theta} - x_i H_{nj}^{\theta} \\ &= \theta_{i-j}(x_{j+1} - x_i) h_{j+3/2} - \theta_{n-j} x_i (x_{j+1} - 1) h_{j+3/2}, \end{aligned} \tag{6.13}$$

and we express Green's function G^{cl} of the Dirichlet problem via Green's function H^{θ} of the initial problem

$$G_{ij}^{\text{cl}} = H_{ij}^{\theta} - x_i H_{nj}^{\theta}. \tag{6.14}$$

We derive expressions for "classical" Green's function

$$G_{ij}^{\mathrm{cl}} = h_{j+3/2}\left(\theta_{i-j}\left(x_{j+1}-x_i\right)+\theta_{n-j}x_i\left(1-x_{j+1}\right)\right)$$
$$= h_{j+3/2}\begin{cases} x_i\left(1-x_{j+1}\right), & i \le j+1, \\ x_{j+1}\left(1-x_i\right), & i \ge j+1, \end{cases} \qquad i \in X,\ j \in \widetilde{X} \tag{6.15}$$

or (see (5.7) and (5.13))

$$\overline{G}_{ij}^{\mathrm{cl}} = h_{j+1/2}\begin{cases} x_i\left(1-x_j\right), & x_i \le x_j, \\ x_j\left(1-x_i\right), & x_i \ge x_j, \end{cases} \qquad i \in X,\ j \in \overline{X}$$
$$\overline{G}_{ij}^{\mathrm{cl},h} = \begin{cases} x_i\left(1-x_j\right), & 0 \le x_i \le x_j \le 1, \\ x_j\left(1-x_i\right), & 0 \le x_j \le x_i \le 1, \end{cases} \qquad i, j \in X. \tag{6.16}$$

Remark 6.2. Note that the index of f on the right-hand side of (6.9) is shifted (cf. (6.1)).

Green's function $\overline{G}^{\mathrm{cl},h}$ is the same as in [10], and it is equal to Green's function

$$\overline{G}^{\mathrm{cl}}(x,y) = \begin{cases} x(1-y), & 0 \le x \le y \le 1, \\ y(1-x), & 0 \le y \le x \le 1 \end{cases} \tag{6.17}$$

for differential problem (6.6) at grid points in the case $\gamma_0 = \gamma_1 = 0$.

For a "nonlocal" problem with the boundary conditions $u_0 = \gamma_0 u_{s_0}$, $u_n = \gamma_1 u_{s_1}$,

$$\vartheta := D(L)[\mathbf{u}] = \begin{vmatrix} \langle L_1, 1\rangle & \langle L_2, 1\rangle \\ \langle L_1, x\rangle & \langle L_2, x\rangle \end{vmatrix} = \begin{vmatrix} 1-\gamma_0\cdot 1 & 1-\gamma_1\cdot 1 \\ x_0-\gamma_0 x_{s_0} & x_n-\gamma_1 x_{s_1} \end{vmatrix}$$
$$= \begin{vmatrix} 1-\gamma_0 & 1-\gamma_1 \\ -\gamma_0\xi_0 & 1-\gamma_1\xi_1 \end{vmatrix} = 1-\gamma_0\left(1-\xi_0\right)-\gamma_1\xi_1+\gamma_0\gamma_1\left(\xi_1-\xi_0\right). \tag{6.18}$$

It follows from (6.5) that

$$\overline{G}_{ij}^{h} = \overline{G}_{ij}^{\mathrm{cl},h} + \gamma_0\frac{1-x_i+\gamma_1\left(x_i-\xi_1\right)}{\vartheta}\overline{G}_{s_0 j}^{\mathrm{cl},h} + \gamma_1\frac{x_i-\gamma_0\left(x_i-\xi_0\right)}{\vartheta}\overline{G}_{s_1 j}^{\mathrm{cl},h} \tag{6.19}$$

if $\vartheta \neq 0$. Green's function does not exist for $\theta = 0$. By substituting Green's function $\overline{G}^{\mathrm{cl},h}$ for the problem with the classical boundary conditions into the above equation, we obtain Green's function for the problem with nonlocal boundary conditions

$$\overline{G}_{ij}^{h} = \begin{cases} x_i(1-x_j), & x_i \le x_j, \\ x_j(1-x_i), & x_i \ge x_j, \end{cases} + \gamma_0 \frac{1 - x_i + \gamma_1(x_i - \xi_1)}{1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 + \gamma_0\gamma_1(\xi_1 - \xi_0)} \begin{cases} \xi_0(1-x_j), & \xi_0 \le x_j, \\ x_j(1-\xi_0), & \xi_0 \ge x_j, \end{cases} + \gamma_1 \frac{x_i - \gamma_0(x_i - \xi_0)}{1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 + \gamma_0\gamma_1(\xi_1 - \xi_0)} \begin{cases} \xi_1(1-x_j), & \xi_1 \le x_j, \\ x_j(1-\xi_1), & \xi_1 \ge x_j. \end{cases} \tag{6.20}$$

