# ON DELAY DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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*Received 21 July 2004*

The monotone iterative method is used to obtain sufficient conditions which guarantee that a delay differential equation with a nonlinear boundary condition has quasisolutions, extremal solutions, or a unique solution. Such results are obtained using techniques of weakly coupled lower and upper solutions or lower and upper solutions. Corresponding results are also obtained for such problems with more delayed arguments. Some new interesting results are also formulated for delay differential inequalities.

## 1. Introduction

In this paper we discuss the boundary value problem

$$\begin{aligned} x'(t) = f\big(t,x(t),x(\alpha(t))\big) \equiv Fx(t), \quad t \in J = [0,T],\ T > 0, \\ 0 = g\big(x(0),x(T)\big), \end{aligned} \tag{1.1}$$

where

($H_1$) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J,J)$, $\alpha(t) \le t$, $t \in J$, and $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

To obtain some existence results for differential problems, someone can apply the monotone iterative technique, for details see, for example, [8]. In recent years, much attention has been paid to the study of ordinary differential equations with different conditions but only a few papers concern such problems with nonlinear boundary conditions, see, for example, [1, 2, 3, 4]. The monotone technique can also be successfully applied to ordinary delay differential problems which are special cases of (1.1), see, for example, [5, 7, 9, 10, 11]. It is known that the monotone method works when a function (appearing on the right-hand side of a differential problem) satisfies a one-sided Lipschitz condition with a corresponding constant (or constants). It is important to indicate that also the authors of the above-mentioned papers obtained their results under such an assumption. In this paper we consider a more general case when constants are replaced by functions. This remark is important when we have differential problems with deviated arguments since in such cases we can obtain less restrictive conditions from corresponding differential inequalities. In this paper we discuss delay problems with nonlinear boundary conditions


Boundary Value Problems 2005:2 (2005) 201–214
DOI: 10.1155/BVP.2005.201

of type (1.1) to obtain quite general existence results. It is the first paper where the monotone technique is applied for delay differential equations when a boundary condition has a nonlinear form. The case when $t \le \alpha(t) \le T$, $t \in J$ is considered in [6].

In Section 2, delay differential inequalities are studied. This part is important when the monotone technique is used with problem (1.1). In the next section we study weakly coupled lower and upper solutions of problem (1.1) formulating corresponding results when problem (1.1) has coupled quasisolutions, extremal solutions, or a unique solution. In Section 4, we formulate corresponding existence results for problem (1.1) using the notion of lower and upper solutions of (1.1). In Section 5, some generalizations of the previous results are formulated when we have more delayed arguments. Examples show how to apply the obtained results.

## 2. Delay differential inequalities

In this chapter we will discuss delay differential inequalities. Such problems are important when we use the monotone iterative technique to obtain existence results for (1.1).

Lemma 2.1. *Let $\alpha \in C(J,J)$, $\alpha(t) \le t$ on $J$. Suppose that $p \in C^1(J,\mathbb{R})$ and*

$$p'(t) \le -N(t)p(\alpha(t)), \quad t \in J, \qquad p(0) \le 0, \tag{2.1}$$

*where a nonnegative function $N$ is integrable on $J$.*

*In addition assume that*

$$\int_0^T N(t)dt \le 1. \tag{2.2}$$

*Then $p(t) \le 0$ on $J$.*

*Proof.* We need to prove that $p(t) \le 0$, $t \in J$. Suppose that the above inequality is not true. Then, we can find $t_0 \in (0,T]$ such that $p(t_0) > 0$. Put

$$p(t_1) = \min_{[0,t_0]} p(t) \le 0. \tag{2.3}$$

Integrating the differential inequality in (2.1) from $t_1$ to $t_0$, we obtain

$$p(t_0) - p(t_1) \le -\int_{t_1}^{t_0} N(t)p(\alpha(t))dt \le -p(t_1)\int_0^T N(t)dt \le -p(t_1). \tag{2.4}$$

It contradicts assumption that $p(t_0) > 0$. This shows that $p(t) \le 0$ on $J$ and the proof is complete. □

Lemma 2.2. *Let $\alpha \in C(J,J)$, $\alpha(t) \le t$ on $J$. Suppose that $K \in C(J,\mathbb{R})$, $q \in C^1(J,\mathbb{R})$, and*

$$q'(t) \le -K(t)q(t) - L(t)q(\alpha(t)), \quad t \in J, \qquad q(0) \le 0, \tag{2.5}$$

*where a nonnegative function $L$ is integrable on $J$.*

*In addition assume that*

($H_2$) $\int_0^T L(t)e^{\int_{\alpha(t)}^t K(s)ds}dt \leq 1$.

