SEMILINEAR PROBLEMS WITH BOUNDED NONLINEAR TERM

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We solve boundary value problems for elliptic semilinear equations in which no asymptotic behavior is prescribed for the nonlinear term.

1. Introduction

Many authors (beginning with Landesman and Lazer [1]) have studied resonance problems for semilinear elliptic partial differential equations of the form

$$-\Delta u - \lambda_\ell u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, λ_ℓ is an eigenvalue of the linear problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

and $f(x,t)$ is a bounded Carathéodory function on $\Omega \times \mathbb{R}$ such that

$$f(x,t) \longrightarrow f_\pm(x) \quad \text{a.e. as } t \longrightarrow \pm\infty. \tag{1.3}$$

Sufficient conditions were given on the functions $f_\pm$ to guarantee the existence of a solution of (1.1). (Some of the references are listed in the bibliography. They mention other authors as well.)

In the present paper, we consider the situation in which (1.3) does not hold. In fact, we do not require any knowledge of the asymptotic behavior of $f(x,t)$ as $|t| \to \infty$. As an example, we have the following.

Theorem 1.1. *Assume that*

$$\sup_{v \in E(\lambda_\ell)} \int_\Omega F(x,v)\,dx < \infty, \tag{1.4}$$

where $E(\lambda_\ell)$ is the eigenspace of λ_ℓ and

$$F(x,t) = \int_0^t f(x,s)\,ds. \tag{1.5}$$

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Assume also that if there is a sequence $\{u_k\}$ *such that*

$$\begin{gathered} \|P_\ell u_k\| \longrightarrow \infty, \qquad \|(I-P_\ell)u_k\| \le C, \\ 2\int_\Omega F(x,u_k)\,dx \longrightarrow b_0, \\ f(x,u_k) \longrightarrow f(x) \quad \text{weakly in } L^2(\Omega), \end{gathered} \tag{1.6}$$

where $f(x) \perp E(\lambda_\ell)$ *and* P_ℓ *is the projection onto* $E(\lambda_\ell)$, *then*

$$b_0 \le (f,u_1) - B_0, \tag{1.7}$$

where $B_0 = \int_\Omega W_0(x)dx$, $W_0(x) = \sup_t[(\lambda_{\ell-1}-\lambda_\ell)t^2 - 2F(x,t)]$, *and* u_1 *is the unique solution of*

$$-\Delta u - \lambda_\ell u = f, \quad u \perp E(\lambda_\ell). \tag{1.8}$$

Then (1.1) has at least one solution. In particular, the conclusion holds if there is no sequence satisfying (1.6).

A similar result holds if (1.4) is replaced by

$$\inf_{v\in E(\lambda_\ell)} \int_\Omega F(x,v)\,dx > -\infty. \tag{1.9}$$

In proving these results we will make use of the following theorem [2].

Theorem 1.2. *Let* N *be a closed subspace of a Hilbert space* H *and let* $M = N^\perp$. *Assume that at least one of the subspaces* M, N *is finite dimensional. Let* G *be a* C^1*-functional on* H *such that*

$$\begin{aligned} m_1 &:= \inf_{w\in M}\sup_{v\in N} G(v+w) < \infty, \\ m_0 &:= \sup_{v\in N}\inf_{w\in M} G(v+w) > -\infty. \end{aligned} \tag{1.10}$$

Then there are a constant $c \in \mathbb{R}$ *and a sequence* $\{u_k\} \subset H$ *such that*

$$m_0 \le c \le m_1, \quad G(u_k) \longrightarrow c, \quad G'(u_k) \longrightarrow 0. \tag{1.11}$$

2. The main theorem

We now state our basic result. Let Ω be a domain in $\mathbb{R}^n$, and let A be a selfadjoint operator on $L^2(\Omega)$ such that the following hold.

(A)

$$\sigma_e(A) \subset (0,\infty). \tag{2.1}$$

(B) There is a function $V(x) > 0$ in $L^2(\Omega)$ such that multiplication by V is a compact operator from $D := D(|A|^{1/2})$ to $L^1(\Omega)$.

(C) If $u \in N(A) \setminus \{0\}$, then $u \ne 0$ a.e. in Ω.

Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying
(D)

$$|f(x,t)| \le V(x). \tag{2.2}$$

Let $\underline{\lambda}(\overline{\lambda})$ be the largest (smallest) negative (positive) point in $\sigma(A)$, and define

$$W_0(x) := \sup_t [\underline{\lambda} t^2 - 2F(x,t)], \tag{2.3}$$

$$W_1(x) := \sup_t [2F(x,t) - \overline{\lambda} t^2], \tag{2.4}$$

where

$$F(x,t) := \int_0^t f(x,s)ds. \tag{2.5}$$

Note that (D) implies

$$-V(x)^2 \underline{\lambda} \le W_0(x), \qquad W_1(x) \le \frac{V(x)^2}{\overline{\lambda}}. \tag{2.6}$$

We also assume
(E)

$$\sup_{v \in N(A)} \int_\Omega F(x,v)dx < \infty. \tag{2.7}$$

(F) If there is a sequence $\{u_k\} \subset D$ such that

$$\begin{gathered} ||P_0 u_k|| \longrightarrow \infty, \qquad ||(I - P_0)u_k|| \le \text{const}, \\ 2\int_\Omega F(x,u_k)dx \longrightarrow b_0, \qquad f(x,u_k) \longrightarrow f(x) \quad \text{weakly in } L^2(\Omega), \end{gathered} \tag{2.8}$$

where $f(x) \in R(A)$ and P_0 is the projection of D onto $N(A)$, then $b_0 \le (f,u_1) - B_0$, where $B_0 = \int_\Omega W_0(x)dx$ and u_1 is the unique solution of

$$Au = f, \quad u \in R(A). \tag{2.9}$$

We have the following.

Theorem 2.1. *Under hypotheses (A)–(F), there is at least one solution of*

$$Au = f(x,u), \quad u \in D. \tag{2.10}$$

Proof. We begin by letting

$$N' = \oplus_{\lambda<0} N(A-\lambda), \quad N = N' \oplus N(A), \qquad M = N^\perp \cap D, \quad M = M' \oplus N(A). \tag{2.11}$$

By hypothesis (A), N', $N(A)$, and N are finite dimensional, and

$$D = M \oplus N' = M' \oplus N. \tag{2.12}$$

It is easily verified that the functional

$$G(u) := (Au,u) - 2\int_\Omega F(x,u)dx \tag{2.13}$$

is continuously differentiable on D. We take

$$\|u\|_D^2 := (|A|u,u) + \|P_0u\|^2 \tag{2.14}$$

as the norm squared on D. We have

$$(G'(u),v) = 2(Au,v) - 2(f(x,u),v), \quad u,v \in D. \tag{2.15}$$

Consequently (2.10) is equivalent to

$$G'(u) = 0, \quad u \in D. \tag{2.16}$$

Note that

$$(Av,v) \leq \underline{\lambda}\|v\|^2, \quad v \in N', \tag{2.17}$$

$$\bar{\lambda}\|w\|^2 \leq (Aw,w), \quad w \in M'. \tag{2.18}$$

By hypothesis (D), (2.5), and (2.13),

$$G(v) \leq \underline{\lambda}\|v\|^2 + 2\|V\| \cdot \|v\| \longrightarrow -\infty \quad \text{as } \|v\| \longrightarrow \infty,\ v \in N'. \tag{2.19}$$

For $w \in M$, we write $w = y + w'$, $y \in N(A)$, $w' \in M'$. Since $|F(x,w) - F(x,y)| \leq V(x)|w'|$ by (D) and (2.5), we have

$$G(w) \geq \bar{\lambda}\|w'\|^2 - 2\int F(x,y)dx - 2\|V\| \cdot \|w'\|. \tag{2.20}$$

In view of (E), (2.19) and (2.20) imply

$$\inf_M G > -\infty, \qquad \sup_{N'} G < \infty. \tag{2.21}$$

We can now apply Theorem 1.2 to conclude that there is a sequence satisfying (1.11). Let

$$u_k = v_k + w_k + \rho_k y_k, \quad v_k \in N',\ w_k \in M',\ y_k \in N(A),\ \|y_k\| = 1,\ \rho_k \geq 0. \tag{2.22}$$