This formula corresponds to the formula of Green's function for differential problem (6.6) (see [4])

$$\overline{G}(x,y) = \begin{cases} x(1-y), & x \le y, \\ x_j(1-x), & x \ge y, \end{cases} + \gamma_0 \frac{1 - x + \gamma_1(x - \xi_1)}{1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 + \gamma_0\gamma_1(\xi_1 - \xi_0)} \begin{cases} \xi_0(1-y), & \xi_0 \le y \\ x_j(1-\xi_0), & \xi_0 \ge y, \end{cases} + \gamma_1 \frac{x - \gamma_0(x - \xi_0)}{1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 + \gamma_0\gamma_1(\xi_1 - \xi_0)} \begin{cases} \xi_1(1-y), & \xi_1 \le y, \\ x_j(1-\xi_1), & \xi_1 \ge y. \end{cases} \tag{6.21}$$

Example 6.3. Let us consider the problem

$$\begin{gathered} -u'' = f(x), \quad x \in (0,1), \\ u(0) = \gamma_0 \int_0^1 \alpha^0(x)u(x)dx, \qquad u(1) = \gamma_1 \int_0^1 \alpha^1(x)u(x)dx, \end{gathered} \tag{6.22}$$

where $\alpha^0, \alpha^1 \in L_1(0,1)$.

Problem (6.22) can be approximated by the difference problem

$$\begin{gathered} -\delta^2 u_i = f_i, \qquad x_i \in \omega^h, \\ u_0 = \gamma_0\left(A^0, u\right)_K, \qquad u_n = \gamma_1\left(A^1, u\right)_K, \end{gathered} \tag{6.23}$$

where A^0, A^1 are approximations of the weight functions α^0, α^1 in integral boundary conditions, $(A,u)_K$ is a quadrature formula for the integral $\int_0^1 A(x)u(x)dx$ approximation (e.g., trapezoidal formula $(A,u)_{\mathrm{trap}} := \sum_{k=0}^{n} A_k u_k h_{k+1/2}$).

The expression of Green's function for the problem with the classical boundary conditions ($\gamma_0 = \gamma_1 = 0$, $u_i^1 = 1$, $u_i^2 = x_i$) is described in Example 6.1. The existence condition of Green's function for problem (6.23) is $\vartheta \neq 0$, where

$$\begin{aligned}\vartheta = D(\mathbf{L})[\mathbf{u}] &= \begin{vmatrix} 1-\gamma_0(A^0,1)_K & 1-\gamma_1(A^1,1)_K \\ -\gamma_0(A^0,x)_K & 1-\gamma_1(A^1,x)_K \end{vmatrix} \\ &= 1-\gamma_0\left(A^0,1-x\right)_K - \gamma_1\left(A^1,x\right)_K + \gamma_0\gamma_1 \begin{vmatrix} (A^0,1-x)_K & (A^0,x)_K \\ (A^1,1-x)_K & (A^1,x)_K \end{vmatrix}\end{aligned} \tag{6.24}$$

(such a condition was obtained for problem (6.23) in [15, 16]) and Green's function is equal to (see Theorem 5.4)

$$\begin{aligned}G_{ij} = G_{ij}^{\mathrm{cl}} &+ \frac{\gamma_0(1-x_i+\gamma_1(x_i(A^1,1)_K-(A^1,x)_K))\left(A^0,G_{\cdot,j}^{\mathrm{cl}}\right)_K}{\vartheta} \\ &+ \frac{\gamma_1(x_i-\gamma_0(x_i(A^0,1)_K-(A^0,x)_K))\left(A^1,G_{\cdot,j}^{\mathrm{cl}}\right)_K}{\vartheta},\end{aligned} \tag{6.25}$$

where G_{ij}^{cl} is defined by (6.15).