*Then $q(t) \leq 0$ on $J$.*

*Proof.* Indeed, the assertion holds if $L(t) = 0$, $t \in J$. Let $\int_0^T L(t)dt > 0$. Put

$$p(t) = e^{\int_0^t K(s)ds}q(t), \quad t \in J. \tag{2.6}$$

This yields $p(0) = q(0) \leq 0$, and

$$p'(t) = e^{\int_0^t K(s)ds}[K(t)q(t) + q'(t)], \tag{2.7}$$

so

$$p'(t) \leq -L(t)e^{\int_{\alpha(t)}^t K(s)ds}p(\alpha(t)), \quad t \in J, \qquad p(0) \leq 0. \tag{2.8}$$

In view of Lemma 2.1, $p(t) \leq 0$ on $J$, by assumption ($H_2$). This also proves that $q(t) \leq 0$ on $J$ and the proof is complete. □

*Remark 2.3.* Note that assumption ($H_2$) holds if $K(t) \geq 0$ on $J$ and

$$\int_0^T L(t)e^{\int_0^t K(s)ds}dt \leq 1. \tag{2.9}$$

We see that condition (2.9) does not depend on $\alpha$. Moreover, if we assume that $K(t) = K > 0$, $L(t) = L > 0$, $t \in J$, and

$$L(e^{KT} - 1) \leq K, \tag{2.10}$$

then condition (2.9) is satisfied.

Note that if $K(t) = K > 0$, $L(t) = L > 0$, $t \in J$, then assumption ($H_2$) takes the form

$$L\int_0^T e^{K[t-\alpha(t)]}dt \leq 1; \tag{2.11}$$

such condition is considered in [11].

## 3. Weakly coupled lower and upper solutions

Here we apply the method of weakly coupled lower and upper solutions to problems of type (1.1). We begin introducing the following definition.

We say that $u, v \in C^1(J,\mathbb{R})$ are called weakly coupled lower and upper solutions of problem (1.1) if

$$\begin{aligned} u'(t) &\leq Fu(t), \quad t \in J, \qquad g(u(0), v(T)) \leq 0, \\ v'(t) &\geq Fv(t), \quad t \in J, \qquad g(v(0), u(T)) \geq 0. \end{aligned} \tag{3.1}$$

We say that $X, Y \in C^1(J,\mathbb{R})$ are called coupled quasisolutions of (1.1) if

$$\begin{aligned} X'(t) &= FX(t), \quad t \in J, \qquad 0 = g(X(0), Y(T)), \\ Y'(t) &= FY(t), \quad t \in J, \qquad 0 = g(Y(0), X(T)). \end{aligned} \tag{3.2}$$

We first formulate conditions when problem (1.1) has coupled quasisolutions.

Theorem 3.1. *Let assumption* ($H_1$) *hold. Let* $y_0, z_0 \in C^1(J,\mathbb{R})$ *be weakly coupled lower and upper solutions of (1.1) and let* $y_0(t) \le z_0(t)$, $t \in J$. *In addition assume that*

($H_3$) *there exist a function* $K \in C(J,\mathbb{R})$ *and a nonnegative function* $L$, *integrable on* $J$, *such that assumption* ($H_2$) *is satisfied and*

$$f(t,u_1,u_2) - f(t,v_1,v_2) \le K(t)[v_1 - u_1] + L(t)[v_2 - u_2] \tag{3.3}$$

*if* $y_0(t) \le u_1 \le v_1 \le z_0(t)$, $y_0(\alpha(t)) \le u_2 \le v_2 \le z_0(\alpha(t))$,

($H_4$) $g$ *is nondecreasing in the second variable and there exists a constant* $M > 0$ *such that*

$$g(u,v) - g(\bar{u},v) \ge -M(\bar{u} - u) \quad \text{if } y_0(0) \le u \le \bar{u} \le z_0(0). \tag{3.4}$$

*Then problem (1.1) has, in the sector* $[y_0,z_0]_*$, *coupled quasisolutions where*

$$[y_0,z_0]_* = \{w \in C^1(J,\mathbb{R}) : y_0(t) \le w(t) \le z_0(t),\ t \in J\}. \tag{3.5}$$

*Proof.* Let

$$\begin{aligned} y'_{n+1}(t) &= \mathcal{F}(t,y_n,y_{n+1}), \quad t \in J, \qquad 0 = g(y_n(0),z_n(T)) + M[y_{n+1}(0) - y_n(0)], \\ z'_{n+1}(t) &= \mathcal{F}(t,z_n,z_{n+1}), \quad t \in J, \qquad 0 = g(z_n(0),y_n(T)) + M[z_{n+1}(0) - z_n(0)] \end{aligned} \tag{3.6}$$

for $n = 0,1,\dots$ with

$$\mathcal{F}(t,a,b) = Fa(t) - K(t)[b(t) - a(t)] - L(t)[b(\alpha(t)) - a(\alpha(t))]. \tag{3.7}$$

Observe that functions $y_1$, $z_1$ are well-defined as initial linear problems (use the Banach fixed point theorem with a corresponding norm). We first show that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \quad t \in J. \tag{3.8}$$

Put $p = y_0 - y_1$, $q = z_1 - z_0$. This and assumptions ($H_3$), ($H_4$) show that

$$\begin{aligned} 0 &= g(y_0(0),z_0(T)) + M[y_1(0) - y_0(0)] \le -Mp(0), \\ 0 &= g(z_0(0),y_0(T)) + M[z_1(0) - z_0(0)] \ge Mq(0), \\ p'(t) &\le Fy_0(t) - \mathcal{F}(t,y_0,y_1) = -K(t)p(t) - L(t)p(\alpha(t)), \\ q'(t) &\le \mathcal{F}(t,z_0,z_1) - Fz_0(t) = -K(t)q(t) - L(t)q(\alpha(t)). \end{aligned} \tag{3.9}$$

