ON WEAK SOLUTIONS OF THE EQUATIONS OF MOTION OF A VISCOELASTIC MEDIUM WITH VARIABLE BOUNDARY

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Received 2 September 2005

The regularized system of equations for one model of a viscoelastic medium with memory along trajectories of the field of velocities is under consideration. The case of a changing domain is studied. We investigate the weak solvability of an initial boundary value problem for this system.

1. Introduction

The purpose of the present paper is an extension of the result of [21] on the case of a changing domain. Let $\Omega_t \in R^n$, $2 \le n \le 4$ be a family of the bounded domains with boundary Γ_t, $Q = \{(t,x) : t \in [0,T],\ x \in \Omega_t\}$, $\Gamma = \{(t,x) : t \in [0,T],\ x \in \Gamma_t\}$. The following initial boundary value problem is under consideration:

$$\begin{gathered}
\rho(v_t + v_i \partial v/\partial x_i) - \mu_1 \operatorname{Div} \int_0^t \exp\Big(-\frac{t-s}{\lambda}\Big)\mathscr{E}(v)(s,z(s;t,x))\,ds \ - \mu_0 \operatorname{Div}\mathscr{E}(v) \\
= -\operatorname{grad} p + \rho\varphi, \qquad \operatorname{div} v = 0, \quad (t,x) \in Q; \qquad \int_{\Omega_t} p\,dx = 0, \quad t \in [0,T]; \\
v(0,x) = v^0(x), \quad x \in \Omega_0, \qquad v(t,x) = v^1(t,x), \quad (t,x) \in \Gamma.
\end{gathered} \tag{1.1}$$

Here $v(t,x) = (v_1,\dots,v_n)$ is a velocity of the medium at location x at time t, $p(t,x)$ is a pressure, ρ, μ_0, μ_1, λ are positive constants, Div means a divergence of a matrix, the matrix $\mathscr{E}(v)$ has coefficients $\mathscr{E}_{ij}(v)(t,x) = (1/2)(\partial v_i(t,x)/\partial x_j + \partial v_j(t,x)/\partial x_i)$. In (1.1) and in the sequel repeating indexes in products assume their summation. The function $z(\tau;t,x)$ is defined as a solution to the Cauchy problem (in the integral form)

$$z(\tau;t,x) = x + \int_t^\tau v(s,z(s;t,x))\,ds, \quad \tau \in [0,T],\ (t,x) \in Q. \tag{1.2}$$

The substantiation of model (1.1) is given in [21]. One can find the details in [12, Chapter 4]. We assume that a domain $Q \subset R^{n+1}$ is defined as an evolution Ω_t, $t \ge 0$ of the volume Ω_0 along the field of velocities of some sufficiently smooth solenoidal vector field $\tilde{v}(t,x)$

Boundary Value Problems 2005:3 (2005) 215–245
DOI: 10.1155/BVP.2005.215

which is defined in some cylindrical domain $\hat{Q}_0 = \{(t,x) : t \in [0,T],\ x \in \hat{\Omega}_0\}$, so that $\Omega_t \subset \hat{\Omega}_0$. This means that $\Omega_t = \tilde{z}(t;0,\Omega_0)$, where $\tilde{z}(\tau;t,x)$ is a solution to the Cauchy problem

$$\tilde{z}(\tau;t,x) = x + \int_t^\tau \tilde{v}(s,\tilde{z}(s;t,x))\,ds, \quad \tau \in [0,T],\ (t,x) \in \hat{Q}. \tag{1.3}$$

Thus, it is clear that the lateral surface Γ of a domain Q and the trace of the function $\tilde{v}(t,x)$ on Γ will be smooth enough, if $\tilde{v}(t,x)$ is smooth enough. We will assume sufficient smoothness of $\tilde{v}(t,x)$, providing validity of embedding theorems for domains Ω_t used below with the common for all t constant.

Let us mention some works which concern the study of the Navier-Stokes equations ((1.1) for $\mu_1 = 0$) in a time-dependent domain (see [2, 5, 8, 13] etc.), by this, different methods are used and various results on existence and uniqueness of both strong and weak solutions are obtained. In the present work, the existence of weak solutions to a regularized initial boundary value problem (1.1) in a domain with a time-dependent boundary Γ_t is established. The approximation-topological methods suggested and advanced in [3, 4] are used in the paper. It assumes replacement of the problem under consideration by an operator equation, approximation of the equation in a weak sense and application of the topological theory of a degree that allows to establish the existence of solutions on the basis of a priori estimates and statements about passage to the limit. Note that in the case of a not cylindrical domain (with respect to t) the necessary spaces of differentiable functions cannot be regarded as spaces of functions of t with values in some fixed functional space. Consequently, the direct application of the method of [21] is not possible. The history of the motion equation from (1.1) is given in details in [21]. On the basis of the rheological relation of Jeffreys-Oldroyd type the existence theorem for weak solutions in a domain with a constant boundary was proved. The purpose of the present paper is to prove a similar result for a domain with changing boundary.

The article is organized as follows. We need a number of auxiliary results about functional spaces for the formulation of the basic results. They are presented in Section 2. We also need some results about the linear problem in a non-cylindrical domain which are given in Section 3. By this the proofs of the part of the results (which require the rather long proofs) are given in Section 8. In Section 4, the main results are formulated, in Sections 5–7 the proofs of the main results are carried out. We will denote constants in inequalities and chains of inequalities by the same M if their values are not important.

2. Auxiliary results

2.1. Functional spaces. Let us introduce necessary functional spaces. Denote norms in $L_2(\Omega_t)$ and $W_2^k(\Omega_t)$ by $|\cdot|_{0,t}$ and $|\cdot|_{k,t}$ accordingly. Denote by $\|\cdot\|_0$ a norm in $L_2(Q)$ or in $L_2(Q_0)$ ($Q_0 = [0,T] \times \Omega_0$), depending on a context. We will denote by $D_{0,t}$ the set of functions, which are smooth, solenoidal and finite on domain Ω_t. We will designate through H_t and V_t a completion of $D_{0,t}$ in the norms $L_2(\Omega_t)$ and $W_2^1(\Omega_t)$ accordingly. We denote by V_t^* the conjugate space to V_t and by $|\cdot|_{-1,t}$ the norm in V_t^*. We denote by $\langle v,h\rangle_t$ an action of the functional $v \in V_t^*$ upon an element $h \in V_t$. Thus, the scalar product $(\cdot,\cdot)_t$ in H_t generates (see, e.g., [7, Chapter 1, page 29]) the dense continuous embeddings $V_t \subset H_t \subset V_t^*$ at every $t \in [0,T]$. It is clear that

$$|\langle u,v\rangle_t| \le |u|_{1,t}|v|_{-1,t}, \quad u \in V_t,\ v \in V_t^*. \tag{2.1}$$

Let D be the set of smooth vector functions on Q, solenoidal and finite on a domain Ω_t for every t. It is easy to show that scalar functions $\varphi(t) = |v(t,x)|_{1,t}$, $\psi(t) = |v(t,x)|_{-1,t}$, $g(t) = |v_t(t,x)|_{-1,t}$, where $v_t(t,x)$ is a derivative with respect to t of function $v(t,x)$, are determined and continuous on $[0,T]$ for every $v \in D$.

We denote by E, E^*, E_1^*, W, W_1, CH, EC, $L_{2,\sigma}(Q)$ the completion of D accordingly in norms

$$\|v\|_E = \left(\int_0^T |v(t,x)|_{1,t}^2 dt\right)^{1/2}, \qquad \|v\|_{E^*} = \left(\int_0^T |v(t,x)|_{-1,t}^2 dt\right)^{1/2},$$

$$\|v\|_{E_1^*} = \int_0^T |v(t,x)|_{-1,t} dt,$$

$$\|v\|_W = \|v\|_E + \|v_t\|_{E^*}, \qquad \|v\|_{W_1} = \|v\|_E + \|v_t\|_{E_1^*}, \qquad \|v\|_{CH} = \max_{t\in[0,T]} |v(t,x)|_{0,t},$$

$$\|v\|_{EC} = \|v\|_E + \|v\|_{CH}, \qquad \|v\|_0 = \left(\int_0^T |v(t,x)|_{0,t}^2 dt\right)^{1/2}. \tag{2.2}$$

Let a sequence $v^n \in D$, $n = 1,2,\dots$ be fundamental on Q in the norm $\|\cdot\|_0 : \int_0^T |v^n(t,x) - v^m(t,x)|_{0,t}^2 dt \to 0$, $n,m \to +\infty$. Then (see [19, page 224]) there exists a subsequence $v^{n_k}(t,x)$ which is fundamental at a.e. t in the norm $|\cdot|_{0,t}$. Let $v(t,x) \in L_2(\Omega_t)$ be the limit of $v^{n_k}(t,x)$. Solenoidality of functions from D implies $v(t,x) \in H_t$ at a.e. t. It implies the possibility to get the completion of D in the norm $\|\cdot\|_0$ as a subspace of usual functions from $L_2(Q)$.

It is similarly shown that an element $v \in E$ is a function $v(t,x)$ at a.e. t, $v(t,x) \in V_t$, $\|v\|_E^2 = \int_0^T |v(t,x)|_{1,t}^2 dt$, and $v \in E^*$ is a function $v(t,x) \in V_t^*$ at a.e. t, $\|v\|_{E^*}^2 = \int_0^T |v(t,x)|_{-1,t}^2 dt$. For $v \in E_1^*$ we have at a.e. t $v(t,x) \in V_t^*$ and $\|v\|_{E_1^*} = \int_0^T |v(t,x)|_{-1,t} dt$.

Lemma 2.1. *Let $v \in E$, $h \in E^*$. The scalar function $\langle v(t,x),h(t,x)\rangle_t = \psi(t)$ is summable and*

$$\int_0^T \langle v(t,x),h(t,x)\rangle_t dt = \langle v,h\rangle. \tag{2.3}$$

Proof. Choose such a sequences $v^n,h^n \subset D$ that $\|v^n - v\|_E \to 0$, $\|h^n - h\|_{E^*} \to 0$ and $|v^n(t,x) - v(t,x)|_{1,t} \to 0$, $|h^n(t,x) - h(t,x)|_{-1,t} \to 0$ at a.e. t. Then the sequence of continuous functions $\psi^n(t) = (v^n(t,x),h^n(t,x))_t$ converges to $\psi(t)$ at a.e. t and, hence, $\psi(t)$ is measurable. As

$$\int_0^T |\psi^n(t)|\, dt \leqslant \|v^n\|_E \|h^n\|_{E^*} \leqslant M, \tag{2.4}$$

where M does not depend on n, then the Fatou's theorem implies summability of $\psi(t)$. From convergence $v^n \to v$ in E, $h^n \to h$ in E^* and the equality $\psi(t) = \langle v,h\rangle$ (2.3) easily follows. The lemma is proved. □

The scalar product $(v,h) = \int_0^T (v(t,x),h(t,x))_t dt$ in $L_{2,\sigma}(Q)$ generates the continuous embeddings $E \subset L_{2,\sigma}(Q) \subset (E)^*$. Here $(E)^*$ is adjoint to E. Denote by $\langle v,h\rangle$ an action of the functional $v \in (E)^*$ upon h. It turns out that $(E)^* = E^*$. Really, E^* is a subspace in $(E)^*$. If E^* does not coincide with $(E)^*$, we can find an element $v_0 \neq 0$ in E for which $\langle h,v_0\rangle = 0$ for all $h \in E^*$. Choosing elements h from the set D dense in E^*, we get that $\langle h,v_0\rangle = (h,v_0) = 0$ for all $h \in D$. This implies $v_0 = 0$. Therefore, $E^* = (E)^*$ and the scalar product (v,h) in $L_{2,\sigma}(Q)$ generate the continuous embeddings $E \subset L_{2,\sigma}(Q) \subset E^*$. Note that $|\langle v,h\rangle| \leqslant \|v\|_E\|h\|_{E^*}$.

In the Banach spaces introduced above it is convenient for us to define equivalent norms by the rule $\|v\|_{k,F} = \|\bar{v}\|_F$, $\bar{v} = \exp(-kt)v$, $k > 0$. Here F is any Banach space of functions defined on Q.

The space E^* is continuously embedded in E_1^*. Below $\langle v,u\rangle$ denotes an action of a functional $v \in E^*$ upon a function $u \in E$. Besides, we need the set CG of functions $z(\tau;t,x)$ defined on $[0,T] \times Q$ which are continuous with respect to all variables and continuously differentiable with respect to x. Moreover, these functions are diffeomorphisms of Ω_t on Ω_τ with the determinants equal to 1. We will consider CG as a metric space with the metrics $\rho(z_1,z_2) = \|z_1 - z_2\|_{CG}$ where $\|z\|_{CG} = \max_\tau \max_t \|z(\tau;t,x)\|_{C(\bar{\Omega}_t)}$.

We denote by $W_2^{l,m}(Q)$ the usual Sobolev spaces of functions $f(t,x)$ on Q, having generalized derivatives up to order l with respect to t and up to order m with respect to x which are square summable. $\|\cdot\|_{l,m}$ stands for their norms.

