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Research Article

On a Perturbed Dirichlet Problem for a Nonlocal Differential Equation of Kirchhoff Type

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We study the existence of positive solutions to the following nonlocal boundary value problem $-K(\|u\|^2)\Delta u = \lambda u^{s-1} + f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where $s \in]1,2[$, $f : \Omega\times\mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function, $K : \mathbb{R}_+ \to \mathbb{R}$ is a positive continuous function, and λ is a real parameter. Direct variational methods are used. In particular, the proof of the main result is based on a property of the infimum on certain spheres of the energy functional associated to problem $-K(\|u\|^2)\Delta u = \lambda u^{s-1}$ in Ω, $u_{|\partial\Omega} = 0$.

1. Introduction

This paper aims to establish the existence of positive solutions in $W_0^{1,2}(\Omega)$ to the following problem involving a nonlocal equation of Kirchhoff type:

$$\begin{gathered} -K\left(\|u\|^2\right)\Delta u = \lambda u^{s-1} + f(x,u), \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{P_λ}$$

Here Ω is an open bounded set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $s \in]1,2[$, $f : \Omega \times [0,+\infty[\to [0,+\infty[$ is a Carathéodory function, $K : \mathbb{R}_+ \to \mathbb{R}$ is a positive continuous function, λ is a real parameter, and $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$ is the standard norm in $W_0^{1,2}(\Omega)$. In what follows, for every real number t, we put $t_+ = (|t| + t)/2$.

By a positive solution of (P_λ), we mean a positive function $u \in W_0^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ which is a solution of (P_λ) in the *weak sense*, that is such that

$$K\left(\|u\|^2\right)\int_\Omega (\nabla u(x)\nabla v(x))dx - \int_\Omega \left(\lambda u(x)^{s-1} + f(x,u(x))\right)v(x)dx = 0 \tag{1.1}$$

for all $v \in W_0^{1,2}(\Omega)$. Thus, the weak solutions of (P_λ) are exactly the positive critical points of the associated energy functional

$$I(u) = \int_0^{\|u\|^2} K(\tau)d\tau - \int_\Omega \left(\lambda u_+(x)^{s-1} + \int_0^{u(x)} f(x,t)dt\right)dx, \quad u \in W_0^{1,2}(\Omega). \tag{1.2}$$

When $K(t) = a + bt$ $(a, b > 0)$, the equation involved in problem (P_λ) is the stationary analogue of the well-known equation proposed by Kirchhoff in [1]. This is one of the motivations why problems like (P_λ) were studied by several authors beginning from the seminal paper of Lions [2]. In particular, among the most recent papers, we cite [3–7] and refer the reader to the references therein for a more complete overview on this topic.

The case $\lambda = 0$ was considered in [3] and [4], where the existence of at least one positive solution is established under various hypotheses on f. In particular, in [3] the nonlinearity f is supposed to satisfy the well-known Ambrosetti-Rabinowitz growth condition; in [4] f satisfies certain growth conditions at 0 and ∞, and $f(x,t)/t$ is nondecreasing on $]0,+\infty[$ for all $x \in \Omega$. Critical point theory and minimax methods are used in [3] and [4]. For $K(t) = a + bt$ and $\lambda = 0$, the existence of a nontrivial solution as well as multiple solutions for problem (P_λ) is established in [5] and [7] by using critical point theory and invariant sets of descent flow. In these papers, the nonlinearity f is again satisfying suitable growth conditions at 0 and ∞. Finally, in [6], where the nonlinearity t_+^{s-1} is replaced by a more general $h(x,t)$ and the nonlinearity f is multiplied by a positive parameter μ, the existence of at least three solutions for all λ belonging to a suitable interval depending on h and K and for all μ small enough (with upper bound depending on λ) is established (see [6, Theorem 1]). However, we note that the nonlinearity t_+^{s-1} does not meet the conditions required in [6]. In particular, condition (a_5) of [6, Theorem 1] is not satisfied by t_+^{s-1}. Moreover, in [6] the nonlinearity f is required to satisfy a subcritical growth at ∞ (and no other condition).

Our aim is to study the existence of positive solution to problem (P_λ), where, unlike previous existence results (and, in particular, those of the aforementioned papers), no growth condition is required on f. Indeed, we only require that on a certain interval $[0, C]$ the function $f(x,\cdot)$ is bounded from above by a suitable constant a, uniformly in $x \in \Omega$. Moreover, we also provide a localization of the solution by showing that for all $r > 0$ we can choose the constant a in such way that there exists a solution to (P_λ) whose distance in $W_0^{1,2}(\Omega)$ from the unique solution of the unperturbed problem (that is problem (P_λ) with $f = 0$) is less than r.