By Lemma 2.2, $y_0(t) \le y_1(t)$, $z_1(t) \le z_0(t)$, $t \in J$. Now, we put $p = y_1 - z_1$. In view of assumption ($H_4$), we have

$$0 = g(y_0(0),z_0(T)) + M[y_1(0) - y_0(0)] - g(z_0(0),y_0(T)) - M[z_1(0) - z_0(0)] \ge Mp(0). \tag{3.10}$$

Moreover,

$$\begin{aligned}
p'(t) &= \mathcal{F}(t, y_0, y_1) - \mathcal{F}(t, z_0, z_1) \\
&\leq K(t)[z_0(t) - y_0(t)] + L(t)[z_0(\alpha(t)) - y_0(\alpha(t))] \\
&\quad - K(t)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\
&\quad - L(t)[y_1(\alpha(t)) - y_0(\alpha(t)) - z_1(\alpha(t)) + z_0(\alpha(t))] \\
&= -K(t)p(t) - L(t)p(\alpha(t)),
\end{aligned} \tag{3.11}$$

by assumption ($H_3$). Lemma 2.2 yields $y_1(t) \leq z_1(t)$ on $J$. It proves (3.8).

In the next step we show that $y_1$, $z_1$ are weakly coupled lower and upper solutions of problem (1.1). Note that

$$\begin{aligned}
y_1'(t) &= \mathcal{F}(t, y_0, y_1) - Fy_1(t) + Fy_1(t) \\
&\leq K(t)[y_1(t) - y_0(t)] + L(t)[y_1(\alpha(t)) - y_0(\alpha(t))] - K(t)[y_1(t) - y_0(t)] \\
&\quad - L(t)[y_1(\alpha(t)) - y_0(\alpha(t))] + Fy_1(t) = Fy_1(t), \\
z_1'(t) &= \mathcal{F}(t, z_0, z_1) - Fz_1(t) + Fz_1(t) \geq Fz_1(t), \\
0 &= g(y_0(0), z_0(T)) + M[y_1(0) - y_0(0)] - g(y_1(0), z_1(T)) + g(y_1(0), z_1(T)) \\
&\geq -M[y_1(0) - y_0(0)] + g(y_1(0), z_1(T)) + M[y_1(0) - y_0(0)] \\
&= g(y_1(0), z_1(T)), \\
0 &= g(z_0(0), y_0(T)) + M[z_1(0) - z_0(0)] - g(z_1(0), y_1(T)) + g(z_1(0), y_1(T)) \\
&\leq g(z_1(0), y_1(T)),
\end{aligned} \tag{3.12}$$

by (3.3) and assumption ($H_4$). This proves that $y_1, z_1$ are weakly coupled lower and upper solutions of problem (1.1).

Using the mathematical induction, we can show that

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq y_{n+1}(t) \leq z_{n+1}(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), \tag{3.13}$$

for $t \in J$ and $n = 0, 1, \ldots$.

Now we will prove that the sequences $\{y_n, z_n\}$ converge to their limit functions $y, z$, respectively. First, we need to show that the sequences are bounded and equicontinuous on $J$. Indeed,

$$A_1 \leq y_0(t) \leq y_n(t) \leq z_n(t) \leq z_0(t) \leq A_2, \quad t \in J,\ n = 0, 1, \ldots, \tag{3.14}$$

so the sequences $\{y_n, z_n\}$ are uniformly bounded. Note that $y_n'$ and $z_n'$ are bounded on $J$ by $W > 0$ because $|\mathcal{F}(t, a, b)|$ is bounded on $J \times [A_1, A_2] \times [A_1, A_2]$. Hence $y_n$, $z_n$ are equicontinuous because for $\epsilon > 0$, $t_1, t_2 \in J$ such that $|t_1 - t_2| < \epsilon/W$, we have

$$|y_n(t_1) - y_n(t_2)| = |y_n'(\xi)|\,|t_1 - t_2| < \epsilon, \qquad |z_n(t_1) - z_n(t_2)| < \epsilon. \tag{3.15}$$

The Arzela-Ascoli theorem guarantees the existence of subsequences $\{y_{n_k}, z_{n_k}\}$ of $\{y_n, z_n\}$, respectively, and continuous functions $y$, $z$ with $y_{n_k}$, $z_{n_k}$ converging uniformly on $J$ to $y$ and $z$, respectively. Note that $y_{n_k}$, $z_{n_k}$ satisfy the integral equations

$$\begin{aligned} y_{n_k+1}(t) &= y_{n_k+1}(0) + \int_0^t \mathcal{F}(s, y_{n_k}, y_{n_k+1})\,ds, \quad t \in J, \\ z_{n_k+1}(t) &= z_{n_k+1}(0) + \int_0^t \mathcal{F}(s, z_{n_k}, z_{n_k+1})\,ds, \quad t \in J, \\ y_{n_k+1}(0) &= y_{n_k}(0) - \frac{1}{M} g(y_{n_k}(0), z_{n_k}(T)), \\ z_{n_k+1}(0) &= z_{n_k}(0) - \frac{1}{M} g(z_{n_k}(0), y_{n_k}(T)). \end{aligned} \tag{3.16}$$