2.2. Regularization operator. Problem (1.1) involves the integral which is calculated along the trajectory $z(\tau;t,x)$ of a particle x in the field of velocities $v(t,x)$ whereby $z(\tau;t,x)$ is a solution to the Cauchy problem (1.2). However, even strong solutions $v(t,x)$ of problem (1.2), having a derivative with respect to t and the second derivatives with respect to x, square summable on Q, do not provide uniquely solvability of problem (1.1). As an exit from this situation in [21] (following [9]) the regularization of the field of velocity with the help of introduction of a linear bounded operator $S_{\delta,t} : H_t \to C^1(\bar{\Omega}_t) \cap V_t$ for $\delta > 0$ such that $S_{\delta,t}(v) \to v$ in H_t at $\delta \to 0$ and fixed t was offered. As far as the boundary Γ_t of a domain Ω_t is concerned, it was assumed to be sufficiently smooth. In the construction of this operator a smooth decomposition of the unit for Ω_t, some homothety transformations in R^n and the operator P_t of orthogonal projection in $L_2(\Omega_t)$ on H_t were used. Let $v(t,x) \in L_{2,\sigma}(Q)$. We define on $L_{2,\sigma}(Q)$ the operator $\widehat{S}_\delta(v) = \widehat{v}$ where $\widehat{v}(t,x) = S_{\delta,t}(v(t,x))$ at $t \geq 0$.

As the operator P_t is constructed by means of solutions of the Dirichlet and Neumann problems for Ω_t (see [18, page 20]) and Ω_t in our case smoothly depends on t, from the structure of the operator $S_{\delta,t}$ at fixed t it follows that the operator $\widehat{S}_\delta$ is a linear bounded operator from $L_{2,\sigma}(Q)$ in $E \cap L_2(0,T;C^1)$, and $\widehat{S}_\delta(v) \to v$ in E at $\delta \to 0$. Here $L_2(0,T;C^1)$ is a Banach space which is a completion of the set of smooth functions on Q in the norm $\|v\|_{L_2(0,T;C^1)} = (\int_0^T \|v(t,x)\|^2_{C^1(\bar{\Omega}_t)} dt)^{1/2}$. Let now $u(t,x) \in L_{2,\sigma}(Q)$, $v(t,x) = u(t,x) + \tilde{v}(t,x)$. Let us define the regularization operator $S_\delta : L_{2,\sigma}(Q) \to L_2(0,T;C^1) \cap E$ by the formula $S_\delta(v) = \widehat{S}_\delta(v - \tilde{v}) + \tilde{v} = \widehat{S}_\delta(u) + \tilde{v}$. It is clear that $S_\delta(v) \to v$ in $L_{2,\sigma}(Q)$ at $\delta \to 0$.

Consider problem (1.2) for $v(t,x) \subset L_2(0,T;C^1)$. The solvability of problem (1.2) for the case of a cylindrical domain Q was established in [10] for $v \in L_2(0,T;C^1)$ vanishing on Γ. In the same place, the estimate was obtained:

$$||z_1(\tau;t,x) - z_2(\tau;t,x)||_{C(\bar{\Omega}_t)} \leq M\left| \int_t^\tau ||v^1(s,x) - v^2(s,x)||_{C^1(\bar{\Omega}_s)} ds \right|, \tag{2.5}$$

where $v^1, v^2 \in L_2(0,T;C^1)$, $t,\tau \in [0,T]$. The same facts are fair and for the case of a non cylindrical Q and for functions coinciding with $\tilde{v}$ on Γ. The proofs are similar to ones resulted in [10] with minor alterations. Inequality (2.5) is required to us in what follows below. We replace (1.2) for (1.1) by the equation

$$z(\tau;t,x) = x + \int_t^\tau S_\delta v(s,z(s;t,x))\,ds, \quad \tau,t \in [0,T],\ x \in \Omega_t. \tag{2.6}$$

For every $v(t,x) \in E$ the function $S_\delta(v) \in L_2(0,T;C^1)$, and, hence, problem (2.6) is uniquely solvable. we designate by $\tilde{Z}_\delta(v)$ the solution to problem (2.6). Note that $S_\delta(v)$ coincides with $\tilde{v}$ on Γ. In particular it means that all trajectories $z(\tau;t,x)$ of problems (2.6) lay in Q.

3. Linear parabolic operator on noncylindrical domain

Consider a linear operator $L : E^* \to E^*$ defined on the set $D(L)$ of smooth solenoidal functions $v(t,x)$ vanishing on Γ and at $t = 0$ by the formula

$$\langle Lv, h\rangle = \int_0^T (v_t - \Delta v, h)_t\,dt. \tag{3.1}$$

Here $h(t,x) \in E$. Obviously, $D(L)$ is dense in E^*.

Let us show that the operator L admits a closure and study its properties. First we establish auxiliary results. The following result is known (see [11, page 8]).

Lemma 3.1. *Let $F(t,x)$ be a smooth scalar function. Then*

$$\frac{d}{dt}\int_{\Omega_t} F(t,x)\,dx = \int_{\Omega_t} F_t(t,x)\,dx + \int_{\Gamma_t} F(t,x)\tilde{v}_n(t,x)\,dx. \tag{3.2}$$

Here $\tilde{v}_n(t,x)$ is the projection of $\tilde{v}(t,x)$ on the direction of the external normal $n(x)$ at a point $x \in \Gamma_t$.

Corollary 3.2. *If $F(t,x) = 0$ for $(t,x) \in \Gamma$ then $(d/dt)\int_{\Omega_t} F(t,x)dx = \int_{\Omega_t} F_t(t,x)dx$.*

Lemma 3.3. *A function $v \in D(L)$ satisfies the inequality:*

$$||v_t||_{E^*} + \sup_{0\leq t\leq T} |v(t,x)|_{0,t} + \|v\|_E \leq M\|Lv\|_{E^*}. \tag{3.3}$$

Proof. Integration by parts and use of Corollary 3.2 yields

$$\begin{aligned}\int_0^t (Lv,v)_s ds &= \int_0^t \Big(\big(v_t(s,x),v(s,x)\big)_s - \big(\triangle v(s,x),v(s,x)\big)_s\Big) ds \\ &= \int_0^t \Big(\frac{1}{2}\frac{d}{ds} \,|\, v(s,x)\,|_{0,s}^2 + \big(\nabla v(s,x),\nabla v(s,x)\big)_s\Big) ds \\ &= \int_0^t \Big(\frac{1}{2}\frac{d}{ds} \,|\, v(s,x)\,|_{0,s}^2 + |\, v(s,x)\,|_{1,s}^2\Big) ds \\ &= \frac{1}{2}\,|\, v(t,x)\,|_{0,t}^2 + \int_0^t |\, v(s,x)\,|_{1,s}^2 ds.\end{aligned} \tag{3.4}$$

From this it follows that $\langle Lv,v\rangle = 1/2|v(T,x)|_{0,T}^2 + \|v\|_E^2$. As $|\langle Lv,v\rangle| \le \|Lv\|_{E^*}\|v\|_E$ we get from (3.4) the inequality $\sup_t |v(t,x)|_{0,t} + \|v\|_E \le M\|Lv\|_{E^*}$. On the other hand,

$$\begin{aligned}||v_t||_{E^*} &= \sup_{h\in E,\|h\|_E=1} \Big| \int_0^T (v_t,h)_t dt\Big| \\ &\le \sup_{h\in E,\|h\|_E=1} \Big| \int_0^T (v_t-\triangle v,h)_t dt\Big| + \sup_{h\in E,\|h\|_E=1} \Big| \int_0^T (\triangle v,h)_t dt\Big| \\ &\le \|Lv\|_{E^*} + \|v\|_E.\end{aligned} \tag{3.5}$$

Thus, $\|v_t\|_{E^*} \le M\|Lv\|_{E^*}$. The last inequalities imply (3.3). The lemma is proved. □

Lemma 3.4. *Let $v\in W$. Then $|v(t,x)|_{0,t}$ is absolutely continuous in t on $[0,T]$, differentiable at a.e. $t\in[0,T]$ and*

$$\frac{1}{2}\frac{d}{dt}\,|\, v(t,x)\,|_{0,t}^2 = \langle v_t(t,x),v(t,x)\rangle_t, \tag{3.6}$$

$$|\, v(t,x)\,|_{0,t} \le \varepsilon||v_t||_{E^*} + M(\varepsilon)\|v\|_E, \quad \varepsilon>0,\ t\in[0,T]. \tag{3.7}$$

Proof. Let $v(t,x)$ be smooth. Then (3.6) follows from Corollary 3.2. Let us prove (3.7). It follows from (3.6) and (2.1) that for $0\le\tau\le T$

$$\begin{aligned}|\, v(t,x)\,|_{0,t}^2 &\le |\, v(\tau,x)\,|_{0,\tau}^2 + 2\int_\tau^t |\, v_s(s,x)\,|_{-1,s}\,|\, v(s,x)\,|_{0,s} ds \\ &\le |\, v(\tau,x)\,|_{0,\tau}^2 + 2||v_t||_{E^*}\|v\|_E\end{aligned} \tag{3.8}$$

is valid. Supposing $t\ge T/4$ and integrating over τ on $[0,T/4]$, we have

$$|\, v(t,x)\,|_{0,t}^2 \le 4/T\int_0^{T/4} |\, v(s,x)\,|_{0,s}^2 ds + 2||v_t||_{E^*}\|v\|_E \le M\Big(\|v\|_{L_2(Q)}^2 + ||v_t||_{E^*}\|v\|_E\Big). \tag{3.9}$$

The same inequality for $t\le 3T/4$ is established by means of integration over τ on $[3T/4,T]$. Using inequality $\|v\|_{L_2(Q)}\le\|v\|_E$ and standard arguments we obtain from this inequality (3.7).

The lemma is proved. □

In the cylindrical case this fact is proved in [18, Lemma 1.2, page 209].

Lemma 3.5. *The space W is embedded in EC and $\|v\|_{EC} \le M\|v\|_W$.*

The proof of the lemma follows from Lemma 3.4.

Let $W_0 = \{v : v \in W,\ v(0,x) = 0\}$. It is not hard to show that W_0 can be easily obtained by means of the closure in the W-norm of the set of smooth on Q functions which are solenoidal on Ω_t at every t and vanish on $\partial\Omega_t$ and Ω_0. In fact, let $v \in W_0$. By the definition of W there exists a sequence of functions v_n smooth on Q and solenoidal at every t such that $\|v - v_n\|_W \to 0$ by $n \to \infty$. Let $\varphi_n(t)$ be a smooth nondecreasing on $[0,T]$ function such that $\varphi_n(t) \equiv 0$ when $t \in [0,T/n]$ and $\varphi_n(t) \equiv 1$ when $t \in [2T/n]$. The function $u_n(t,x) = \varphi_n(t)v_n(t,x)$ vanishes at $t = 0$ and on $\partial\Omega_t$ at every t. Obviously, u_n converges to v by $n \to \infty$ in the W-norm.

Theorem 3.6. *The operator L admits a closure $\bar{L} : E^* \to E^*$ with $D(\bar{L}) = W_0$, its range $R(\bar{L})$ is closed and $\bar{L}$ is invertible on $R(\bar{L})$.*

Proof. From (3.3) it follows that L admits a closure $\bar{L}$. Its domain $D(\bar{L})$ consists of those $v \in E^*$ for which there exists such a sequence $v^n \in D(L)$ that $v^n \to v$ and $Lv^n \to u$ in E^*. Then by definition $\bar{L}v = u$. Let us show that $D(\bar{L}) \subseteq W_0$. Let $v \in D(\bar{L})$. Then there exists such a sequence $v^n \in D(L)$ that $v^n \to v$ in E and $Lv^n \to u$ in E^*. Then by means of passing to the limit we have from (3.3) for v^n that $v \in W_0$ and the inequality holds:

$$\|v_t\|_{E^*} + \sup_t |v(t,x)|_{0,t} + \|v\|_E \le M\|\bar{L}v\|_{E^*}. \tag{3.10}$$

Thus, $D(\bar{L}) \subseteq W_0$.

Let us show that $D(\bar{L}) \supseteq W_0$. Let $v \in W_0$ and $v^n \to v$, $v^n \in D$, in the W-norm that $v \in W_0$. From (3.1) it follows that for $h \in E$ $\langle \bar{L}v^n, h\rangle = \langle Lv, h\rangle = \int_0^T (\nabla v^n(t,x), \nabla h(t,x))_t dt + \langle v_t^n, h\rangle$ takes place. The passage to the limit gives the validity of $\langle \bar{L}v, h\rangle = \langle v_t, h\rangle + \int_0^T (\nabla v(t,x), \nabla h(t,x))_t dt$ for $v \in D(\bar{L})$. From the obtained above it follows that the right-hand side part defines an element $u \in E^*$ for any $v \in W_0$. By this $v \in D(\bar{L})$ and $\bar{L}v = u$. Thus, $W_0 \subseteq D(\bar{L})$ and consequently $W_0 = D(\bar{L})$.

From (3.10) it follows that $R(\bar{L})$ is closed and $\bar{L}$ is invertible on $R(\bar{L})$. The theorem is proved. □

Remark 3.7. From (3.4) established for smooth v by means of the passage to the limit and the differentiation with respect to t it is easy to show that the scalar function $(\bar{L}v, v)_t$ for $v \in D(L)$ and a.e. t satisfies the relation

$$(\bar{L}v, v)_t = \frac{1}{2}\frac{d}{dt}|v(t,x)|_{0,t}^2 + |\nabla v(t,x)|_{0,t}^2. \tag{3.11}$$

Theorem 3.8. *The range $R(\bar{L})$ of the operator $\bar{L}$ is dense in E^*.*

We give the proof of this theorem in Section 8.

From Theorems 3.6 and 3.8 the next result follows.