Green's function for differential problem (6.22) was derived in [8]. For this problem

$$\begin{aligned}\vartheta &= 1-\gamma_0\int_0^1 \alpha_0(x)(1-x)dx - \gamma_1\int_0^1 \alpha_1(x)x\,dx \\ &\quad -\gamma_0\gamma_1\iint_0^1 \alpha_0(x)\alpha_1(y)(x-y)dx\,dy, \\ G(x,y) &= \overline{G}^{\mathrm{cl}}(x,y) + \frac{\gamma_0\left(1-x+\gamma_1\int_0^1\alpha_1(t)(x-t)dt\right)}{\vartheta}\cdot\int_0^1 \alpha_0(t)\overline{G}^{\mathrm{cl}}(t,y)dt \\ &\quad + \frac{\gamma_1\left(x-\gamma_0\int_0^1\alpha_0(t)(x-t)dt\right)}{\vartheta}\cdot\int_0^1 \alpha_1(t)\overline{G}^{\mathrm{cl}}(t,y)dt\end{aligned} \tag{6.26}$$

if $\vartheta \neq 0$, where $\overline{G}^{\mathrm{cl}}(x,y)$ is defined by formula (6.17).

Remark 6.4. We could substitute (6.15) into (6.25) and obtain an explicit expression of Green's function. However, it would be quite complicated, and we will not write it out. Note that, if $(A^0,u)_K = u_{s_0}$, $(A^1,u)_K = u_{s_1}$, then discrete problem (6.23) is the same as (6.7)-(6.8). For example, it happens if a trapezoidal formula is used for the approximation α^l, $l = 0,1$ and we take $A_i^l = \delta_i^{s_l}/h_{s_l+1/2}$. It is easy to see that we could obtain the same expression for Green's function (6.19) in this case.

Example 6.5. Let us consider a difference problem

$$-\delta^2 u_i = f_i, \qquad x_i \in \omega^h,$$
$$u_0 = \alpha_0 u_1 + \gamma_0 u_{n-1}, \qquad u_n = \alpha_1 u_1 + \gamma_1 u_{n-1}. \tag{6.27}$$

A condition for the existence of the Green's function (fundamental system $\{1 - x, x\}$) is

$$\vartheta := \mathrm{D}(\mathbf{L})[\mathbf{u}] = \begin{vmatrix} 1-\alpha_0(1-h_1)-\gamma_0 h_n & -\alpha_1(1-h_1)-\gamma_1 h_n \\ -\alpha_0 h_1 - \gamma_0(1-h_n) & 1-h_n-\alpha_1 h_1 - \gamma_1(1-h_n) \end{vmatrix}$$
$$= \begin{vmatrix} 1-\alpha_0 & \gamma_0 \\ \alpha_1 & 1-\gamma_1 \end{vmatrix} + h_1 \begin{vmatrix} \alpha_0 & 1-\gamma_0 \\ \alpha_1 & 1-\gamma_1 \end{vmatrix} + h_n \begin{vmatrix} 1-\alpha_0 & \gamma_0 \\ 1-\alpha_1 & \gamma_1 \end{vmatrix} \neq 0. \tag{6.28}$$

We consider three types ($\alpha_0 = \gamma_1 = 0$, $\gamma_0 = \alpha_1 = 1$; $\alpha_0 = \alpha_1 = (1 + h_1/h_n)^{-1}$, $\gamma_0 = \gamma_1 = (1 + h_n/h_1)^{-1}$; $\alpha_0 = 0$, $\gamma_0 = 1$, $\alpha_1 = h_n/h_1$, $\gamma_1 = (1 - h_n/h_1)$ of discrete boundary conditions

$$u_0 = u_{n-1}, \qquad u_1 = u_n,$$
$$u_0 = u_n, \qquad \delta u_{1/2} = \delta u_{n-1/2}, \tag{6.29}$$
$$u_0 = u_{n-1}, \qquad \delta u_{1/2} = \delta u_{n-1/2}.$$

All the cases yield $\vartheta = 0$. Consequently, Green's function for the three problems does not exist.

7. Conclusions

Green's function for problems with additional conditions is related with Green's function of a similar problem, and this relation is expressed by formulae (5.26). Green's function exists if $\vartheta = D(\mathbf{L})[\mathbf{u}] \neq 0$. If we know Green's function for the problem with additional conditions and the fundamental basis of a homogeneous difference equation, then we can obtain Green's function for a problem with the same equation but with other additional conditions. It is shown by a few examples for problems with NBCs that but formulae (5.26) can be applied to a very wide class of problems with various boundary conditions as well as additional conditions.

All the results of this paper can be easily generalized to the n-order difference equation with n additional functional conditions. The obtained results are similar to a differential case [8, 17].

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