If $n_k \to \infty$, then from the above relations, we have

$$\begin{aligned} y(t) &= y(0) + \int_0^t Fy(s)\,ds, \quad t \in J, \qquad g(y(0), z(T)) = 0, \\ z(t) &= z(0) + \int_0^t Fz(s)\,ds, \quad t \in J, \qquad g(z(0), y(T)) = 0, \end{aligned} \tag{3.17}$$

because $f$ and $g$ are continuous. Thus $y, z \in C^1(J)$ and

$$y'(t) = Fy(t), \quad z'(t) = Fz(t), \quad t \in J. \tag{3.18}$$

It proves that $y$, $z$ are coupled quasisolutions of problem (1.1). It ends the proof. □

*Remark 3.2.* If $f$ is nondecreasing with respect to the last two variables, then assumption $(H_3)$ holds with $K(t) = L(t) = 0$ on $J$.

*Remark 3.3.* Note that if $g$ is nonincreasing with respect to the first variable, then condition (3.4) holds.

Our next two theorems concern the case when the boundary problem of type (1.1) has a unique solution.

THEOREM 3.4. *Assume that all assumptions of Theorem 3.1 are satisfied. In addition assume that*

$(H_5)$ *there exist constants* $M_1$, $M_2$ *such that* $M \geq M_1 > 0$, $M_2 \geq 0$, *and*

$$g(u, \bar{v}) - g(\bar{u}, v) \leq -M_1(\bar{u} - u) + M_2(\bar{v} - v) \tag{3.19}$$

*if* $y_0(0) \leq u \leq \bar{u} \leq z_0(0)$, $y_0(T) \leq v \leq \bar{v} \leq z_0(T)$,

$(H_6)$ $f$ *is nonincreasing in the last argument, there exists an integrable on* $J$ *a function* $Q$ *such that* $K(t) + Q(t) \geq 0$, $t \in J$,

$$f(t, u, v) - f(t, \bar{u}, v) \geq -Q(t)[\bar{u} - u] \quad \text{if } y_0(t) \leq u \leq \bar{u} \leq z_0(t), \tag{3.20}$$

$$M_2 e^{\int_0^T Q(s)ds} < M_1. \tag{3.21}$$

*Then problem (1.1) has, in the sector* $[y_0, z_0]_*$, *a unique solution.*

*Proof.* Theorem 3.1 guarantees that functions $y, z$ are coupled quasisolutions of problem (1.1) and $y_0(t) \le y(t) \le z(t) \le z_0(t)$, $t \in J$. We first show that $y(t) = z(t)$, $t \in J$. Put $p = y - z$. Then

$$\begin{gathered} 0 = g(y(0), z(T)) - g(z(0), y(T)) \le M_1 p(0) - M_2 p(T), \\ p'(t) = Fy(t) - Fz(t) \ge Q(t)p(t). \end{gathered} \tag{3.22}$$

It yields

$$p(t) \ge p(0)e^{\int_0^t Q(s)ds}, \quad t \in J, \tag{3.23}$$

so

$$[M_1 - M_2 e^{\int_0^T Q(s)ds}]p(0) \ge 0. \tag{3.24}$$

In view of (3.21), $y(t) \ge z(t)$, $t \in J$. It proves that $y = z$, so problem (1.1) has a solution.

It remains to show that $y = z$ is a unique solution of (1.1) in the sector $[y_0, z_0]_*$. Let $w \in [z_0, y_0]_*$ be any solution of (1.1). We assume that $y_m(t) \le w(t) \le z_m(t)$, $t \in J$ for some $m$. Let $p = y_{m+1} - w$, $q = w - z_{m+1}$. Then,

$$\begin{gathered} 0 = g(y_m(0), z_m(T)) + M[y_{m+1}(0) - y_m(0)] - g(w(0), w(T)) \\ \ge -M[w(0) - y_m(0)] + M[y_{m+1}(0) - y_m(0)] = Mp(0), \\ 0 = g(z_m(0), y_m(T)) + M[z_{m+1}(0) - z_m(0)] - g(w(0), w(T)) \le -Mq(0), \\ p'(t) = \mathcal{F}(t, y_m, y_{m+1}) - Fw(t) \le -K(t)p(t) - L(t)p(\alpha(t)), \\ q'(t) = Fw(t) - \mathcal{F}(t, z_m, z_{m+1}) \le -K(t)q(t) - L(t)q(\alpha(t)), \end{gathered} \tag{3.25}$$

by assumption $(H_4)$ and condition (3.3). This, in view of Lemma 2.2, gives $y_{m+1}(t) \le w(t) \le z_{m+1}(t)$, $t \in J$. By induction, $y_n(t) \le w(t) \le z_n(t)$, $t \in J$, $n = 0, 1, \ldots$. If $n \to \infty$, then $y = z = w$ which proves the assertion of our theorem. □

*Remark 3.5.* Observe that if $f$ satisfies the Lipschitz condition with respect to the first variable, so

$$|f(t,u,v) - f(t,\bar{u},v)| \le K(t)|u - \bar{u}|, \quad K \in C(J, \mathbb{R}_+), \tag{3.26}$$

then $Q(t) = K(t)$, $t \in J$.