THEOREM 3.9. *For every $f \in E^*$ the equation $\bar{L}v = f$ has a unique solution v and the estimate holds:*

$$||v_t||_{E^*} + \sup_t |v(t,x)|_{0,t} + \|v\|_E \leq M\|f\|_{E^*}. \tag{3.12}$$

Let $k > 0$. Everywhere below we set $\bar{v} = \exp(-kt)v$. It is easy to show that $\exp(-kt)\bar{L}(v) = \bar{L}(\bar{v}) + k\bar{v}$. From here it follows that if $L(v) = f$ then $\bar{L}(\bar{v}) + k\bar{v} = \bar{f}$.

COROLLARY 3.10. *For the solution v of the equation $\bar{L}v = f$ by any $k > 0$ the estimate holds:*

$$||v_t||_{E^*,k} + \|v\|_{EC,k} \leq M\|f\|_{E^*,k}. \tag{3.13}$$

To prove it is enough to make the change $\bar{v} = \exp(-kt)v$ and take advantage of Theorem 3.6. From Corollary 3.10 and from Theorem 3.9 there follows the following theorem.

THEOREM 3.11. *For every $\bar{f} \in E^*$ the equation $\bar{L}(\bar{v}) + k\bar{v} = \bar{f}$ has a unique solution $\bar{v}$ and the estimate holds:*

$$||\bar{v}_t||_{E^*} + \sup_t |\bar{v}(t,x)|_{0,t} + \|\bar{v}\|_E + k\|\bar{v}\|_0 \leq M\|\bar{f}\|_{E^*}. \tag{3.14}$$

4. Formulation of the main results

We are interested in the solvability of the regularized problem (1.1). By this we suppose without loss of generality $\mu_0 = \mu_1 = \rho = 1$, replace $z(s;t,x)$ by $Z_\delta(v)$ and restrict ourselves with the case $v^0(x) = \tilde{v}(0,x)$, $x \in \Omega_0$, $v^1(t,x) = \tilde{v}(t,x)$, $(t,x) \in \Gamma$. Thus, we get the problem

$$\begin{gathered} v_t + v_k \partial v/\partial x_k - \operatorname{Div}\mathscr{E}(v) - \operatorname{Div}\int_0^t \exp(s-t)\mathscr{E}(v)(s,\tilde{Z}_\delta(v))(s;t,x)ds = -\nabla p + \Phi, \\ \operatorname{div} v(t,x) = 0, \quad (t,x) \in Q; \qquad \int_{\Omega_t} p(t,x)dx = 0, \quad t \in [0,T]; \\ v(0,x) = \tilde{v}(0,x), \quad x \in \Omega_0, \qquad v(t,x) = \tilde{v}(t,x), \quad (t,x) \in \Gamma. \end{gathered} \tag{4.1}$$

One should mark that the case of smooth and satisfying accordance conditions functions $v^0(x)$ and $v^1(t,x)$ in (1.1) can be reduced to the conditions in (4.1). Let $\Phi(t,x) \in V_t^*$ at a.e. t. Let $\widehat{w}(t,y) = w(t,z(t;0,y))$ for an arbitrary function $w(t,x)$ defined on Q.

Definition 4.1. A function $v(t,x) = \tilde{v}(t,x) + w(t,x)$,

$$w(t,x) \in E, \qquad \widehat{w} \in L_2(0,T;W_2^1(\Omega_0)) \bigcap W_1^1(0,T;W_2^{-1}(\Omega_0)) \tag{4.2}$$

is called a weak solution of problem (4.1) if for any $h(t,x) \in D$, $h(T,x) = 0$ the identity holds:

$$
\begin{aligned}
-\int_0^T &(v(t,x),h_t(t,x))_t dt + \int_0^T (v_i(t,x)v_j(t,x),\partial h_i(t,x)/\partial x_j)_t dt \\
&+ \int_0^T (\mathcal{E}_{ij}(v(t,x)),\mathcal{E}_{ij}(h(t,x)))_t dt \\
&+ \int_0^T \left(\int_0^t \exp(s-t)\mathcal{E}_{ij}(v)(s,\widetilde{Z}_\delta(v)(s;t,x))ds,\mathcal{E}_{ij}(h(t,x))\right)_t dt \\
&= \langle \Phi(t,x),h(t,x)\rangle - \langle \tilde{v}(0,x),h(0,x)\rangle_0.
\end{aligned}
\tag{4.3}
$$

The following main result takes place.

Theorem 4.2. *Let $\Phi = f_1 + f_2, f_1 \in E_1^*$, $f_2 \in E^*$. Then the problem (4.1) has at least one weak solution.*

The proof of the theorem is organized as follows. Following [21], we need to consider a family of approximating operator equations with a more weak nonlinearity. Alongside with the operator $\bar{L} : W_0 \to E^*$ introduced above we will consider the operators

$$
\begin{aligned}
K_t^i : V_t \longrightarrow V_t^*, \quad i = 1,2,3, \qquad &\langle K_t^3(w),h\rangle_t = (w_i w_j,\partial h_i/\partial x_j)_t, \quad w,h \in V_t; \\
&\langle K_t^1(w),h\rangle_t = (w_i \tilde{v}_j,\partial h_i/\partial x_j)_t, \quad w,h \in V_t; \\
&\langle K_t^2(w),h\rangle_t = (\tilde{v}_i w_j,\partial h_i/\partial x_j)_t, \quad w,h \in V_t;
\end{aligned}
\tag{4.4}
$$

the functional

$$
\tilde{g} \in E^* : \langle \tilde{g},h\rangle = \int_0^T (\tilde{v}_i\tilde{v}_j,\partial h/\partial x_j)_t dt, \quad h \in E;
\tag{4.5}
$$

the functional

$$
\tilde{V} \in E^* : \langle \tilde{V},h\rangle = \int_0^T (\tilde{v}_t - \triangle\tilde{v},h)_t dt, \quad h \in E;
\tag{4.6}
$$

the operator $A_t : E \to E^*$: $\langle A_t(v),h\rangle_t = (\mathcal{E}_{ij}(v),\mathcal{E}_{ij}(h))_t$, $v,h \in V_t$; the operator $C_t : E \times CG \to V_t^*$

$$
\langle C_t(v,z),h\rangle_t = \left(\int_0^t \exp(s-t)\mathcal{E}_{ij}(v+\tilde{v})(s,z(s;t,x)ds,\mathcal{E}_{ij}(h))\right)_t, \quad v \in E,\ z \in CG.
\tag{4.7}
$$

The operators A_t and C_t naturally generate the operators $A : E \to E^*$, and $C : E \times CG \to E^*$:

$$\begin{aligned} \langle A(w),h\rangle &= \int_0^T \langle A_t(w),h\rangle_t dt \quad h \in E^*,\ w \in E; \\ \langle C(v,z),h\rangle &= \int_0^T \langle C_t(v,z),h\rangle_t dt, \quad v \in E,\ z \in CG,\ h \in E. \end{aligned} \tag{4.8}$$

Alongside with the operators K_t^i we will consider for $\varepsilon > 0$ the operators $K_{t,\varepsilon}^i : V_t \to V_t^*$ and the operators $K_\varepsilon^i : E \to E^*$ generated by them:

$$\begin{aligned} \langle K_{t,\varepsilon}^1(u),h\rangle_t &= \left(\frac{u_i\tilde{v}_j}{1+\varepsilon|u|^2}, \frac{\partial h_i}{\partial x_j}\right)_t, \quad \langle K_{t,\varepsilon}^2(u),h\rangle_t = \left(\frac{\tilde{v}_i u_j}{1+\varepsilon|u|^2}, \frac{\partial h_i}{\partial x_j}\right)_t, \\ \langle K_{t,\varepsilon}^3(u),h\rangle_t &= \left(\frac{u_i u_j}{1+\varepsilon|u|^2}, \frac{\partial h_i}{\partial x_j}\right)_t, \quad u,h \in V_t; \\ \langle K_\varepsilon^i(u),h\rangle &= \int_0^T \langle K_{t,\varepsilon}^i(u),h(t,x)\rangle_t dt, \quad u,h \in E. \end{aligned} \tag{4.9}$$

Let $K_\varepsilon(v) = \sum_{i=1}^3 K_\varepsilon^i(v)$. Let $Z_\delta(w) = \tilde{Z}_\delta(\tilde{v}+w)$ for $w \in E$. Consider for $\varepsilon > 0$ the operator equation

$$\bar{L}w - K_\varepsilon(w) + C(w, Z_\delta(w)) = f_\varepsilon. \tag{4.10}$$

Theorem 4.3. *For any $f_\varepsilon \in E^*$ (4.10) has at least one solution $w_\varepsilon \in W_0$.*

Let us approximate a function $f_1 \in E_1^*$ by $f_{1,\varepsilon} \to f_1$ at $\varepsilon \to 0$, $f_{1,\varepsilon} \in L_{2,\sigma}(Q)$. Let $v^\varepsilon = w^\varepsilon + \tilde{v}$, where w^ε is a solution to (4.10). Using the passage to the limit at $\varepsilon \to 0$ we establish that functions v^ε converge to the function v which is a weak solution to problem (4.1), that is, we get the assertion of Theorem 4.2.

The proof of Theorem 4.3 is carried out in Section 6. The operator terms involved in (4.10) are investigated in Section 5.

5. Investigation of properties of operators

To investigate the operator terms of (4.10) we need the additional properties of functional spaces.

Let $\tilde{z}$ be a solution to the Cauchy problem (1.3) and $u(t,y) = \tilde{z}(t;0,y)$, $y \in \Omega_0$. It is clear that u maps Q_0 in Q, $Q_0 = [0,T] \times \Omega_0$ and $u(t,\Omega_0) = \Omega_t$. Let $U(t,x) = \tilde{z}(0;t,x)$. The maps $u(t,y)$ and $U(t,x)$ at fixed t are mutually inverse and

$$U(t,u(t,y)) = y, \quad y \in \Omega_0, \qquad u(t,U(t,x)) = x, \quad x \in \Omega_t. \tag{5.1}$$

Moreover, solenoidality of $\tilde{v}$ implies

$$|U_x(t,x)| = |u_y(t,y)| = 1. \tag{5.2}$$

Here U_x, u_y are the Jacobs matrixes and $|U_x|$, $|u_y|$ are their determinants.

Let $v(t,x)$ be a function smooth on Q. Define the operator Υ as $\hat{v} = \Upsilon v$, $\hat{v}(t,y) = v(t,u(t,y))$. It is clear that $v(t,x) = \hat{v}(t,U(t,x))$. Using differentiation, we have

$$\hat{v}_t(t,y) = v_t(t,u(t,y)) + v_x(t,u(t,y))u_t(t,y) = v_t(t,u(t,y)) + v_x(t,u(t,y))\tilde{v}(t,u(t,y)); \tag{5.3}$$

$$\hat{v}_y(t,y) = v_x(t,u(t,y))u_y(t,y); \tag{5.4}$$

$$v_t(t,x) = \hat{v}_t(t,U(t,x)) + \hat{v}_y(t,U(t,x))U_x(t,x); \qquad v_x(t,x) = \hat{v}_y(t,U(t,x))U_x(t,x). \tag{5.5}$$

From the smoothness of $\tilde{v}(t,x)$ the smoothness of $u(t,y)$, $U(t,x)$ and their derivatives follow.

Denote by $W^{1,-1}_{k,2}(Q)$, $k = 1,2$ a closure of the set of functions smooth both with respect to t and with respect to x and finite on Ω_t for all t in the norm

$$\|v\|_{W^{1,-1}_{k,2}(Q)} = \left(\int_0^T \|v(t,x)\|^k_{W^1_2(\Omega_t)}dt\right)^{1/k} + \left(\int_0^T \|v_t(t,x)\|^k_{W^{-1}_2(\Omega_t)}dt\right)^{1/k}. \tag{5.6}$$

Spaces $W^{1,-1}_{k,2}(Q_0)$, $k = 1,2$ are defined on analogy. Let $W^{0,m}_{k,2}(Q)$, $k = 1,2$, $m = 1,-1$ be a closure of the set of the same functions in the norm

$$\|v\|_{W^{0,m}_{k,2}(Q)} = \left(\int_0^T \|v(t,x)\|^k_{W^m_2(\Omega_t)}dt\right)^{1/k}. \tag{5.7}$$

Spaces $W^{0,m}_{k,2}(Q_0)$ are defined similarly. Let $\hat{F} = \Upsilon F$ where F is some space of functions on Q.