We claim that

$$\|u_k\|_D \leq C. \tag{2.23}$$

To see this, note that (1.11) and (2.15) imply

$$(Au_k,h) - (f(x,u_k),h) = o(\|h\|). \tag{2.24}$$

Taking $h = v_k$, we see that $\|v_k\|^2 = O(\|v_k\|)$ in view of (2.17) and (D). Thus $\|v_k\|_D$ is bounded. Similarly, taking $h = w_k$, we see that $\|w_k\|_D \le C$. Suppose

$$\rho_k \longrightarrow \infty. \tag{2.25}$$

There is a renamed subsequence such that $y_k \to y$ in $N(A)$. Clearly $\|y\| = 1$. Thus by hypothesis (D), $y \neq 0$ a.e. This means that $\|\rho_k y_k\| \to \infty$. Hence (2.8) holds. Let $u_k' = v_k + w_k \in N(A)^\perp = R(A)$. Then $\|u_k'\|_D \le C$. Thus there is a renamed subsequence such that $u_k' \to u_1$ weakly in D. By hypothesis (B), there is a renamed subsequence such that $Vu_k' \to Vu_1$ strongly in $L^1(\Omega)$. Since $V(x) > 0$, there is another renamed subsequence such that $u_k' \to u_1$ a.e. in Ω. On the other hand, since $f_k(x) = f(x, u_k(x))$ is uniformly bounded in $L^2(\Omega)$ by hypothesis (D), there is an $f(x) \in L^2(\Omega)$ such that for a subsequence

$$f_k(x) \longrightarrow f(x) \quad \text{weakly in } L^2(\Omega). \tag{2.26}$$

Since

$$(Au_k', h) - (f_k(x), h) = o(\|h\|_D), \quad h \in D, \tag{2.27}$$

we see in the limit that u_1 is a solution of (2.9), and consequently that $f \in R(A)$. Moreover, we see by (2.27) that

$$(A[u_k' - u_1], h) - (f_k - f, h) = o(\|h\|_D), \quad h \in D. \tag{2.28}$$

Write $u_1 = v_1 + w_1$, and take h successively equal to $v_k - v_1$ and $w_k - w_1$. Then

$$\begin{gathered} ||v_k - v_1||_D^2 \le 2||V[v_k - v_1]||_1 + o(||v_k' - v_1||_D), \\ ||w_k - w_1||_D^2 \le 2||V[w_k - w_1]||_1 + o(||w_k - w_1||_D). \end{gathered} \tag{2.29}$$

Hence $u_k' \to u_1$ in D. Consequently,

$$(Au_k, u_k) = (Au_k', u_k') = (f_k, u_k') + o(||u_k'||) \longrightarrow (f, u_1), \tag{2.30}$$

$$2\int F(x, u_k)\,dx = (Au_k, u_k) - G(u_k) \longrightarrow (f, u_1) - c, \tag{2.31}$$

where $m_0 \le c \le m_1$. By (2.3)

$$G(v) \le (Av, v) - \underline{\lambda}\|v\|^2 + B_0, \quad v \in N'. \tag{2.32}$$

Thus $m_1 \le B_0$. Consider first the case $m_1 < B_0$. Then (2.31) implies $b_0 = (f, u_1) - c$, and consequently, $m_0 \le (f, u_1) - b_0 \le m_1 < B_0$. Thus $b_0 > (f, u_1) - B_0$, contradicting (1.7). This shows that the assumption (2.25) is not possible. Consequently (2.23) holds, and we have a renamed subsequence such that $u_k \to u$ strongly in D and a.e. in Ω. It now follows from (2.27) that

$$(Au, h) = (f(x, u), h), \quad h \in D, \tag{2.33}$$

showing that (2.10) indeed has a solution. Assume now that $m_1 = B_0$. Let v_k be a maximizing sequence in N' such that $G(v_k) \to m_1$. By (2.19), $\|v_k\|_D \le C$, and there is a renamed subsequence such that $v_k \to v_0$ in N'. By continuity $G(v_k) \to G(v_0)$. Hence $G(v_0) = m_1 = B_0$. Thus

$$\underline{\lambda}\|v_0\|^2 \le 2\int F(x,v_0)\,dx + B_0 = (Av_0,v_0) \le \underline{\lambda}\|v\|^2. \tag{2.34}$$

Consequently, $(Av_0,v_0) = \underline{\lambda}\|v_0\|^2$ and $Av_0 = \underline{\lambda}v_0$. We also have

$$\int_\Omega [2F(x,v_0) - \underline{\lambda}v_0^2 + W_0(x)]\,dx = 0. \tag{2.35}$$