*Remark 3.6.* Take $g(x,y) = x - h(y)$, where $h \in C([y_0(T), z_0(T)], \mathbb{R})$ and

$$0 \le h(u) - h(v) \le M_3(v - u) \quad \text{if } y_0(T) \le u \le v \le z_0(T). \tag{3.27}$$

Then assumptions $(H_4)$, $(H_5)$ are satisfied with $M = M_1 = 1$, $M_2 = M_3$.

*Example 3.7.* Consider the problem

$$
\begin{aligned}
x'(t) &= 2e^{-x(t)} - (\sin t)e^{-et}x\left(\frac{1}{2}t\right) - 1 \equiv Fx(t), \quad t \in J = [0,1], \\
0 &= \beta x(0) + x^2(0) + x(1) - 1 \equiv g(x(0),x(1)),
\end{aligned}
\tag{3.28}
$$

where $\beta > 1$. In this example $\alpha(t) = (1/2)t$.

Put $y_0(t) = -t$, $z_0(t) = 1$, $t \in J$. Then

$$
\begin{gathered}
Fy_0(t) = 2e^t + \frac{1}{2}t(\sin t)e^{-et} - 1 > -1 = y_0'(t), \\
Fz_0(t) = 2e^{-1} - (\sin t)e^{-et} - 1 < 0 = z_0'(t), \\
g(y_0(0),z_0(1)) = g(0,1) = 0, \\
g(z_0(0),y_0(1)) = g(1,-1) = \beta - 1 > 0.
\end{gathered}
\tag{3.29}
$$

It shows that $y_0, z_0$ are weakly coupled lower and upper solutions of problem (3.28). Note that $K(t) = 2e$, $L(t) = (\sin t)e^{-et}$, and

$$
\int_0^1 L(t)e^{\int_{\alpha(t)}^t K(s)ds}\,dt = 1 - \cos 1 < 1,
\tag{3.30}
$$

so assumption ($H_3$) holds. Similarly, assumption ($H_5$) is satisfied with $M_1 = \beta$, $M_2 = 1$. It is easy to see that for $-t \le u \le \bar{u} \le 1$, we have

$$
f(t,u,v) - f(t,\bar{u},v) = 2\left[e^{-u} - e^{-\bar{u}}\right] \ge 0 = -0(\bar{u} - u),
\tag{3.31}
$$

so $Q(t) = 0$, $t \in J$. Moreover

$$
M_2 e^{\int_0^1 Q(s)ds} = 1 < \beta.
\tag{3.32}
$$

All assumptions of Theorem 3.4 hold, so problem (3.28) has, in the segment $[y_0,z_0]_*$, a unique solution.

Theorem 3.8. *Let assumptions ($H_1$), ($H_6$) hold. Assume that problem (1.1) has, in the sector $[y_0,z_0]_*$, at least one solution. In addition assume that*

($H_7$) *there exist constants $M_1$, $M_2$ such that $M_1 > 0$, $M_2 \ge 0$, and*

$$
g(\bar{u},\bar{v}) - g(u,v) \le -M_1(\bar{u} - u) + M_2(\bar{v} - v)
\tag{3.33}
$$

*if $y_0(0) \le u \le \bar{u} \le z_0(0)$, $y_0(T) \le v \le \bar{v} \le z_0(T)$.*

*Then problem (1.1) has, in the sector $[y_0,z_0]_*$, a unique solution.*

*Proof.* Let $y, z \in [y_0,z_0]_*$ be arbitrary solutions of problem (1.1). Put $p(t) = y(t) - z(t)$, $t \in J$. We distinguish two cases.

*Case 1.* Let $y(t) \ne z(t)$ for all $t \in J$. Without the loss of generality, we can assume that $p(t) > 0$, $t \in J$. It yields

$$
0 = g(y(0),y(T)) - g(z(0),z(T)) \le -M_1 p(0) + M_2 p(T),
\tag{3.34}
$$

by assumption ($H_7$). Moreover

$$p'(t) = Fy(t) - Fz(t) \le Q(t)p(t), \tag{3.35}$$

by assumption ($H_6$). Hence

$$\begin{gathered} p(t) \le e^{\int_0^t Q(s)ds} p(0), \quad t \in J, \\ M_1 p(0) \le M_2 p(T) \le M_2 e^{\int_0^T Q(s)ds} p(0), \end{gathered} \tag{3.36}$$

so

$$p(0)\left[M_1 - M_2 e^{\int_0^T Q(s)ds}\right] \le 0. \tag{3.37}$$

Because $p(0) > 0$, it yields

$$M_1 - M_2 e^{\int_0^T Q(s)ds} \le 0. \tag{3.38}$$

In view of (3.21), it is a contradiction.
*Case 2.* Assume that there exists $t_0 \in J$ such that $y(t_0) = z(t_0)$.
*Subcase 2.1.* Let $t_0 = T$. Then $p(T) = 0$. It yields

$$0 = g(y(0), y(T)) = g(z(0), y(T)). \tag{3.39}$$

We show that in this case also $y(0) = z(0)$, so $p(0) = 0$. Assume that it is not true. This means that $p(0) > 0$ or $p(0) < 0$. If $p(0) > 0$, then

$$0 = g(y(0), y(T)) - g(z(0), z(T)) \le -M_1 p(0), \tag{3.40}$$

this is a contradiction. If $p(0) < 0$, then

$$0 = g(y(0), y(T)) - g(z(0), z(T)) \ge M_1[z(0) - y(0)] = -M_1 p(0), \tag{3.41}$$

and this is a contradiction too. It shows that $p(0) = 0$.