Lemma 5.1. *Let $v(t,x)$ be a smooth solenoidal on Q vector function, $\hat{v}(t,y) = v(t,u(t,y))$. Then*

$$\|\hat{v}\|_{L_2(Q_0)} = \|v\|_{L_2(Q)}, \tag{5.8}$$

$$\|\hat{v}_t\|_{W^{0,-1}_{2,2}(Q_0)} \le M\left(\|v_t\|_{E^*} + \|v\|_{W^{0,1}_{2,2}(Q)}\right), \qquad \|v_t\|_{E^*} \le M\left(\|\hat{v}_t\|_{W^{0,-1}_{2,2}(Q_0)} + \|\hat{v}\|_{W^{0,1}_{2,2}(Q_0)}\right), \tag{5.9}$$

$$\|\hat{v}_t\|_{W^{0,-1}_{1,2}(Q_0)} \le M\left(\|v_t\|_{E_1^*} + \|v\|_{W^{0,1}_{2,2}(Q)}\right), \qquad \|v_t\|_{E_1^*} \le M\left(\|\hat{v}_t\|_{W^{0,-1}_{2,2}(Q_0)} + \|\hat{v}\|_{W^{0,1}_{2,2}(Q_0)}\right), \tag{5.10}$$

Proof. Using the change of the variable $x = u(t,y)$ and (5.5), we have

$$\begin{aligned}
|v_t(t,x)|_{-1,t} &= \sup_{|h|_{1,t}=1} \left|(v_t(t,x),h(x))_t\right| \\
&= \sup_{|h|_{1,t}=1} \left|\int_{\Omega_t} v_t(t,x)h(x)dx\right| \\
&= \sup_{|h|_{1,t}=1} \left|\int_{\Omega_0} v_t(t,u(t,y))h(u(t,y))dy\right| \\
&\leq \sup_{|h|_{1,t}=1} \left|\int_{\Omega_0} \hat{v}_t(t,y)h(u(t,y))dy\right| \\
&\quad + \sup_{|h|_{1,t}=1} \left|\int_{\Omega_0} \hat{v}_y(t,y)u_y^{-1}(t,y)\tilde{v}(t,u(t,y))h(u(t,y))dy\right| \\
&= J_1 + J_2.
\end{aligned} \tag{5.11}$$

Further, using the smoothness of $u(t,y)$, we have

$$\begin{aligned}
J_1 &= \sup_{|h|_{1,t}=1} \left|\int_{\Omega_0} \hat{v}_t(t,y)\frac{h(u(t,y))}{|h(u(t,y))|_{1,0}}dy\right| |h(u(t,y))|_{1,0} \\
&\leq M \sup_{|\hat{h}|_{1,0}=1} \left|\int_{\Omega_0} \hat{v}_t(t,y)\hat{h}(y)dy\right| \sup_{|h|_{1,t}=1} |h(u(t,y))|_{1,0} \leq M|\hat{v}_t|_{-1,0}, \\
J_2 &\leq \sup_{|h|_{1,t}=1} \left(\left|\int_{\Omega_0} \hat{v}_y(t,y)u_y^{-1}(t,y)\tilde{v}(t,u(t,y))\frac{h(u(t,y))}{|h(u(t,y))|_{1,0}}dy\right| |h(u(t,y))|_{1,0}\right) \\
&\leq M \sup_{|\hat{h}|_{1,0}=1} \left|\int_{\Omega_0} \hat{v}_y(t,y)u_y^{-1}(t,y)\tilde{v}(t,u(t,y))\hat{h}(y)dy\right| \sup_{|h|_{1,0}=1} |h(u(t,y))|_{1,0} \\
&\leq M|\hat{v}_y(t,y)u_y^{-1}(t,y)\tilde{v}(t,u(t,y))|_{-1,0}.
\end{aligned} \tag{5.12}$$

By similar reasonings from the smoothness of $u(t,y)$ and $\tilde{v}(t,x)$ it follows that

$$|\hat{v}_y(t,y)u_y^{-1}(t,y)\tilde{v}(t,u(t,y))|_{-1,0} \leq M|\hat{v}(t,y)|_{W_2^1(\Omega_0)}. \tag{5.13}$$

Thus, we get

$$|v_t(t,x)|_{-1,t} \leq M\left(|\hat{v}_t(t,y)|_{-1,0} + |\hat{v}(t,y)|_{W_2^1(\Omega_0)}\right). \tag{5.14}$$

From the last three inequalities the second inequality (5.9) follows. Other inequalities (5.8)–(5.10) are proved in the same way more easily. Lemma 5.1 is proved. □

Let Υ_t be the restriction of the operator Υ on the space of functions on Ω_t. From the density of the set of smooth functions in the spaces mentioned below and the proof of Lemma 5.1 the next lemma follows.

Lemma 5.2. *The linear operator Υ_t is bounded and boundedly invertible as an operator*

$$\Upsilon_t : L_2(\Omega_t) \longrightarrow L_2(\Omega_0), \qquad \Upsilon_t : W_2^1(\Omega_t) \longrightarrow W_2^1(\Omega_0), \qquad \Upsilon_t : W_2^{-1}(\Omega_t) \longrightarrow W_2^{-1}(\Omega_0). \tag{5.15}$$

The operator Υ is bounded and boundedly invertible as an operator

$$\begin{aligned} &\Upsilon : W_{k,2}^{0,m}(Q) \longrightarrow W_{k,2}^{0,m}(Q_0), \quad k = 1,2,\ m = 1,-1; \\ &\Upsilon : W_{k,2}^{1,-1}(Q) \longrightarrow W_{k,2}^{1,-1}(Q_0), \quad k = 1,2. \end{aligned} \tag{5.16}$$

In the cylindrical case the continuous embedding takes place $W \subset C(0,T;H_0)$ ([6, Theorem 1.17, page 177]). From this and Lemmas 5.1-5.2 the next fact follows.

Lemma 5.3. *For any $v \in W_1$ the function $\hat{v}(t,y)$ is weakly continuous as a function with values in $L_2(\Omega_0)$.*

Proof of Lemma 5.3. From the boundedness of the orthogonal projection operator $P_t : W_2^1(\Omega_t) \to V_t$ (see [18, page 24]) which is uniform with respect to t thanks to the smoothness of $\tilde{v}(t,x)$ it follows that

$$\|P_t h\|_{W_2^1(\Omega_t)} \le \|h\|_{W_2^1(\Omega_t)}, \quad h \in W_2^1(\Omega_t). \tag{5.17}$$

Let $v \in W_1$ and $\|v^n - v\|_{W_1} \to 0$, $v^n \in D$ at $n \to \infty$. Using (5.17) and the obvious relation $P_t v_t^n = v_t^n$ for $v^n \in D$ it is not difficult to show that

$$\|\hat{v}_t^n\|_{W_2^{-1}(\Omega_t)} \le M \|v_t^n\|_{V_t^*}. \tag{5.18}$$

By means of the change of the variable $x = u(t,y)$ from (5.18) and Lemma 5.2 it follows that the sequence $\hat{v}_t^n$ is fundamental in $W_{1,2}^{0,-1}(Q_0)$ or that is the same in $L_1(0,T; W_2^{-1}(\Omega_0))$. Since v^n is fundamental in E then $\hat{v}^n$ is fundamental in $L_2(0,T;W_2^1(\Omega_0))$ and moreover in $L_2(0,T;W_2^{-1}(\Omega_0))$. Thus,

$$\hat{v} \in W_1^1(0,T;W_2^{-1}(\Omega_0)) \cap L_2(0,T;W_2^1(\Omega_0)). \tag{5.19}$$

From this and [18, Lemmas III.1.1 and III.1.4], the assertion of Lemma 5.3 follows. Lemma 5.3 is proved. □

Remark 5.4. By means of the change of the variable $y = U(t,x)$ from Lemma 5.3 it follows that for $v(t,x) \in W_1$ there exists a trace belonging to H_t at all $t \in [0,T]$ and the inclusion $|v(t,x)|_{0,t} \in L_\infty[0,T]$.

Examine the operator terms of (4.10).

Lemma 5.5. *Let $n \le 4$. Then the operators K_ε^i, $\varepsilon > 0$, $i = 1,2,3$, are bounded and continuous as operators $K_\varepsilon^i : E \to E^*$, the operators $K_0^i = K^i$ are bounded and continuous as operators $K^i : E \to E_1^*$, and the inequalities hold:*

$$\|K_\varepsilon^i(v)\|_{E^*} \le M_1/\varepsilon, \quad \varepsilon > 0; \qquad \|K_\varepsilon^i(v)\|_{E_1^*} \le M_1 \|v\|_E^2, \quad \varepsilon \ge 0. \tag{5.20}$$

Here M_1 depends on $\|\tilde{v}\|_{C(Q)}$. Besides, the operators $K_\varepsilon^i : W \to E^*$, $\varepsilon > 0$, $i = 1,2,3$ are completely continuous.

The proof of Lemma 5.5 for $i = 3$ is similar to the proof for the cylindrical case ([3, Lemma 2.1 and Theorem 2.2]). For $i = 1,2$ the proof is easier because of the smoothness of $\tilde{v}$.

Lemma 5.6. *For any $v \in E$, $z \in CG$ the inclusion $C(v,z) \in E^*$ is valid and the map $C : E \times CG \to E^*$ is continuous and bounded.*

The proof repeats the proof of Lemma 2.2 in [21] for the cylindrical case which is fit for the non cylindrical case as well.

Lemma 5.7. *The map $Z_\delta : W_1 \to CG$ is continuous and for every weakly converging sequence $\{v_l\}$, $v_l \in W_1$, $v_l \rightharpoonup v_0$, there exists a subsequence $\{v_{l_k}\}$ such that $Z_\delta(v_{l_k}) \to Z_\delta(v_0)$ in the space CG.*

For the proof of Lemma 5.6 it is enough to repeat the proof of Lemma 3.2 in [21] and take advantage of Lemma 5.6 and (2.5).

Lemma 5.8. *For any $z \in CG$ and $u,v \in E$ the estimates hold:*

$$\begin{gathered} \|C(v,z) - C(u,z)\|_{k,E^*} \leq \sqrt{M/k}\|u - v\|_{k,E}; \\ \|C(v,z)\|_{k,E^*} \leq \sqrt{M/k}\left(\|v\|_{k,E} + \|\tilde{v}\|_{k,E}\right). \end{gathered} \tag{5.21}$$

Proof of estimates (5.21) repeats the proof of Lemma 2.4 of [21] with the change of Ω_0 by Ω_t in the calculations.

Let γ_k be a Kuratovsky's noncompactness measure (see [1]) in E^* with the norm $\|\cdot\|_{k,E^*}$. Let $G(v) = C(v, Z_\delta(v))$.

Theorem 5.9. *For sufficiently large k the map $G : W_0 \to E^*$ is $\bar{L}$-condensing with respect to γ_k.*

The definition of the $\bar{L}$-condensing map is given in [3].

The proof of theorem repeats the proof of Theorem 2.2 in [21] on the strength of Lemmas 5.6–5.8 and the inequality $\|u - v\|_{k,E} \leq M\|\bar{L}u - \bar{L}v\|_{k,E^*}$, $v,u \in W_0$ following from (3.13).

6. Proof of Theorem 4.3

Introduce an auxiliary family of operator equations

$$\bar{L}v - \lambda\sum_{i=1}^{3} K_\varepsilon^i(v) - \lambda C(v, Z_\delta(v)) = f_\varepsilon, \quad f_\varepsilon \in E^*,\ \lambda \in [0,1]. \tag{6.1}$$

At $\lambda = 1$ (6.1) coincides with (4.10). Following [21], we will obtain a priori estimates of solutions to this family.

Theorem 6.1. *For any solution $v \in W_0$ to problem (4.10) the estimates*

$$\|v\|_{EC} \leq M\Big(1+\|f_\varepsilon\|_{E^*}\Big), \qquad \|v_t\|_{E^*} \leq M\Big(1+\|f_\varepsilon\|_{E^*}\Big) \tag{6.2}$$

hold. Here M does not depend on λ but depends on ε and on $\|\tilde{v}\|_{C(Q)}$.

Proof. Let $v \in W_0$ be a solution to (6.1). From (6.1) and (3.13) it follows that

$$\|v\|_{k,EC} \leq M\left(\sum_{i=1}^{3}\|K_\varepsilon^i(v)\|_{k,E^*} + \|C(v,Z_\delta(v))\|_{k,E^*} + \|f_\varepsilon\|_{k,E^*}\right). \tag{6.3}$$

From Lemma 5.8 it follows that

$$\|C(v,Z_\delta(v))\|_{K,E^*} \leq \sqrt{M/k}\left(\|v\|_{k,E} + \|\tilde{v}\|_{k,E}\right). \tag{6.4}$$

Taking into account (5.20) and the last inequalities, we get

$$\|v\|_{k,EC} \leq M\Big(\varepsilon^{-1} + \sqrt{M/k}\left(\|v\|_{k,E} + \|\tilde{v}\|_{k,E}\right) + \|f_\varepsilon\|_{k,E^*}\Big), \tag{6.5}$$

where M depends on $\|\tilde{v}\|_{C(Q)}$.

As $\|v\|_{k,E} \leq M\|v\|_{k,EC}$, we get from this

$$\|v\|_{k,E} \leq M\Big(\varepsilon^{-1} + \|f_\varepsilon\|_{k,E^*} + \|\tilde{v}\|_{k,E}\Big) \tag{6.6}$$

for sufficiently large k. Now, taking into account the equivalence of the norms $\|\cdot\|_{k,EC}$ and $\|\cdot\|_{EC}$, $\|\cdot\|_{k,E^*}$ and $\|\cdot\|_{E^*}$, we come to the first estimate (6.2). The second estimate (6.2) follows from the first estimate (6.2) and the boundedness in E of the maps A, K_ε, and C. Theorem 6.1 is proved. □

Let us go over to the proof of Theorem 4.3. On the strength of Theorem 5.9 and Lemma 5.1 maps $\lambda(\sum_{i=1}^3 K_\varepsilon^i - G)$ are $\bar{L}$-condensing with respect to the Kuratovsky's noncompactness measure γ_k as maps from $W \times [0,T]$ in E^*. Moreover, from a priori estimates (6.2) it follows that every equation from (6.1) at $\lambda \in [0,1]$ has no solutions on the boundary of the ball $\bar{B}_R \subset W_0$ of a sufficiently large radius R with the center in zero. Hence, for every $\lambda \in [0,1]$ the degree of a map $\deg_2(\bar{L} - (\sum_{i=1}^3 K_\varepsilon^i - G), \bar{B}_R, f_\varepsilon)$ (see [20]) is defined. As a degree of a map does not change by the change of λ, then $\deg_2(\bar{L} - \lambda\sum_{i=1}^3 K_\varepsilon^i + G, \bar{B}_R, f_\varepsilon) = \deg_2(\bar{L}, \bar{B}_R, f_\varepsilon)$. The map $\bar{L}$ is invertible, therefore the equation $\bar{L}v = f_\varepsilon$ has a unique solution $v_0 \in W_0$ and v_0 satisfies estimates (6.2). Then $v_0 \in B_R$ and $\deg_2(\bar{L}, \bar{B}_R, f_\varepsilon) = 1$. Therefore, $\deg_2(\bar{L} - \sum_{i=1}^3 K_\varepsilon^i + G, \bar{B}_R, f_\varepsilon) = 1$. The difference from zero of a degree of a map implies the existence of solutions of the operator equation (6.1) and consequently the existence of a solution of (4.10) ((6.1) at $\lambda = 1$). Theorem 4.3 is proved.