2. Results

Our first main result gives some conditions in order that the energy functional associated to the unperturbed problem (P_λ) has a unique global minimum.

Theorem 2.1. *Let $s \in]1,2[$ and $\lambda > 0$. Let $K : [0,+\infty[\to \mathbb{R}$ be a continuous function satisfying the following conditions:*

(a_1) $\inf_{t\geq 0} K(t) > 0$;

(a_2) *the function* $t \to (1/2)\int_0^t K(\tau)d\tau - (1/s)K(t)t$ *is strictly monotone in* $[0,+\infty[$;

(a_3) $\liminf_{t\to+\infty} t^{-2\alpha}\int_0^t K(\tau)d\tau > 0$ *for some* $\alpha \in](s/2), 1[$.

Then, the functional

$$\Psi(u) = \frac{1}{2}\int_0^{\|u\|^2} K(\tau)d\tau - \frac{\lambda}{s}\int_\Omega u_+^s dx, \quad u \in W_0^{1,2}(\Omega) \tag{2.1}$$

admits a unique global minimum on $W_0^{1,2}(\Omega)$.

Proof. From condition (a_3) we find positive constants C_1, C_2 such that

$$\frac{1}{2}\int_0^{\|u\|^2} K(\tau)d\tau \geq C_1\|u\|^{2\alpha} - C_2, \quad \text{for every } u \in W_0^{1,2}(\Omega). \tag{2.2}$$

Therefore, by Sobolev embedding theorems, there exists a positive constant C_3 such that

$$\Psi(u) \geq C_1\|u\|^{2\alpha} - C_2 - C_3\|u\|^s, \quad \text{for every } u \in W_0^{1,2}(\Omega). \tag{2.3}$$

Since $s \in]0, 2\alpha[$, from the previous inequality we obtain

$$\lim_{\|u\| \to +\infty} \Psi(u) = +\infty. \tag{2.4}$$

By standard results, the functional

$$u \in W_0^{1,2}(\Omega) \longrightarrow \frac{1}{s}\int_\Omega u_+^s dx \tag{2.5}$$

is of class C^1 and sequentially weakly continuous, and the functional

$$u \in W_0^{1,2}(\Omega) \longrightarrow \frac{1}{2}\int_0^{\|u\|^2} K(\tau)d\tau \tag{2.6}$$

is of class C^1 and sequentially weakly lower semicontinuous. Then, in view of the coercivity condition (2.4), the functional Ψ attains its global minimum on $W_0^{1,2}(\Omega)$ at some point $u_0 \in W_0^{1,2}(\Omega)$.

Now, let us to show that

$$\inf_{W_0^{1,2}(\Omega)} \Psi < 0. \tag{2.7}$$

Indeed, fix a nonzero and nonnegative function $v \in C_0^\infty(\Omega)$, and put $v_\varepsilon = \varepsilon v$. We have

$$\Psi(\varepsilon v) \leq \varepsilon^2 \max_{t \in [0, \varepsilon^2\|v\|^2]} K(t)\|v\|^2 - \frac{\lambda\varepsilon^s}{s}\int_\Omega v^s dx. \tag{2.8}$$

Hence, taking into account that $s < 2\alpha < 2$, for ε small enough, one has $\Psi(v_\varepsilon) < 0$. Thus, inequality (2.7) holds.

At this point, we show that u_0 is unique. To this end, let $v_0 \in W_0^{1,2}(\Omega)$ be another global minimum for Ψ. Since Ψ is a C^1 functional with

$$\Psi'(u)(v) = K\left(\|u\|^2\right)\int_\Omega \nabla u \nabla v\, dx - \int_\Omega u_+^{s-1} v\, dx \tag{2.9}$$

for all $u, v \in W_0^{1,2}(\Omega)$, we have that $\Psi'(u_0) = \Psi'(v_0) = 0$. Thus, u_0 and v_0 are weak solutions of the following nonlocal problem:

$$\begin{gathered} -K\left(\|u\|^2\right)\Delta u = \lambda u_+^{s-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.10}$$