Without the loss of generality, we can assume that $p(t) > 0$, $t \in (0, t_1]$ for some $t_1 \le T$. In view of assumption ($H_6$), we have

$$p'(t) = Fy(t) - Fz(t) \le Q(t)p(t). \tag{3.42}$$

This yields

$$p(t) \le p(0)e^{\int_0^t Q(s)ds} = 0, \quad t \in [0, t_1]. \tag{3.43}$$

It is a contradiction.
*Subcase 2.2.* Let $t_0 = 0$. Then $p(0) = 0$ and we have the case considered above.

*Subcase 2.3.* Let $t_0 \in (0,T)$. Then $p(t_0) = 0$. Without the loss of generality, we can assume that $p(t) > 0$, $t \in (t_0, t_1]$ for some $t_1 \le T$. In view of assumption $(H_6)$, we have

$$p'(t) = Fy(t) - Fz(t) \le Q(t)p(t). \tag{3.44}$$

This gives

$$p(t) \le p(t_0)e^{\int_{t_0}^{t} Q(s)ds} = 0, \quad t \in [t_0, t_1]. \tag{3.45}$$

It is a contradiction. Hence we have $p(t) = 0$, $t \in [t_0, T]$, so $p(T) = 0$. This case was considered in Subcase 2.1. This ends the proof. □

*Remark 3.9.* Take $g(x,y) = -x + h(y)$, where $h \in C([y_0(T), z_0(T)], \mathbb{R})$ and there exists a positive constant $M_3$ such that

$$h(v) - h(u) \le M_3(v-u) \quad \text{if } y_0(T) \le u \le v \le z_0(T). \tag{3.46}$$

Then assumption $(H_7)$ holds with $M_1 = 1$, $M_2 = M_3$. Such case was discussed in [4] for the case when $f$ does not depend on the last argument.

## 4. Lower and upper solutions

We say that $u \in C^1(J, \mathbb{R})$ is called a lower solution of (1.1) if

$$u'(t) \le Fu(t), \quad t \in J, \qquad g(u(0), u(T)) \le 0, \tag{4.1}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

*Remark 4.1.* Note that if $u, v \in C^1(J, \mathbb{R})$ are lower and upper solutions of (1.1), respectively, then $g(u(0), u(T)) \le 0 \le g(v(0), v(T))$. In case we have an initial problem, so if $g(x,y) = x - c$, $c \in \mathbb{R}$, then the above condition reduces to $u(0) \le c \le v(0)$.

THEOREM 4.2. *Let assumption $(H_1)$ hold. Let $y_0, z_0 \in C^1(J, \mathbb{R})$ be lower and upper solutions of (1.1), respectively, and let $y_0(t) \le z_0(t)$, $t \in J$. In addition assume that assumption $(H_3)$ holds and*

$(H_4')$ *$g$ is nonincreasing in the second variable and there exists a constant $M > 0$ such that condition (3.4) is satisfied.*

*Then problem (1.1) has, in the sector $[y_0, z_0]_*$, minimum and maximum solutions.*

*Proof.* Let

$$\begin{aligned} y_{n+1}'(t) &= \mathcal{F}(t, y_n, y_{n+1}), \quad t \in J, \qquad 0 = g(y_n(0), y_n(T)) + M[y_{n+1}(0) - y_n(0)], \\ z_{n+1}'(t) &= \mathcal{F}(t, z_n, z_{n+1}), \quad t \in J, \qquad 0 = g(z_n(0), z_n(T)) + M[z_{n+1}(0) - z_n(0)] \end{aligned} \tag{4.2}$$

for $n = 0, 1, \ldots$ with $\mathcal{F}$ defined as in the proof of Theorem 3.1. Repeating the proof of Theorem 3.1, we can show that $y_n$, $z_n$ converge, respectively, to solutions $y$, $z$ of problem (1.1) and $y_0(t) \le y(t) \le z(t) \le z_0(t)$, $t \in J$.

Now we need to show that $y$, $z$ are extremal solutions of (1.1) in the sector $[y_0, z_0]_*$. Let $w \in [y_0, z_0]_*$ be any solution of (1.1). We assume that $y_m(t) \le w(t) \le z_m(t)$, $t \in J$ for some $m$. Let $p = y_{m+1} - w$, $q = w - z_{m+1}$. Then,

$$\begin{aligned} 0 &= g(y_m(0), y_m(T)) + M[y_{m+1}(0) - y_m(0)] - g(w(0), w(T)) \ge Mp(0), \\ 0 &= g(z_m(0), z_m(T)) + M[z_{m+1}(0) - z_m(0)] - g(w(0), w(T)) \le -Mq(0), \\ p'(t) &= \mathcal{F}(t, y_m, y_{m+1}) - Fw(t) \le -K(t)p(t) - L(t)p(\alpha(t)), \\ q'(t) &= Fw(t) - \mathcal{F}(t, z_m, z_{m+1}) \le -K(t)q(t) - L(t)q(\alpha(t)), \end{aligned} \tag{4.3}$$

by assumption ($H_4$) and condition (3.3). This and Lemma 2.2 give $y_{m+1}(t) \le w(t) \le z_{m+1}(t)$, $t \in J$. By induction, $y_n(t) \le w(t) \le z_n(t)$, $t \in J$, $n = 0, 1, \dots$. If $n \to \infty$, then we have the assertion. This ends the proof. □