7. Proof of Theorem 4.2

Let us establish some auxiliary facts. Approximate $f_1 \in E_1^*$ by means of $f_{1,\varepsilon} \to f_1$ as $\varepsilon \to 0$, $f_{1,\varepsilon} \in L_{2,\sigma}(Q)$. Let $f_2 \in E^*$, $f_\varepsilon = f_{1,\varepsilon} + f_2$. It is clear that $f_{1,\varepsilon} \in E^*$ and consequently $f_\varepsilon \in E^*$.

Theorem 7.1. *Any solution of problem (4.10) for $\varepsilon > 0$ satisfies the estimate*

$$\|w\|_{EL} \le M\Big(1 + \|f_{1,\varepsilon}\|_{E_1^*} + \|f_2\|_E\Big), \tag{7.1}$$

$$\|w_t\|_{E_1^*} \le M\Big(1 + \|f_{1,\varepsilon}\|_{E_1^*} + \|f_2\|_{E^*}\Big), \tag{7.2}$$

where M does not depend on ε but depends on $\tilde{v}$.

Here $\|w\|_{EL} = \sup_t |w(t,x)|_{0,t} + \|w\|_E$.

Proof. Let w be a solution to problem (4.10) for some $\varepsilon > 0$. Then $\bar{L}(w) - K_\varepsilon(w) + C(w, Z_\delta(w)) = f_{1,\varepsilon} + f_2$. Multiplying the equation by e^{-kt}, we get

$$\bar{L}(\overline{w}) + k\overline{w} - \overline{K}_\varepsilon(\overline{w}) + \overline{C}(\overline{w}, Z_\delta(e^{kt}\overline{w}(t))) = \overline{f}_{1,\varepsilon} + \overline{f}_2, \tag{7.3}$$

where

$$w(t) = e^{kt}\overline{w}(t), \quad \overline{K}_\varepsilon(\overline{w})(t) = e^{-kt}K_\varepsilon(w(t)), \quad \overline{C}(\overline{w}, Z_\delta(e^{kt}\overline{w}(t)))(t) = e^{-kt}C(w, Z_\delta(w))(t),$$
$$\overline{f}_{1,\varepsilon}(t) = e^{-kt}f_{1,\varepsilon}(t), \qquad \overline{f}_2(t) = e^{-kt}f_2(t). \tag{7.4}$$

Operator $k\overline{w}$ is defined by the equality $\langle k\overline{w}, h\rangle = k(\overline{w}, h)$ for $h \in E$.

Consider an action of the functionals in the left-hand side part of (7.3) upon the function $\overline{w}$:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\overline{w}(t)|_{0,t}^2 + k|\overline{w}(t)|_{0,1}^2 + |\nabla\overline{w}(t)|_{0,t}^2 - (\overline{K}_\varepsilon(\overline{w}(t)), \overline{w}(t))_t \\ &\quad = -(\overline{C}(\overline{w}, Z_\delta(e^{kt}\overline{w}(t)))(t), \overline{w}(t))_t + (\overline{f}_{1,\varepsilon}(t), \overline{w}(t))_t + (\overline{f}_2(t), \overline{w}(t))_t. \end{aligned} \tag{7.5}$$

It is known that $(\overline{K}_\varepsilon^3(\overline{w}(t)), \overline{w}(t))_t = 0$ for all $t \in [0,T]$ (see [3]). Therefore, the integration of both parts of the identity on $[0,t]$ yields

$$\begin{aligned} &(1/2)\|\overline{w}(t)\|_H^2 + k\|\overline{w}\|_0^2 + \|\overline{w}\|_E^2 \\ &\quad = -\int_0^t (\overline{C}(\overline{w}, Z_\delta(e^{k\tau}\overline{w}(\tau)))(\tau), \overline{w}(\tau))_\tau d\tau + \int_0^t (\overline{f}_{1,\varepsilon}(\tau), \overline{v}(\tau))_t d\tau \\ &\qquad + \int_0^t (\overline{f}_2(\tau), \overline{w}(\tau))_\tau d\tau + \sum_{i=1}^2 \int_0^t (\overline{K}_\varepsilon^i(\overline{w}(t)), \overline{w}(\tau))_\tau d\tau. \end{aligned} \tag{7.6}$$

Taking into account the smoothness of $\tilde{w}$, we have

$$\begin{aligned} \left|\sum_{i=1}^2 \int_0^t (\overline{K}_\varepsilon^i(\overline{w}(t)), \overline{w}(\tau))_\tau d\tau\right| &\le \sum_{i=1}^2 \int_0^t |(\overline{K}_\varepsilon^i(\overline{w}(t)), \overline{w}(\tau))_\tau| d\tau \\ &\le M\|\tilde{w}\|_{C(Q)} \int_0^t \|\overline{w}(\tau,x)\|_{L_2(\Omega_\tau)} \|\overline{w}(\tau,x)\|_{W_2^1(\Omega_\tau)} d\tau \\ &\le M\|\tilde{v}\|_{C(Q)} \|\overline{w}(\tau,x)\|_{L_2(Q)} \|\overline{w}(\tau,x)\|_E. \end{aligned} \tag{7.7}$$

Then using the Cauchy inequality and estimate (5.21), we get the inequality

$$\begin{aligned}(1/2)\|\overline{w}(t)\|_H^2 + k\|\overline{w}\|_{L_2(Q)}^2 + \|\overline{w}\|_E^2 \\ \leq \sqrt{M/k}\|\overline{w}\|_E^2 + M\|\tilde{v}\|_{C(Q)}\|\overline{w}(\tau,x)\|_{L_2(Q)}\|\overline{w}(\tau,x)\|_E \\ + \sqrt{M/k}\|\tilde{v}\|_E\|\overline{w}(\tau,x)\|_{L_2(Q)} + \|\overline{f}_{1,\varepsilon}\|_{E_1^*}\|\overline{w}\|_{CH} + \|\overline{f}_2\|_{E^*}\|\overline{w}\|_E + \sqrt{M/k}\|\tilde{v}\|_E\|\overline{w}\|_E.\end{aligned} \tag{7.8}$$

It easy to see that

$$\begin{aligned}\|\overline{w}(\tau,x)\|_{L_2(Q)}\|\overline{w}(\tau,x)\|_E \leq \varepsilon_1\|\overline{w}(\tau,x)\|_E^2 + C(\varepsilon_1)\|\overline{w}(\tau,x)\|_{L_2(Q)}^2, \quad \varepsilon_1 > 0, \\ \|\tilde{v}\|_E\|\overline{w}(\tau,x)\|_{L_2(Q)} \leq \varepsilon_2 + C(\varepsilon_2)\|\overline{w}(\tau,x)\|_{L_2(Q)}^2, \quad \varepsilon_2 > 0.\end{aligned} \tag{7.9}$$

Taking k sufficiently large, ε_1 and ε_2 sufficiently small and using the last inequalities, we obtain the estimate

$$\|\overline{w}\|_{CH}^2 + k\|\overline{w}\|_{L_2(Q)}^2 + \|\overline{w}\|_E^2 \leq M\left(1 + \|\overline{f}_{1,\varepsilon}\|_{E_1^*}^2 + \|\overline{f}_2\|_{E^*}^2\right), \tag{7.10}$$

from which the required estimate (7.1) follows.

Let us prove estimate (7.2). Repeat the arguments of the proof of estimate (6.2). Let w be a solution to problem (4.10) for some $\varepsilon > 0$. From (4.10) it follows that $w_t = -A(w) - K_\varepsilon(w) - C(v, Z_\delta(w)) + f_\varepsilon$. Consequently

$$\|w_t\|_{E_1^*} \leq M\left(\|A(w)\|_{E^*} + \|K_\varepsilon(v)\|_{E_1^*} + \|C(v,Z_\delta(w))\|_{E^*} + \|f_{1,\varepsilon}\|_{E_1^*} + \|f_2\|_{E^*}\right). \tag{7.11}$$

Establish the estimate $\|K_\varepsilon(w)\|_{E_1^*}$ in the similar to [3] way. For $n \leq 4$ the inclusion $V_t \subset L_4(\Omega_t)$ is uniformly continuous with respect to t. Then we have by definition of K_ε^3 for $w \in V_t$ the inequality

$$|K_\varepsilon^3(w)|_{-1,t} \leq \max_{i,j}\|w_iw_j/(1+\varepsilon|w|^2)\|_{L_2(\Omega_t)} \leq \max_{i,j}\|w_iw_j\|_{L_2(\Omega_t)} \leq M\|w\|_{L_4(\Omega_t)}^2. \tag{7.12}$$

The expressions K_ε^i, $i = 1,2$ are estimated in the same manner. Thus, $\|K_\varepsilon(w)\|_{E_1^*} \leq M\|w\|_{L_2(0,T;L_4)}^2$. Here we denote the completion of the set of smooth on Q functions in the norm $\|w\|_{L_2(0,T;L_4)} = (\int_0^T \|w(t,x)\|_{L_4}^2 dt)^{1/2}$ by $L_2(0,T;L_4)$. In virtue of the mentioned continuity of the inclusion $V_t \subset L_4$ the inclusion $E \subset L_2(0,T;L_4)$ is continuous as well. Then $\|w\|_{L_2(0,T;L_4)} \leq M\|v\|_E$ and consequently $\|K_\varepsilon(w)\|_{E_1^*} \leq C\|w\|_E^2$. This estimate easily proved inequality $|\triangle(w)|_{-1,t} \leq M|w|_{1,t}$, the boundedness of the map C on E and estimate (7.1) admit to get the required estimate (7.2) from (7.11). The theorem is proved. □

Let us go over to the proof of Theorem 4.2. Let $v = w + \tilde{v}$. It is easy to show that v is a weak solution to problem (4.1) if and only if w satisfies the integral identity

$$
\begin{aligned}
&-\int_0^T (w(t,x),h_t(t,x))_t dt + \int_0^T (w_i(t,x)w_j(t,x),\partial h_i(t,x)/\partial x_j)_t dt \\
&\quad + \int_0^T (w_i(t,x)\tilde{v}_j(t,x),\partial h_i(t,x)/\partial x_j)_t dt + \int_0^T (\tilde{v}_i(t,x)w_j(t,x),\partial h_i(t,x)/\partial x_j)_t dt \\
&\quad + \int_0^T (\mathcal{E}_{ij}(w)(t,x),\mathcal{E}_{ij}(h)(t,x))_t dt \\
&\quad + \int_0^T \left(\int_0^t \exp(s-t)\mathcal{E}_{ij}(\tilde{v}+w)(s,Z_\delta(w)(s;t,x))ds,\mathcal{E}_{ij}(h(t,x))\right)_t dt \\
&= \int_0^T \langle \Phi(t,x),h(t,x)\rangle_t dt + \langle \tilde{V}+\tilde{g},h\rangle - \langle \tilde{v}(0,x),h(0,x)\rangle_0
\end{aligned}
\tag{7.13}
$$

for h satisfying the conditions of Definition 4.1. Let $f_{1,\varepsilon} \to f_1$ in E_1^* as $\varepsilon \to 0$, $f_{1,\varepsilon} \in L_{2,\sigma}(Q)$. Consider (4.10) by $f_\varepsilon = f_{\varepsilon,1} + \tilde{f}_2$ where $\tilde{f}_2 = \tilde{V} + \tilde{g} + f_2$. It is clear that $\tilde{f}_\varepsilon \in E^*$ at any $\varepsilon > 0$. In virtue of Theorem 4.3 there exists a solution w^ε to problem (4.10). It is not difficult to show that w^ε satisfies the identity

$$
\begin{aligned}
&-\int_0^T (w^\varepsilon(t,x),h_t(t,x))_t dt + \langle K_\varepsilon^1(w^\varepsilon),h\rangle + \langle K_\varepsilon^2(w^\varepsilon),h\rangle + \int_0^T \langle K_{t,\varepsilon}^3(u),h\rangle_t dt \\
&\quad + \int_0^T (\mathcal{E}_{ij}(w^\varepsilon)(t,x)ds,\mathcal{E}_{ij}(h)(t,x))_t dt + \langle C(w^\varepsilon,Z_\delta(w^\varepsilon))\rangle \\
&= \int_0^T \langle f_{1,\varepsilon},h\rangle_t dt + \langle \tilde{f}_2,h\rangle - \langle \tilde{v}(0,x),h(0,x)\rangle_0
\end{aligned}
\tag{7.14}
$$

for h satisfying the conditions in Definition 4.1.