Moreover, in view of (2.7), u_0 and v_0 are nonzero. Therefore, from the Strong Maximum Principle, u_0 and v_0 are positive in Ω as well. Now, it is well known that, for every $\mu > 0$, the problem

$$\begin{gathered} -\Delta u = \mu u_+^{s-1}, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.11}$$

admits a unique positive solution in $W_0^{1,2}(\Omega)$ (see, e.g., [8, Lemma 3.3]). Denote it by u_μ. Then, it is easy to realize that for every couple of positive parameters μ_1, μ_2, the functions u_{μ_1}, u_{μ_2} are related by the following identity:

$$u_{\mu_1} = \left(\frac{\mu_1}{\mu_2}\right)^{1/(s-1)} u_{\mu_2}. \tag{2.12}$$

From (2.12) and condition (a_1), we infer that u_0 and v_0 are related by

$$u_0 = \left(\frac{K\left(\|v_0\|^2\right)}{K\left(\|u_0\|^2\right)}\right)^{1/(s-1)} v_0. \tag{2.13}$$

Now, note that the identities

$$\Psi'(u_0)(u_0) = \Psi'(v_0)(v_0) = 0 \tag{2.14}$$

lead to

$$K\left(\|u_0\|^2\right)\|u_0\|^2 = \lambda \int_\Omega u_0^s dx, \qquad K\left(\|v_0\|^2\right)\|v_0\|^2 = \lambda \int_\Omega v_0^s dx \tag{2.15}$$

which, in turn, imply that

$$\Psi(u_0) = \frac{1}{2}\int_0^{\|u_0\|^2} K(\tau)d\tau - \frac{1}{s}K\left(\|u_0\|^2\right)\|u_0\|^2,$$
$$\Psi(v_0) = \frac{1}{2}\int_0^{\|v_0\|^2} K(\tau)d\tau - \frac{1}{s}K\left(\|v_0\|^2\right)\|v_0\|^2. \tag{2.16}$$

Now, since u_0 and v_0 are both global minima for Ψ, one has $\Psi(u_0) = \Psi(v_0)$. It follows that

$$\frac{1}{2}\int_0^{\|u_0\|^2} K(\tau)d\tau - \frac{1}{s}K\left(\|u_0\|^2\right)\|u_0\|^2 = \frac{1}{2}\int_0^{\|v_0\|^2} K(\tau)d\tau - \frac{1}{s}K\left(\|v_0\|^2\right)\|v_0\|^2. \tag{2.17}$$

At this point, from condition (a_2) and (2.17), we infer that

$$K\left(\|u_0\|^2\right) = K\left(\|v_0\|^2\right) \tag{2.18}$$

which, in view of (2.13), clearly implies $u_0 = v_0$. This concludes the proof. □

Remark 2.2. Note that condition (a_2) is satisfied if, for instance, K is nondecreasing in $[0, +\infty[$ and so, in particular, if $K(t) = a + bt$ with $a, b > 0$.

From now on, whenever the function K satisfies the assumption of Theorem 2.1, we denote by u_s the unique global minimum of the functional Ψ defined in (2.1). Moreover, for every $u \in W_0^{1,2}(\Omega)$ and $r > 0$, we denote by $B_r(u)$ the closed ball in $W_0^{1,2}(\Omega)$ centered at u with radius r. The next result shows that the global minimum u_s is *strict* in the sense that the infimum of Ψ on every sphere centered in u_s is strictly greater than $\Psi(u_s)$.

Theorem 2.3. *Let K, λ, and s be as Theorem 2.1. Then, for every $r > 0$ one has*

$$\inf_{\|v\|=r} \Psi(u_s + v) > \Psi(u_s). \tag{2.19}$$

Proof. Put $\widetilde{K}(t) = (1/2)\int_0^t K(\tau)d\tau$ for every $t \geq 0$, and let $r > 0$. Assume, by contradiction, that

$$\inf_{\|v\|=r} \Psi(u_s + v) = \Psi(u_s). \tag{2.20}$$

Then,

$$\inf_{W_0^{1,2}(\Omega)} \Psi = \Psi(u_s) = \inf_{\|v\|=r}\left[\widetilde{K}\left(r^2 + \|u_s\|^2 + 2\langle u_s, v\rangle\right) - \frac{\lambda}{s}\int_\Omega (u_s + v)_+^s dx\right]. \tag{2.21}$$