Theorem 4.3. *Let all assumptions of Theorem 4.2 be satisfied. In addition assume that assumptions ($H_7'$) and ($H_6$) hold, where*

($H_7'$) *there exist constants $M_1$, $M_2$ such that $M \ge M_1 > 0$, $M_2 \ge 0$, and*

$$g(u, v) - g(\bar{u}, \bar{v}) \le -M_1(\bar{u} - u) + M_2(\bar{v} - v) \tag{4.4}$$

*if $y_0(0) \le u \le \bar{u} \le z_0(0)$, $y_0(T) \le v \le \bar{v} \le z_0(T)$.*

*Then problem (1.1) has, in the sector $[y_0, z_0]_*$, a unique solution.*

The proof is similar to the proof of Theorem 3.4 and therefore it is omitted.

*Remark 4.4.* Assume that $g(x, y) = -x - h(y)$, where $h \in C([y_0(T), z_0(T)], \mathbb{R})$ and

$$h(u) - h(v) \le M_3(v - u) \quad \text{if } y_0(T) \le u \le v \le z(T). \tag{4.5}$$

Then assumption ($H_7$) is satisfied with $M_1 = 1$, $M_2 = M_3$.

*Example 4.5.* Consider the problem

$$\begin{gathered} x'(t) = \beta(\sin t)x(t) - 2\beta(\sin t)x(\alpha(t)) - \beta \sin t \equiv Fx(t), \quad t \in J = [0, \pi], \\ 0 = x(0) - e^{-1}x(\pi), \end{gathered} \tag{4.6}$$

where $0 \le \beta \le 1/4$, $\alpha \in C(J, J)$, $\alpha(t) \le t$ on $J$.

Put $y_0(t) = -1$, $z_0(t) = 0$, $t \in J$. Then

$$\begin{gathered} Fy_0(t) = 0 = y_0'(t), \qquad Fz_0(t) = -\beta \sin t \le 0 = z_0'(t), \\ g(y_0(0), y_0(\pi)) = g(-1, -1) < 0, \qquad g(z_0(0), z_0(\pi)) = g(0, 0) = 0. \end{gathered} \tag{4.7}$$

It proves that $y_0$, $z_0$ are lower and upper solutions of problem (4.6), respectively. Note that $K(t) = 0$, $L(t) = 2\beta \sin t$, and

$$\int_0^T L(t) e^{\int_{\alpha(t)}^t K(s)ds} dt = 4\beta \le 1, \tag{4.8}$$

so assumption ($H_3$) holds. Assumption ($H_7'$) holds with $M_1 = 1$, $M_2 = e^{-1}$. Moreover $Q(t) = \beta \sin t$, and

$$M_2 e^{\int_0^T Q(s)ds} = e^{2\beta - 1} < 1. \tag{4.9}$$

By Theorem 4.3, problem (4.6) has, in the sector $[y_0, z_0]_*$, a unique solution.

## 5. Generalizations

In this section we consider a boundary value problem of the form

$$\begin{gathered} x'(t) = f(t, x(t), x(\alpha_1(t)), \dots, x(\alpha_r(t))) \equiv Gx(t), \quad t \in J = [0, T], \\ 0 = g(x(0), x(T)). \end{gathered} \tag{5.1}$$

We formulate only corresponding results using the notions of lower and upper (or weakly lower and upper) solutions (or coupled quasisolutions) of problem (5.1) which are the same as before with the operator $G$ instead of operator $F$.

We introduce three assumptions:

($H_8$) $f \in C(J \times \mathbb{R}^{r+1}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha_i \in C(J, J)$, $\alpha_i(t) \le t$ on $J$ for $i = 1, 2, \dots, r$,

($H_9$) there exist a function $K \in C(J, \mathbb{R})$ and nonnegative integrable on $J$ functions $L_i$, $i = 1, 2, \dots, r$ such that the following condition holds:

$$\sum_{i=1}^{r} \int_0^T L_i(t) e^{\int_{\alpha_i(t)}^t K(s)ds} dt \le 1, \tag{5.2}$$

and moreover

$$f(t, u_0, u_1, \dots, u_r) - f(t, v_0, v_1, \dots, v_r) \le K(t)[v_0 - u_0] + \sum_{i=1}^{r} L_i(t)[v_i - u_i] \tag{5.3}$$

if $t \in J$, $z_0(\alpha_i(t)) \le u_i \le v_i \le y_0(\alpha_i(t))$, $i = 0, 1, \dots, r$ with $\alpha_0(t) = t$,

($H_{10}$) $f$ is nonincreasing in the last $r$ variables, there exists an integrable on $J$ a function $Q$ such that $K(t) + Q(t) \ge 0$, $t \in J$, condition (3.21) holds (for some constants $M_1$, $M_2$), and

$$f(t, u, v_1, \dots, v_r) - f(t, \bar{u}, v_1, \dots, v_r) \ge -Q(t)[\bar{u} - u] \tag{5.4}$$

if $y_0(t) \le u \le \bar{u} \le z_0(t)$.