Let us show that there exists such a number sequence ε_l, $l = 1,2,\ldots$, that $w^{\varepsilon_l} \equiv w^l$ converges to $w \in E$ which satisfies identity (7.13).

Choose the sequence of positive numbers $\{\varepsilon_l\}$ converging to zero. As $f_{1,\varepsilon_l} \to f_1$ in E_1^* by $l \to \infty$ then the sequence $\|f_{1,\varepsilon_l}\|_{E_1^*}$ is bounded. For each number ε_l the corresponding problem (4.10) has at least one solution $w^l \in W_0$. In virtue of estimates (7.1) the sequence $\{w^l\}$ is bounded in the norm $\|\cdot\|_{EL}$. Then without loss of generality we will assume that $w^l \to w^*$ weakly in E, $w^l \to w^*$ $*$-weakly in E_1^*. Moreover, $w^l \to w^*$ strongly in $L_2(Q)$. In fact, since both $\|w_t^l\|_{E_1^*}$ and $\|w^l\|_E$ are bounded in virtue of (7.1)-(7.2), then as in the proof of Lemma 5.3 we can suppose that $w^l \to w^*$ strongly in $L_2(Q)$.

Show that

$$
\langle C(w^l,Z_\delta(w^l)) - C(w^*,Z_\delta(w^*)),h\rangle \longrightarrow 0, \quad l \longrightarrow \infty. \tag{7.15}
$$

Using Lemma 5.7, without loss of generality we will assume that

$$Z_\delta(w^l) \longrightarrow Z_\delta(w^*) \quad \text{in the norm of the space } CG. \tag{7.16}$$

Let $h \in E$ be an arbitrary function. Consider

$$\begin{aligned}\langle C(w^l,Z_\delta(w^l)) - C(w^*,Z_\delta(w^*)),h\rangle &= \langle C(w^l,Z_\delta(w^l)) - C(w^*,Z_\delta(w^l)),h\rangle \\ &\quad + \langle C(w^*,Z_\delta(w^l)) - C(w^*,Z_\delta(w^*)),h\rangle.\end{aligned} \tag{7.17}$$

The second term converges to zero by virtue of the assumption (7.16) and the continuity of the map C with respect to z. By means of the change of the variable $z = Z_\delta(w^l)(s;t,x)$ (the inverse change looks like $x = Z_\delta(w^l)(t;s,z)$) in the first term we get

$$\begin{aligned}&\langle C(w^l,Z_\delta(w^l)) - C(w^*,Z_\delta(w^l)),h\rangle \\ &\quad = \int_0^T \int_{\Omega_t} \int_0^t [\mathscr{E}_{ij}(w^l)(s,Z_\delta(w^l)(s;t,x)) \\ &\qquad\qquad - \mathscr{E}_{ij}(v^*)(s,Z_\delta(w^l)(s;t,x))]ds\mathscr{E}_{ij}(h)(t,x)dx\,dt \\ &\quad = \int_0^T \int_0^t \int_{\Omega_0} [\mathscr{E}_{ij}(w^l)(s,z) - \mathscr{E}_{ij}(w^*)(s,y)]\mathscr{E}_{ij}(h)(t,Z_\delta(w^l)(t;s,y))dy\,ds\,dt.\end{aligned} \tag{7.18}$$

Under the assumption given above the first bracket converges to zero weakly in $L_2(Q)$. The map $(w,y) \mapsto \mathscr{E}(w)(s,z(s;t,y))$ from the space $E \times CG$ in the space $L_2([0,T]\times[0,T], L_2(\Omega_0))$ is continuous. From this and (7.16) we get the strong $L_2([0,T]\times[0,T], L_2(\Omega_0))$ convergence of the second factor in the last integral. Then the whole expression converges to zero as $l \to \infty$ as well. This yields the proof of convergence (7.15).

The possibility of the passage to the limit for the terms with the exception of that ones containing K_ε^i follows from the facts given above. In [3], the convergence $\int_0^T (\widehat{K}_{\varepsilon_l}^3(w^l),\widehat{h})_t dt$ to $\int_0^T (\widehat{K}^3(w^*),\widehat{h})_t dt$ was shown. The similar fact for $\widehat{K}_{\varepsilon_l}^i$, $i = 1,2$ is proved easier. Using the inverse to $x = u(t,x)$ change of variable, we get the convergence for the terms in (7.14) containing K_ε^i.

To finish the proof of the theorem we pass to the limit as $l \to \infty$ in (7.14). Taking into account the convergences mentioned above and passing to the limit in (7.14) we get that w^* satisfies the identity (7.13).

Besides, from estimate (7.1), (7.2), and Lemma 5.1 it follows that $\|\widehat{w}^l\|_{L_2(0,T;W_2^1(\Omega_0))}$ and $\|\widehat{w}^l\|_{W_1^1(0,T;W_2^{-1}(\Omega_0))}$ are bounded. Without loss of generality we can believe that $\widehat{w}^l$ converges $*$-weakly to $\widehat{w}$ in $L_2(0,T;W_2^1(\Omega_0)) \bigcap W_1^1(0,T;W_2^{-1}(\Omega_0))$.

The theorem is proved.

8. Proof of Theorem 3.8

The basic moment here is the proof of the solvability of a corresponding linear Stokes problem in a non cylindrical domain. For this purpose using the change of the variable $x = u(t,y) = \tilde{z}(\tau;t,x)$, we reduce this problem to the corresponding problem in a cylindrical domain. By this the methods of [17] are essentially used.

Let us show now that the range $R(\bar{L})$ of the operator $\bar{L}$ is dense in E^*. For this purpose we establish the solvability in the class $W_2^{1,2}(Q)$ of the problem

$$v_t - \triangle v + \nabla p = g, \qquad \nabla \cdot v = 0, \qquad (t,x) \in Q, \quad \int_{\Omega_t} p(t,x)dx = 0, \quad 0 \le t \le T; \tag{8.1}$$

$$v(0,x) = 0, \quad x \in \Omega_0; \qquad v(t,x) = 0, \quad (t,x) \in \Gamma. \tag{8.2}$$

Here $\nabla \cdot v = \operatorname{div} v$. The pair of functions $v \in W_2^{1,2}(Q)$, $p \in W_2^{0,1}(Q)$ is called a solution to problem (8.1)-(8.2) if it satisfies a.e. (8.1) and conditions (8.2).

Using the change of the variable

$$x = u(t,y), \quad u(t,y) = y + \int_0^t \tilde{v}(s,y)dy, \; x = \tilde{z}(t,0,y) \tag{8.3}$$

and (5.3)–(5.5) we reduce the problem (8.1)-(8.2) to the problem

$$\begin{gathered} \hat{v}_t + D(\hat{v}) - \hat{\triangle}\hat{v} + \hat{\nabla}\hat{p} = \hat{g}, \qquad \hat{\nabla} \cdot \hat{v} = 0, \qquad (t,y) \in Q_0; \qquad \int_{\Omega_0} \hat{p}(t,y)dy = 0; \\ \hat{v}(0,y) = 0, \quad y \in \Omega_0; \qquad \hat{v}(t,y) = 0, \quad (t,y) \in \Gamma_0. \end{gathered} \tag{8.4}$$

Here $D(\hat{v}) = \hat{v}_y C$, $C(t,y)$ is a vector function with smooth coefficients which are expressed by means of the coefficients of the Jacobs matrix $\tilde{v}_y(t,y)$ and

$$\hat{\Delta}\hat{v} = \Upsilon(\Delta v), \qquad \hat{\nabla}\hat{p} = \Upsilon(\nabla p), \qquad \hat{\nabla} \cdot \hat{v} = \Upsilon(\nabla \cdot v), \qquad \Upsilon v(t,x) = \hat{v}(t,y) = v(t,u(t,y)). \tag{8.5}$$

Rewrite (8.4) in the form

$$\begin{gathered} \hat{v}_t - \triangle\hat{v} + \nabla\hat{p} = f(t,y), \qquad \nabla \cdot \hat{v} = \rho(t,y); \qquad \int_{\Omega_0} \hat{p}(t,y)dy = 0; \\ \hat{v}(0,y) = 0; \qquad \hat{v}(t,y) = 0, \quad (t,y) \in \Gamma_0. \end{gathered} \tag{8.6}$$

Here

$$f = \hat{g} - D(\hat{v}) + (\widehat{\triangle} - \triangle)\hat{v} + (\nabla - \hat{\nabla})\hat{p}, \qquad \rho = (\nabla - \hat{\nabla}) \cdot \hat{v}. \tag{8.7}$$

Note that

$$\int_{\Omega_0} \rho(t,y)dy = 0. \tag{8.8}$$

In fact, using the Ostrogradsky formula and the inverse to (8.3) change of the variable $y = U(t,x) = \tilde{z}(0,t,x)$, we have

$$\int_{\Omega_0} \rho(t,y)dy = \int_{\Omega_0} \nabla \cdot \hat{v}dy - \int_{\Omega_0} \hat{\nabla} \cdot \hat{v}dy = -\int_{\Gamma_0} v_n(\tilde{t},y)dy + \int_{\Omega_t} \nabla \cdot v(t,x)dx = 0. \tag{8.9}$$

Let us establish the solvability of problem (8.6). At first consider an auxiliary problem with an arbitrary $f \in L_2(Q_0)$ and $\rho(t,y) \in L_2(Q_0)$, satisfying some additional conditions which we will impose below:

$$u_t - \triangle u + \nabla p = f(t,y), \quad \nabla \cdot u(t,y) = \rho(t,y), \quad (t,y) \in Q_0;$$
$$u(0,y) = 0, \quad y \in \Omega_0, \qquad u(t,y)\mid_\Gamma = 0, \qquad \int_{\Omega_t} p(t,x)dy = 0. \tag{8.10}$$

Let us reduce the question of the solvability of problem (8.10) to the solvability of a similar problem with $\rho \equiv 0$. By this we assume the condition (8.8) is fulfilled. For this purpose we will consider an auxiliary problem for $t \in [0,T]$

$$-\triangle \Phi = \rho(t,y), \quad y \in \Omega_0, \quad \Phi\mid_{\Gamma_0} = 0. \tag{8.11}$$

If $\rho(t,y) \in L_2(\Omega_0)$ at a.e. t, then problem (8.11) is uniquely solvable (see [15]) and its solution has the form $\Phi = R^{-1}\rho$ and

$$|\Phi(t,y)|_{2,0} \leqslant M|\rho(t,y)|_{0,0} \tag{8.12}$$

is valid. From this it follows that $\|\Phi\|_{0,2} \leqslant M\|\rho\|_0$. Here R is a selfadjoint operator with the domain $W^2_{2,0}(\Omega_0)$, positively defined in $L_2(\Omega_0)$, R^{-1} is an operator inverse to R. It is clear that

$$|R^{1/2}u|^2_{0,0} = (Ru,u)_0 = -(\triangle u,u)_0 = (\nabla \cdot u, \nabla \cdot u)_0 = |u|^2_{1,0}, \quad u \in W^2_{2,0}(\Omega_0). \tag{8.13}$$

In virtue of this we get the boundedness in $L_2(\Omega_0)$ of the following operators and the estimates

$$\left|\frac{\partial}{\partial y_j}R^{-1/2}u\right|_{0,0} \leqslant M|u|_{0,0}, \qquad \left|\overline{R^{-1/2}\frac{\partial}{\partial y_i}}u\right|_{0,0} \leqslant M|u|_{0,0},$$
$$\left|\overline{\frac{\partial}{\partial y_i}R^{-1}\frac{\partial}{\partial y_j}}u\right|_{0,0} \leqslant M|u|_{0,0}. \tag{8.14}$$

Here the bar above means the closure of an operator.

Let $u^1 = -\nabla\Phi$, where Φ is a solution to problem (8.11). Then $u^1 \in W_2^{0,1}(Q_0)$ and $\nabla u^1 = \rho$. Let $\Psi(t,y)$ be a solution to problem

$$\triangle\Psi = 0, \qquad \nabla_n\Psi\mid_{\Gamma_0} = \nabla_n\Phi\mid_{\Gamma_0}. \tag{8.15}$$

Here ∇_n is a normal derivative on Γ_0. As $\Phi(t,y) \in W_2^{3/2}(\Gamma_0)$ at a.e. t and in virtue of (8.8)

$$\int_{\Gamma_0} \nabla_n\Phi(t,y)dy = -\int_{\Omega_0} \triangle\Phi(t,y)dy = -\int_{\Omega_0} \rho(t,y)dy = 0, \tag{8.16}$$

is valid then problem (8.15) is uniquely solvable in $W_2^2(\Omega)$ (see [15]).

Its solution has the form $\Psi = N^{-1}\gamma_n\nabla\Phi$. Here N^{-1} is a linear bounded operator acting from $W_2^{1/2}(\Gamma_0)$ into $W_2^2(\Omega_0)$ (see [17]), and γ_n is the operator of taking of the normal component of a trace on the boundary of a function defined on $\overline{\Omega}_0$. The operator γ_n

is bounded as an operator from $W_2^1(\Omega_0)$ in $W_2^{1/2}(\Gamma_0)$. From here it follows that if $\Phi \in W_2^{0,2}(Q_0)$, then $\Psi \in W_2^{0,2}(Q_0)$, and $\|\Psi\|_{0,2} \leqslant M\|\Phi\|_{0,2}$.