Now, it is easy to check that the functional

$$J(u) = \widetilde{K}\left(r^2 + \|u_s\|^2 + 2\langle u_s, u\rangle\right) - \frac{\lambda}{s}\int_\Omega (u_s + u)_+^s dx, \qquad u \in W_0^{1,2}(\Omega) \tag{2.22}$$

is sequentially weakly continuous in $W_0^{1,2}(\Omega)$. Moreover, by the Eberlein-Smulian Theorem, every closed ball in $W_0^{1,2}(\Omega)$ is sequentially weakly compact. Consequently, J attains its global minimum in $B_r(0)$, and

$$\inf_{\|u\|\le r} J(u) = \inf_{\|u\|=r} J(u). \tag{2.23}$$

Let $v_0 \in B_r(0)$ be such that $J(v_0) = \inf_{\|u\|=r} J(u)$. From assumption (a_1), $\widetilde{K}$ turns out to be a strictly increasing function. Therefore, in view of (2.21), one has

$$\Psi(u_s) = J(v_0) \ge \widetilde{K}\Big(\|v_0\|^2 + \|u_s\|^2 + 2\langle u_s, u\rangle\Big) - \frac{\lambda}{s}\int_\Omega (u_s+u)_+^s dx = \Psi(u_s+v_0). \tag{2.24}$$

This inequality entails that u_s+v_0 is a global minimum for Ψ. Thus, thanks to Theorem 2.1, v_0 must be identically 0. Using again the fact that $\widetilde{K}$ is strictly increasing, from inequality (2.24), we would get

$$\Psi(u_s) = J(v_0) > \Psi(u_s+v_0) \tag{2.25}$$

which is impossible. □

Whenever the function K is as in Theorem 2.1, we put

$$\mu_r = \inf_{\|v\|=r} \Psi(u_s+v) - \Psi(u_s) \tag{2.26}$$

for every $r > 0$. Theorem 2.3 says that every μ_r is a positive number.

Before stating our existence result for problem (P_λ), we have to recall the following well-known Lemma which comes from [9, Theorems 8.16 and 8.30] and the regularity results of [10].

Lemma 2.4. *For every $h \in L^\infty(\Omega)$, denote by u_h the (unique) solution of the problem*

$$\begin{aligned} -\Delta u &= h(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.27}$$

Then, $u_h \in C^1(\overline{\Omega})$, and

$$\sup_{h\in L^\infty(\Omega)\setminus\{0\}} \frac{\max_{\overline{\Omega}}|u_h|}{\|h\|_{L^\infty(\Omega)}} \overset{\text{def}}{=} C_0 < \infty, \tag{2.28}$$

where the constant C_0 depends only on N, $|\Omega|$.

Theorem 2.5 below guarantees, for every $r > 0$, the existence of at least one positive solution u_r for problem (P_λ) whose distance from u_s is less than r provided that the perturbation term f is sufficiently small in $\Omega \times [0, C]$ with

$$C > \widetilde{C}_0 \overset{\text{def}}{=} \left(\frac{\lambda C_0}{M}\right)^{1/(2-s)}. \tag{2.29}$$

Here C_0 is the constant defined in Lemma 2.4 and $M = \inf_{t\geq 0} K(t) > 0$. Note that no growth condition is required on f.

Theorem 2.5. *Let K, λ, and s be as in Theorem 2.3. Moreover, fix any $C > \widetilde{C}_0$. Then, for every $r > 0$, there exists a positive constant a_r such that for every Carathéodory function $f : \Omega \times [0, +\infty[\to [0, +\infty[$ satisfying*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,C]} f(x,t) < a_r \overset{\text{def}}{=} \min\left\{\lambda \frac{C^{s-1}}{\widetilde{C}_0^{2-s}}\left(C^{2-s} - \widetilde{C}_0^{2-s}\right), \frac{\mu_r}{\gamma r}\right\}, \tag{2.30}$$

where μ_r is the constant defined in (2.26) and γ is the embedding constant of $W_0^{1,2}(\Omega)$ in $L^1(\Omega)$, problem (P_λ) admits at least a positive solution $u \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ such that $\|u_r - u_s\| < r$.

Proof. Fix $C > \widetilde{C}_0$. For every fixed $r > 0$ which, without loss of generality, we can suppose such that $r \leq \|u_s\|$, let a_r be the number defined in (2.30). Let $f : \Omega \times [0, +\infty[\to [0, +\infty[$ be a Carathéodory function satisfying condition (2.30), and put

$$f_C(x,t) = \begin{cases} f(x,0), & \text{if}(x,t) \in \Omega \times]-\infty, 0[, \\ f(x,t), & \text{if}(x,t) \in \Omega \times [0, C], \\ f(x,C), & \text{if}(x,t) \in \Omega \times]C, +\infty[, \end{cases} \tag{2.31}$$

as well as

$$a = \operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,C]} f(x,t). \tag{2.32}$$