The next two lemmas are natural generalizations of Lemmas 2.1 and 2.2.

Lemma 5.1. *Let $\alpha_i \in C(J, J)$, $\alpha_i(t) \le t$ on $J$ for $i = 1, 2, \dots, r$. Suppose that $p \in C^1(J, \mathbb{R})$ and*

$$p'(t) \le -\sum_{i=1}^{r} N_i(t) p(\alpha_i(t)), \quad t \in J, \qquad p(0) \le 0, \tag{5.5}$$

*where nonnegative functions $N_i$ are integrable on $J$.*

*In addition assume that*

$$\sum_{i=1}^{r}\int_0^T N_i(t)dt \le 1. \tag{5.6}$$

*Then $p(t) \le 0$ on $J$.*

Lemma 5.2. *Let $\alpha_i \in C(J,J)$, $\alpha_i(t) \le t$ on $J$, $i = 1,2,\dots,r$. Suppose that $K \in C(J,\mathbb{R})$, $q \in C^1(J,\mathbb{R})$, and*

$$q'(t) \le -K(t)q(t) - \sum_{i=1}^{r} L_i(t)q(\alpha_i(t)), \quad t \in J, \qquad q(0) \le 0, \tag{5.7}$$

*where nonnegative functions $L_i$ are integrable on $J$. In addition assume that condition (5.2) holds.*

*Then $q(t) \le 0$ on $J$.*

Now we formulate similar results to Theorems 3.1, 3.4, and 3.8, respectively.

Theorem 5.3. *Let assumption ($H_8$) be satisfied. Let $y_0, z_0 \in C^1(J,\mathbb{R})$ be weakly coupled lower and upper solutions of (5.1) and let $y_0(t) \le z_0(t)$ on $J$. In addition assume that assumptions ($H_4$), ($H_9$) are satisfied.*

*Then problem (5.1) has, in the sector $[y_0,z_0]_*$, coupled quasisolutions.*

In the proof use the sequences $\{y_n, z_n\}$ defined by

$$\begin{aligned}
y'_{n+1}(t) &= Gy_n(t) - K(t)[y_{n+1}(t) - y_n(t)] - \sum_{i=1}^{r} L_i(t)[y_{n+1}(\alpha_i(t)) - y_n(\alpha_i(t))],\\
0 &= g(y_n(0), z_n(T)) + M[y_{n+1}(0) - y_n(0)],\\
z'_{n+1}(t) &= Gz_n(t) - K(t)[z_{n+1}(t) - z_n(t)] - \sum_{i=1}^{r} L_i(t)[z_{n+1}(\alpha_i(t)) - z_n(\alpha_i(t))],\\
0 &= g(z_n(0), y_n(T)) + M[z_{n+1}(0) - z_n(0)]
\end{aligned} \tag{5.8}$$

for $t \in J$, $n = 0,1,\dots$.

Theorem 5.4. *Assume that all assumptions of Theorem 5.3 are satisfied. In addition suppose that assumptions ($H_5$), ($H_{10}$) hold.*

*Then problem (5.1) has, in the sector $[y_0,z_0]_*$, a unique solution.*

Theorem 5.5. *Let assumptions ($H_1$), ($H_7$), ($H_{10}$) hold. In addition assume that problem (5.1) has, in the sector $[y_0,z_0]_*$, at least one solution.*

*Then problem (5.1) has, in the sector $[y_0,z_0]_*$, a unique solution.*

The next two theorems correspond to Theorems 4.2 and 4.3, respectively.

Theorem 5.6. *Let assumption ($H_8$) hold. Let $y_0, z_0 \in C^1(J,\mathbb{R})$ be lower and upper solutions of problem (5.1), respectively, and let $y_0(t) \le z_0(t)$, $t \in J$. In addition suppose that assumptions ($H'_4$), ($H_9$) hold.*

*Then problem (5.1) has, in the sector $[y_0,z_0]_*$, extremal solutions.*

In the proof use the sequences $\{y_n, z_n\}$ defined by

$$
\begin{aligned}
y'_{n+1}(t) &= Gy_n(t) - K(t)[y_{n+1}(t) - y_n(t)] - \sum_{i=1}^{r} L_i(t)[y_{n+1}(\alpha_i(t)) - y_n(\alpha_i(t))], \\
0 &= g(y_n(0), y_n(T)) + M[y_{n+1}(0) - y_n(0)], \\
z'_{n+1}(t) &= Gz_n(t) - K(t)[z_{n+1}(t) - z_n(t)] - \sum_{i=1}^{r} L_i(t)[z_{n+1}(\alpha_i(t)) - z_n(\alpha_i(t))], \\
0 &= g(z_n(0), z_n(T)) + M[z_{n+1}(0) - z_n(0)]
\end{aligned}
\tag{5.9}
$$

for $t \in J$, $n = 0, 1, \ldots$.

Theorem 5.7. *Let all assumptions of Theorem 5.6 be satisfied. In addition suppose that assumptions ($H'_7$), ($H_{10}$) hold.*

*Then problem (5.1) has, in the sector $[y_0, z_0]_*$, a unique solution.*

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