Let now $u^2 = \nabla\Psi$ and $u = u^1 + u^2 + v$. It is clear that $\nabla \cdot v = 0$, $v\,|_{\Gamma_0} = (u^1 + u^2)\,|_{\Gamma_0}$, $\gamma_n(u^1 + u^2) = 0$. The function v is a solution to the problem

$$\begin{gathered} v_t - \triangle v + \nabla p = w, \qquad \nabla \cdot v = 0, \\ v(0,y) = 0, \qquad v(t,y)\,|_{\Gamma} = (u^1 + u^2)\,|_{\Gamma_0}, \qquad \gamma_n(v) = 0. \end{gathered} \tag{8.17}$$

Here $w = f - u_t^1 - u_t^2 + \triangle u^1 + \triangle u^2$. If

$$u^1 \in W_2^{1,2}(Q_0), \tag{8.18}$$

then from the relation $u^2 = \nabla N^{-1}\gamma_n u^1$ by virtue of the boundedness of operators N^{-1} and γ_n it follows that $u^2 \in W_2^{1,2}(Q_0)$, $\|u^2\|_{1,2} \leqslant M\|u^1\|_{1,2}$ and, hence, $w \in L_2(Q_0)$. From the continuity of the embedding $W_2^{1,2}(Q_0) \subset W_2^{3/4,3/2}(S_0)$, $S_0 = [0,T] \times \Gamma_0$ it follows that $(u^1 + u^2)|_\Gamma \in W_2^{3/4,3/2}(S_0)$. Therefore (see [15]), the following theorem takes place.

Theorem 8.1. *Let $u^1 \in W_2^{1,2}(Q_0)$. Then problem (8.17) has a unique solution v, p and the estimate holds:*

$$\|v\|_{1,2} + \|p\|_{0,1} \le M\Big(\|w\|_0 + \big|\big|u^1 + u^2\big|\big|_{W_2^{3/4,3/2}(S_0)}\Big) \le M\big|\big|u^1\big|\big|_{1,2}. \tag{8.19}$$

Let us find out the conditions under which on ρ condition (8.18) is fulfilled.

Theorem 8.2. *Let*

$$R^{-1/2}\rho \in W_2^1(0,T;L_2(\Omega_0)), \quad \rho \in L_2(0,T;W_2^1(\Omega_0)). \tag{8.20}$$

Then $u^1 = \nabla R^{-1}\rho \in W_2^{1,2}(Q_0)$ and the estimate holds:

$$\big|\big|u^1\big|\big|_{1,2} \leqslant M\left(\left|\left|\frac{d}{dt}R^{-1/2}\rho\right|\right|_0 + \|\rho\|_{0,1}\right). \tag{8.21}$$

Proof of Theorem 8.2. Supposing $u^1 = \nabla R^{-1/2}R^{-1/2}\rho$ and using the boundedness of the operator $\nabla R^{-1/2}$ in $L_2(\Omega_0)$ and (8.20), we get

$$\begin{gathered} \frac{d}{dt}u^1 = \frac{d}{dt}(\nabla R^{-1/2}R^{-1/2}\rho) = \nabla R^{-1/2}\frac{d}{dt}R^{-1/2}\rho \in L_2(Q_0), \\ \left|\frac{d}{dt}u^1(t,y)\right|_{0,0} \leqslant M\left|\frac{d}{dt}R^{-1/2}\rho\right|_{0,0}. \end{gathered} \tag{8.22}$$

Similarly,

$$\triangle u^1 = \triangle\nabla R^{-1}\rho = \triangle\nabla R^{-3/2}(R^{1/2}\rho). \tag{8.23}$$

By virtue of the boundedness of the operator $\nabla\nabla R^{-3/2}$ in $L_2(\Omega_0)$ and the conditions of Theorem 8.2 we have the estimate

$$\begin{aligned}||\triangle u^1|| \leqslant M||R^{1/2}\rho||_0 &= M\Big(\int_0^T |R^{1/2}\rho(t,x)|_{0,0}^2 dt\Big)^{1/2} \\ \leq M\Big(\int_0^T |\rho(t,x)|_{1,0}^2 dt\Big)^{1/2} &= M\|\rho\|_{0,1}.\end{aligned} \tag{8.24}$$

From estimates (8.22) and (8.24) the statement of Theorem 8.2 follows. Theorem 8.2 is proved. □

THEOREM 8.3. *Let a function $\rho(t,y) \in W_2^{0,1}(Q_0)$ satisfy the integral identity*

$$(\rho(t,y),\phi(y))_0 = \int_0^t (b(s,y),\nabla\phi(y))_0 ds + \int_0^t (r(s,y),\phi(y))_0 dy, \tag{8.25}$$

where $r,b \in L_2(Q_0)$ for an arbitrary function ϕ smooth and finite in Ω_0. Then $u^1 \in W_2^{1,2}(Q_0)$ and the inequality holds:

$$||u^1||_{1,2} + \|p\|_{0,1} \leqslant M(\|r\|_0 + \|b\|_0 + \|\rho\|_{0,1}). \tag{8.26}$$

Proof of Theorem 8.3. Let us show that under the conditions of Theorem 8.3 inclusions (8.20) take place. As ϕ in (8.25) belongs to $D(R)$ then from (8.25) it follows that

$$(R^{-1/2}\rho, R^{1/2}\phi)_0 = \int_0^t (R^{-1/2}\nabla b, R^{1/2}\phi)_0 ds + \int_0^t (R^{-1/2}r, R^{1/2}\phi)_0 ds. \tag{8.27}$$

As the set of functions $R^{-1/2}\phi$ is dense in $L_2(\Omega_0)$ then from (8.27) it follows that $R^{-1/2}\rho \in W_2^1(0,T;L_2(\Omega_0))$ and

$$\frac{d}{dt}R^{-1/2}\rho = \overline{R^{-1/2}\nabla} b + R^{-1/2}r. \tag{8.28}$$

The first inclusion (8.20) is established. The second one follows from the inclusion $\rho \in W_2^{0,1}(Q_0)$. Thus, the conditions of Theorem 8.2 are fulfilled. Therefore, $u^1 \in W_2^{1,2}(Q)$. Estimate (8.26) follows from (8.28) and $\rho \in W_2^{0,1}(Q_0)$. Theorem 8.3 is proved. □

From Theorems 8.1 and 8.3, there follows the following theorem.

THEOREM 8.4. *Let ρ satisfy the conditions of Theorem 8.3. Then problem (8.10) has a unique solution and the inequality holds:*

$$\|v\|_{1,2} + \|p\|_{0,1} \leq M(\|f\|_0 + \|b\|_0 + \|r\|_0 + \|\rho\|_{0,1}). \tag{8.29}$$

Establish the solvability of problem (8.6)-(8.7). We will construct the approximations $\widehat{v}^n$, $\widehat{p}^n$, using the auxiliary iterative process

$$\begin{aligned}\widehat{v}_t^{n+1} - \triangle\widehat{v}^{n+1} + \nabla\widehat{p}^{n+1} &= w^n(t,y), &\int_{\Omega_0}\widehat{p}^{n+1}(t,y)dy = 0,\\ \nabla\widehat{v}^{n+1} = \rho^n(t,y), \quad y &\in \Omega_0; &\widehat{v}^{n+1}|_{\Gamma_0} = 0.\end{aligned} \tag{8.30}$$

Here $n = 0,1,\ldots,\widehat{v}^0 = 0$, $\widehat{p}^0 = 0$, and

$$\begin{gathered} w^n(t,y) = \widehat{g} - D(\widehat{v}^n) - (\triangle - \hat{\triangle})\widehat{v}^n + (\nabla - \widehat{\nabla})\widehat{p}^n, \\ \rho^n(t,y) = (\nabla - \widehat{\nabla})\widehat{v}^n. \end{gathered} \tag{8.31}$$

Show that for any n problem (8.30) is solvable. If $\widehat{v}^n \in W_2^{1,2}(Q_0)$, $\widehat{p}^n \in W_2^{0,1}(Q_0)$ then $w^n \in L_2(Q_0)$, $\rho^n \in W_2^{0,1}(Q_0)$. To take advantage of Theorem 8.3 it is enough to check the fulfillment of condition (8.25).

Let

$$a^{ij}(t,y) = \frac{\partial \tilde{z}^i}{\partial y_j}(t;0,y), \qquad \beta_{ij}(t,x) = \frac{\partial \tilde{z}^i}{\partial x_j}(t;0,x), \qquad b_{ij}(t,y) = \beta_{ij}(t,z(t;0,y)). \tag{8.32}$$

As

$$\tilde{z}(\tau;t,y) = y + \int_t^\tau \tilde{v}(s,z(s;t,y))\,ds, \tag{8.33}$$

then

$$\begin{aligned} a_{ij}(t,y) &= \delta_{ij} + \int_0^t \frac{\partial}{\partial x_k}\tilde{v}_i\left(s,\tilde{z}(s;0,y)\frac{\partial \tilde{z}^k}{\partial y_j}(s;0,y)\right)ds = \delta_{ij} + \tilde{a}_{ij}(t,y), \\ b_{ij}(t,y) &= \delta_{ij} - \int_0^t \frac{\partial \tilde{v}_i}{\partial x_k}(s,\tilde{z}(s;t,\tilde{z}(t;0,y)))\frac{\partial \tilde{z}^k}{\partial x_i}(s;t,\tilde{z}(t,0,y))\,ds \\ &= \delta_{ij} - \int_0^t \frac{\partial \tilde{v}_i}{\partial x_k}(s,\tilde{z}(0;s,y))\frac{\partial \tilde{z}^k}{\partial x_i}(0;s,y)ds = \delta_{ij} + \tilde{b}_{ij}(t,y). \end{aligned} \tag{8.34}$$

It is clear that at $i \neq j$ $b_{ij} = -\tilde{b}_{ij}$. Using (8.5), we get

$$\begin{gathered} \widehat{\nabla}\widehat{v}(t,y) = b_{ji}(t,y)\frac{\partial \widehat{v}}{\partial y_j}(t,y), \\ \widehat{\nabla}\widehat{p}(t,y) = \left\{b_{ji}(t,y)\frac{\partial \widehat{p}}{\partial y_j}(t,y)\right\}_{i=1}^n, \\ \hat{\triangle}\widehat{v}(t,y) = \left\{b_{ji}\frac{\partial}{\partial y_j}\left[b_{li}\frac{\partial \widehat{v}_k}{\partial y_l}\right]\right\}_{k=1}^n = \left\{b_{ji}b_{li}\frac{\partial^2 \widehat{v}^k}{\partial y_i \partial y_l} + b_{ji}\frac{\partial}{\partial y_j}b_{li}\frac{\partial \widehat{v}_k}{\partial y_l}\right\}_{k=1}^n. \end{gathered} \tag{8.35}$$

From (8.32) and (8.34) it follows that

$$(\nabla - \widehat{\nabla})\widehat{v} = \tilde{b}_{ij}(t,y)\frac{\partial \widehat{v}^i}{\partial y_j}, \tag{8.36}$$

$$(\triangle - \hat{\triangle})\widehat{v} = \tilde{b}_{ji}\tilde{b}_{li}\frac{\partial^2 \widehat{v}}{\partial y_j \partial y_l} + (\tilde{b}_{jl} + b_{lj})\frac{\partial^2 \widehat{v}}{\partial y_j}\partial y_l + b_{ji}\frac{\partial}{\partial y_j}\tilde{b}_{li}\frac{\partial \widehat{v}}{\partial y_l}, \tag{8.37}$$

$$(\nabla - \widehat{\nabla})\widehat{p} = \left\{\tilde{b}_{ji}\frac{\partial \widehat{p}}{\partial y_j}\right\}_{i=1}^n. \tag{8.38}$$

From (8.36) it follows that

$$\rho^n = (\nabla - \widehat{\nabla})\widehat{v}^n = \tilde{b}_{ij}\frac{\partial \widehat{v}_i^n}{\partial y_j},$$
$$(\rho,\phi)_0^n = \left(\tilde{b}_{ij}\frac{\partial \widehat{v}_i^n}{\partial y_j},\phi\right)_0 = -\left(\widehat{v}_i^n,\frac{\partial}{\partial y_j}(\tilde{b}_{ij}\phi)\right)_0. \tag{8.39}$$

Denote $\tilde{b}_{ij,k} = (\partial/\partial y_k)\tilde{b}_{ij}$ and $(u)_{y_k} = \partial u/\partial y_k$. Differentiating with respect to t, we have

$$\begin{aligned}\frac{d}{dt}(\rho^n,\phi) &= -\left(\frac{d}{dt}\widehat{v}_i^n,(\tilde{b}_{ij}\phi)_{y_j}\right)_0 - \left(\widehat{v}_i^n,\left(\frac{d}{dt}\tilde{b}_{ij}\phi\right)_{y_j}\right)_0\\ &= -\left(\widehat{g}_i,(\tilde{b}_{ij}\phi)_{y_j}\right)_0 + \left((D(v^n))_i,(\tilde{b}_{ij}\phi)_{y_j}\right)_0 + \left((\triangle-\hat{\triangle})\widehat{v}_i^n,(\tilde{b}_{ij}\phi)_{y_j}\right)_0\\ &\quad - \left(((\nabla-\widehat{\nabla})\widehat{p}^n)_i,(\tilde{b}_{ij}\phi)_{y_j}\right)_0 - \left(\widehat{v}_i^n,\left(\frac{d}{dt}\tilde{b}_{ij}\phi\right)_{y_j}\right)\\ &= \sum_{i=1}^5 Z_i.\end{aligned} \tag{8.40}$$

It is easy to see that

$$(\tilde{b}_{ij}\phi)_{y_j} = \tilde{b}_{ij,j}\phi + \tilde{b}_{ij}\phi_{y_j},$$
$$\frac{d}{dt}\tilde{b}_{ij} = -\tilde{v}_{x_k}(t,\tilde{z}(0;t,y))\tilde{z}_{y_i}^k(0;t,y). \tag{8.41}$$

From formulas (4.7), (8.41) it follows that

$$\frac{d}{dt}(\rho^n,\phi)_0 = (q^n,\nabla\phi)_0 + (r^n,\phi)_0. \tag{8.42}$$

Here the vector functions q^n and r^n are products of smooth functions. By this they are expressed by means of $\tilde{v}$ multiplied by the derivatives up to the second order of $\widehat{v}^n$ and the first order of $\widehat{p}^n$ with respect to the spatial variables and items which are products of smooth functions and components of $\widehat{g}$. Therefore, $q^n, r^n \in L_2(Q_0)$. Besides, $\int_{\Omega_0}\rho^n(t,y)dy = 0$. This equality can be checked in the way as in (8.9), using the Ostrogradsky formula and taking into account that $\widehat{v}^n(t,y) = 0$ on Γ_0.