In view of (2.3), the integrand is nonnegative. Hence

$$2F(x,v_0) \equiv \underline{\lambda}v_0^2 - W_0(x). \tag{2.36}$$

Let

$$\Phi(u) = \int_\Omega [2F(x,u) - \underline{\lambda}u^2]\,dx. \tag{2.37}$$

Then

$$\begin{gathered} \Phi(u) \ge \Phi(v_0), \quad u \in D, \\ (\Phi'(u),y) = 2(f(x,u),h) - 2\underline{\lambda}(u,h). \end{gathered} \tag{2.38}$$

Thus

$$\Phi'(v_0) = 2f(x,v_0) - 2\underline{\lambda}v_0 \equiv 0. \tag{2.39}$$

This implies

$$Av_0 = \underline{\lambda}v_0 = f(x,v_0), \tag{2.40}$$

and v_0 is a solution of (2.10). This completes the proof. □

THEOREM 2.2. *In Theorem 2.1, replace hypotheses (E), (F) by*

(E')

$$\inf_{v\in N(A)} \int_\Omega F(x,v)\,dx > -\infty, \tag{2.41}$$

(F') *if (2.8) hold with $f(x) \in R(A)$, then*

$$b_0 \ge (f,u_1) + B_1. \tag{2.42}$$

Then (2.10) has at least one solution.

Proof. We modify the proof of Theorem 2.1. This time we use the second decomposition in (2.12). For $v \in N$ we write $v = v' + v_0$, where $v' \in N'$ and $v_0 \in N(A)$. By (D) and (2.5),

$$\int_\Omega F(x, v_0)\,dx \leq \int_\Omega F(x, v)\,dx + \|V\| \cdot \|v'\|. \tag{2.43}$$

Hence

$$G(v) \leq \underline{\lambda}\|v'\|^2 + 2\|V\| \cdot \|v'\| - 2\int F(x, v_0)\,dx, \quad v \in N. \tag{2.44}$$

Consequently,

$$m_1 = \sup_N G < \infty. \tag{2.45}$$

On the other hand

$$G(w) \geq \bar{\lambda}\|w\|^2 - 2\|V\| \cdot \|w\|, \quad w \in M', \tag{2.46}$$

so that

$$m_0 = \inf_{M'} G > -\infty. \tag{2.47}$$

It now follows from Theorem 1.2 that there is a sequence $\{u_k\} \subset D$ satisfying (1.11). We now follow the proof of Theorem 2.1 from (2.22) to (2.31). By (2.4),

$$G(w) \geq (Aw, w) = \bar{\lambda}\|w\|^2 - B_1, \quad w \in M', \tag{2.48}$$

where $B_1 = \int_\Omega W_1(x)\,dx$. Thus $m_0 \geq -B_1$. Assume first that $m_0 > -B_1$. Then (1.11) and (2.31) imply

$$-B_1 < m_0 \leq (f, u_1) - b_0, \tag{2.49}$$

contradicting (2.42). Thus (2.25) cannot hold, and we obtain a solution of (2.10) as in the proof of Theorem 2.1. If $m_0 = -B_1$, let $\{w_k\} \subset M'$ be a minimizing sequence such that $w_k \to w_0$ weakly in D, $Vw_k \to Vw_0$ in $L^1(\Omega)$ and a.e. in Ω. By hypothesis (D),

$$\int_\Omega [F(x, w_k) - F(x, w_0)]\,dx = \int_\Omega \int_0^1 f(x, w_0 + \theta(w_k - w_0))(w_k - w_0)\,d\theta\,dx \longrightarrow 0. \tag{2.50}$$

Thus G is weakly lower semicontinuous, and

$$G(w_0) \leq \lim G(w_k) = m_0 - B_1. \tag{2.51}$$

Hence

$$\bar{\lambda} w_0 = f(x, w_0) \leq 2\int F(x, w_0) - B_1 \leq \bar{\lambda}\left|\left|w_0\right|\right|^2, \tag{2.52}$$

and we proceed as before to show that

$$Aw_0 = \overline{\lambda} w_0 = f(x, w_0). \tag{2.53}$$

The proof is complete. □

References

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