Moreover, for every $u \in W_0^{1,2}(\Omega)$, put $\Phi(u) = \int_\Omega (\int_0^{u(x)} f_C(x,t)dt)dx$. By standard results, the functional Φ is of class C^1 in $W_0^{1,2}(\Omega)$ and sequentially weakly continuous. Now, observe that thanks to (2.30), one has

$$\begin{aligned}\sup_{\|v\|\le r}(\Phi(u_s+v)-\Phi(u_s)) &= \sup_{\|v\|\le r}\int_\Omega\left(\int_{u_s(x)}^{u_s(x)+v(x)} f_C(x,t)dt\right)dx\\ &\le \sup_{\|v\|\le r}\int_\Omega\left(\int_{u_s(x)}^{u_s(x)+|v(x)|} f_C(x,t)dt\right)dx\\ &\le a\sup_{\|v\|\le r}\int_\Omega |v(x)|dx < a_r\gamma r = \mu_r.\end{aligned} \tag{2.33}$$

Then, we can fix a number

$$\sigma \in \left]\Psi(u_s),\Psi(u_s)+\mu_r\right[\tag{2.34}$$

in such way that

$$\frac{\sup_{\|v\|\le r}(\Phi(u_s+v)-\Phi(u_s))}{\sigma-\Psi(u_s)} < 1. \tag{2.35}$$

Applying [11, Theorem 2.1] to the restriction of the functionals Ψ and $-\Phi$ to the ball $B_r(u_s)$, it follows that the functional $\Psi-\Phi$ admits a global minimum on the set $B_r(u_s)\cap\Psi^{-1}(]-\infty,\sigma[)$. Let us denote this latter by u_r. Note that the particular choice of σ forces u_r to be in the interior of $B_r(u_s)$. This means that u_r is actually a local minimum for $\Psi-\Phi$, and so $(\Psi-\Phi)'(u_r)=0$. In other words, u_r is a weak solution of problem (P_λ) with f_C in place of f. Moreover, since $r \le \|u_s\|$ and $\|u_s-u_r\| < r$, it follows that u_r is nonzero. Then, by the Strong Maximum Principle, u_r is positive in Ω, and, by [10], $u_r \in C^1(\overline{\Omega})$ as well. To finish the proof is now suffice to show that

$$\max_{\overline{\Omega}} u \le C. \tag{2.36}$$

Arguing by contradiction, assume that

$$\max_{\overline{\Omega}} u > C. \tag{2.37}$$

From Lemma 2.4 and condition (2.30) we have

$$\max_{\overline{\Omega}} u \le \frac{C_0}{K\left(\|u\|^2\right)}\left(\lambda\max_{\overline{\Omega}} u^{s-1}+a_r\right). \tag{2.38}$$

Therefore, using (2.30) (and recalling the notation $M = \inf_{t\geq 0} K(t) > 0$), one has

$$\max_{\overline{\Omega}} u^{2-s} \leq \frac{C_0}{M}\left(\lambda + \frac{a_r}{\max_{\overline{\Omega}}\ u^{s-1}}\right) \leq \frac{C_0}{M}\left(\lambda + \frac{a_r}{C^{s-1}}\right) \leq C^{2-s}, \tag{2.39}$$

that is absurd. The proof is now complete. □

Remarks 2.6. To satisfy assumption (2.30) of Theorem 2.5, it is clearly useful to know some lower estimation of a_r. First of all, we observe that by standard comparison results, it is easily seen that

$$C_0 = \max_{x\in\overline{\Omega}} u_0(x), \tag{2.40}$$

where u_0 is the unique positive solution of the problem

$$\begin{aligned} -\Delta u = 1, &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega. \end{aligned} \tag{2.41}$$

When Ω is a ball of radius $R > 0$ centered at $x_0 \in \mathbb{R}^N$, then $u_0(x) = (1/2N)(R^2 - |x - x_0|^2)$, and so $C_0 = R^2/2N$. More difficult is obtaining an estimate from below of μ_r: if $r > \|u_s\|$, one has

$$\inf_{\|v\|=r} \Psi(u_s + v) \geq \frac{1}{2}\inf_{t\geq 0} K(t)(r - \|u_s\|)^2 - \frac{\lambda}{s}\gamma_s^s r^s, \tag{2.42}$$

where γ_s is the embedding constant of $L^s(\Omega)$ in $W_0^{1,2}(\Omega)$. Therefore, μ_r grows as r^2 at $+\infty$. If $r \leq \|u_s\|$, it seems somewhat hard to find a lower bound for μ_r. However, with regard to this question, it could be interesting to study the behavior of μ_r on varying of the parameter λ for every fixed $r > 0$. For instance, how does μ_r behave as $\lambda \to +\infty$? Another question that could be interesting to investigate is finding the exact value of μ_r at least for some particular value of r (for instance $r = \|u_s\|$) even in the case of $K \equiv 1$.

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