Thus, from Theorem 8.3 it follows that the consecutive approximations $\widehat{v}^n$, $\widehat{p}^n$ are defined for all n.

Establish the convergence of $\widehat{v}^n$, $\widehat{p}^n$. Let $\phi^{n+1} = \widehat{v}^{n+1} - \widehat{v}^n$, $\psi^{n+1} = \widehat{p}^{n+1} - \widehat{p}^n$. Then

$$\phi_y^{n+1} - \triangle\phi^{n+1} + \nabla p^{n+1} = -D(\phi^n) - (\triangle - \hat{\triangle})\phi^n + (\nabla - \widehat{\nabla})\psi^n,$$
$$\nabla\phi^{n+1} = (\nabla - \widehat{\nabla})\phi^n, \qquad \phi(0,y) = 0, \qquad \phi(t,y)|_{\Gamma_0}, \qquad \phi^n|_{\Gamma_0} = 0. \tag{8.43}$$

It is easy to see that

$$D(\phi^n) = \phi_{y_j}^{n+1}c_j, \qquad c_j(t,y) = \frac{d}{dt}z_j(0,t;x), \quad x = z(t,0,y). \tag{8.44}$$

Using Theorem 8.3, we get

$$||\phi^{n+1}||_{1,2} + ||\psi^{n+1}||_{1,2} \leqslant M\Big(||g^n||_0 + ||r^n||_0\Big). \tag{8.45}$$

Here g^n and r^n are the functions from the formula

$$\frac{d}{dt}(\rho^n,\phi)_0 = (g^n,\nabla\phi)_0 + (z^n,\phi)_0, \tag{8.46}$$

which arose in a way similar to (8.40) for $\rho^n = (\nabla - \hat{\nabla})\phi^n$.

Let us consider $g^n, z^n \in L_2(Q_0)$ in details. All terms involved in g^n which are defined by Z_3 and Z_5 from formulas (4.7) contain derivatives up to the second order of ϕ^n and the first derivatives of ψ^n multiplied by $\tilde{b}_{ij}$ or $\tilde{b}_{ij,j}$. Therefore, they are estimated by

$$||\phi^n||_{1,2} + ||\psi^n||_{0,1} \leq \max_t \max_{i,j} |\tilde{b}_{ij}|, \qquad |\tilde{b}_{ij,j}| \leqslant MT\Big(||\phi^n||_{1,2} + ||\psi^n||_{0,1}\Big), \tag{8.47}$$

because both $\tilde{b}_{ij}$ and $\tilde{b}_{ij,j}$ are integrals of smooth functions. The terms which are defined by Z_2 and Z_4, are estimated by

$$M||\phi^n||_{0,1}. \tag{8.48}$$

Using now the known estimate

$$\|u\|_{0,1} \leqslant \varepsilon\|u\|_{1,2} + c(\varepsilon)\|u\|_0, \quad \varepsilon > 0, \tag{8.49}$$

for $u \in W_2^{1,2}(Q_0)$ and (8.45) and (8.47), we get that

$$||g^n||_0 \leqslant \varepsilon||\phi^n||_{1,2} + c(\varepsilon)||\phi^n||_0 + \varepsilon||\psi^n||_{0,1}. \tag{8.50}$$

Taking into account that for $u(t,x) \in W_2^{1,2}(Q_0)$ with zero conditions on Γ_0 and at $t = 0$ the inequality (see [14])

$$|u(t,y)|_{1,t} \leqslant M\|u\|_{1,2}, \quad M \neq M(t),\ 0 \leqslant t \leqslant T, \tag{8.51}$$

takes place and integrating with respect to t on $[0,T]$, we get the estimate

$$\|u\|_{0,1} \leqslant MT^{1/2}\|u\|_{1,2}. \tag{8.52}$$

Choosing T small enough, we get the inequality

$$||g^n||_0 \leqslant \varepsilon||\phi^n||_{1,2} \tag{8.53}$$

with small $\varepsilon > 0$. The analogous estimate for z^n takes place as well.

Thus, using (8.45), (8.50), and (8.53), we have the following estimate:

$$||\phi^{n+1}||_{1,2}+||\psi^{n+1}||_{0,1}\leqslant\varepsilon\Big(||\phi^{n}||_{1,2}+||\psi^{n}||_{0,1}\Big) \tag{8.54}$$

for some $0<\varepsilon<1$. From this here the convergence of the approximations $\hat{v}^n$ to $\hat{v}\in W_1^{1,2}(Q_0)$, $\hat{p}^n$ to $\hat{p}\in W_q^{0,1}(Q_0)$, follows. Moreover, the pair $\hat{v}$, $\hat{p}$ is the solution to the problem (8.6)-(8.7), if T is small enough.

Let us establish the solvability of problem (8.6)-(8.7) on an arbitrary $[0,T]$. Let the solvability on $[0,T_0]$ be established at small $T_0>0$. Let $\hat{v}^1(t,y)$ be a solution on $[0,T_0]$.

Consider the function $v^2(t)\in W_2^{1,2}(Q_1)$, $Q_1=[T_0,2T_0]\times\Omega_0$ which is a solution to the parabolic problem on $[T_0,2T_0]$:

$$v_t^2-\triangle v^2=0,\quad (t,y)\in Q_1,\qquad v(T_0,y)=\hat{v}^1(T_0,y),\qquad v(t,y)\,|_{\Gamma_0}=0. \tag{8.55}$$

Consider problem (8.6)-(8.7) on $[T_0,2T_0]$ in which the zero condition at $t=0$ is replaced by the condition $\hat{v}(T_0,y)=\hat{v}^1(T_0,y)$. Believing $\hat{v}=\hat{v}^3+v^2$, we have that $\hat{v}^3$ is a solution to problem (8.6)-(8.7) on $[T_0,2T_0]$ with the zero boundary and initial conditions and the right-hand side part $\hat{g}-v_t^2+\triangle v^2$. Its solvability is established in a similar to the case $[0,T_0]$ way, using the change $t=t'+T_0$. Supposing $\hat{v}=\hat{v}^3+v^2$ on $[T_0,2T_0]$ and $\hat{v}=\hat{v}^1$ on $[0,T_0]$, we get the solution $\hat{v}(t,y)$ on $[0,2T_0]$. To going on with this procedure on the whole interval $[0,T]$, we get the solvability of the problem on $[0,T]$.

Thus, we have proved the following fact.

Theorem 8.5. *Let $\hat{g}\in L_2(Q_0)$. Then problem (8.4) has a unique solution and the estimate holds:*

$$\|\hat{v}\|_{1,2}+\|\hat{p}\|_{0,1}\leqslant M\|\hat{g}\|_0. \tag{8.56}$$

By means of the change of the variable (8.3) from this theorem the following theorem follows.

Theorem 8.6. *Let $g\in L_2(Q)$. Then problem (8.1)-(8.2) has a unique solution and the estimate holds:*

$$\|v\|_{1,2}+\|p\|_{0,1}\leqslant M\|g\|_0. \tag{8.57}$$

The norms in the last inequality are taken for the domain Q.

Let us establish now the density of $R(\bar{L})$ in E^*. By virtue of the density of $L_{2,\sigma}(Q)$ in E^* it is sufficiently to show that for any $f\in L_{2,\sigma}(Q)$ there exist such a sequence $f^n=\bar{L}v^n$ that $\|f^n-f\|_{E^*}\to 0$ as $n\to+\infty$. Let $f\in L_{2,\sigma}(Q)$. By virtue of Theorem 8.6 the problem (8.1)-(8.2) has a unique solution (v,p) and $v\in W_2^{1,2}(Q)$. Let Q be a star domain with respect to some point $(t^*,x^*)\in Q$ and σ_ε be a homothety transformation with the coefficient $1-\varepsilon$ of Q into itself with respect to (t^*,x^*), so that $\sigma_\varepsilon(t,x)=\varepsilon(t^*,x^*)+(1-\varepsilon)(t,x)$. We will assume that the functions f, v are defined on a domain Q_1 more wide with respect to t which is obtained as the evolution of $\widehat{\Omega}_0$ alongside the trajectories of the field of velocities $\tilde{v}$ on an interval $T_1=T+\delta$, $\delta>0$. As the values of the functions f, v at $t>T$ are not important for us we assume that $v(t,x)=0$ at $t>T+\delta/2$. One can get it easily by the multiplying v by a suitable "cutting" function. Expand by means of zero the function v outside Q_1 on R^{n+1}, preserving the previous denotation for the expansion. Let $v^\gamma(t,x) = v(\sigma_\gamma(t,x))$. Define the regularization $v^{\varepsilon,\gamma}$ of the functions v^γ by the formula

$$\begin{aligned} v^{\varepsilon,\gamma}(t,x) &= \int_{R^{n+1}} \varepsilon^{-1}K\big(\varepsilon^{-1}(t-\tau),\varepsilon^{-1}(x-y)\big)v^\gamma(\tau,y)d\tau\,dy \\ &\equiv \int_{R^{n+1}} K^\varepsilon(t-\tau,x-y)v^\gamma(\tau,y)d\tau\,dy. \end{aligned} \tag{8.58}$$

Here $K(t,x)$ is an infinitely differentiable kernel of averaging with a support in the unite ball of R^{n+1},

$$\int_{R^{n+1}} K(t,x)dt\,dx = 1. \tag{8.59}$$

It is clear that at every t the support of $v^{\varepsilon,\gamma}(t,x)$ lays in Ω_t and the functions $v^{\varepsilon,\gamma}(t,x)$ and $v^\gamma(t,x)$ are solenoidal if $\gamma > 0$ and $\varepsilon > 0$ are small enough. Besides, $v^\gamma \in W_2^{1,1}(Q)$ and $v^{\varepsilon,\gamma}$ are infinitely differentiable.

Using differentiation and integration by parts, we have

$$\begin{aligned} \frac{\partial}{\partial x_i}v^{\varepsilon,\gamma}(t,x) &= \int_{R^{n+1}} \frac{\partial}{\partial x_i}K^\varepsilon(t-\tau,x-y)v^\gamma(\tau,y)d\tau\,dy \\ &= -\int_{R^{n+1}} \frac{\partial}{\partial y_j}K^\varepsilon(t-\tau,x-y)v^\gamma(\tau,y)d\tau\,dy \\ &= \int_{R^{n+1}} K^\varepsilon(t-\tau,x-y)\frac{\partial}{\partial y_j}v^\gamma(\tau,y)d\tau\,dy. \end{aligned} \tag{8.60}$$

On analogy (here the condition $v(x,T_1) = 0$ is used)

$$v_t^{\varepsilon,\gamma}(t,x) = \int_{R^{n+1}} K^\varepsilon(t-\tau,x-y)v_\tau^\gamma(\tau,y)d\tau\,dy. \tag{8.61}$$

It is easy to show that

$$||v - v^\gamma||_{0,1} = O(\gamma), \qquad ||v_t - v_t^\gamma||_{0,-1} = O(\gamma). \tag{8.62}$$

From the convergence of the averagings it follows that

$$||v_t^{\varepsilon,\gamma} - v_t^\gamma||_{E^*} = O(\varepsilon), \qquad ||v^{\varepsilon,\gamma} - v^\gamma||_{1,0} = O(\varepsilon). \tag{8.63}$$

Let $\varepsilon_n \to 0$ as $n \to +\infty$, $v^{n,\gamma} \equiv v^{\varepsilon_n,\gamma}$, $Lv^{n,\gamma} = f^{n,\gamma}$. Then for $h \in E$

$$\langle Lv^{n,\gamma},h\rangle = (v_t^{n,\gamma} - \triangle v^{n,\gamma},h) = (v_t^{n,\gamma},h) + (\nabla v^{n,\gamma},\nabla h^{n,\gamma}). \tag{8.64}$$

Define $f^\gamma \in E^*$ by the formula $\langle f^\gamma, h\rangle = (v_t^\gamma, h) + (\nabla v^\gamma, \nabla h)$, $h \in E$. Here $(,)$ means the scalar product in $L_2(Q)$. Then

$$\begin{aligned} ||f^{n,\gamma} - f^\gamma||_{E^*} &= \sup_{\|h\|_E=1} \left| (v_t^{n,\gamma} - v_t^\gamma, h) + (\nabla v^{n,\gamma} - v^\gamma, h) \right| \\ &\leqslant ||v_t^{n,\gamma} - v_t^\gamma||_{E^*} + ||v^{n,\gamma} - v^\gamma||_{0,1}. \end{aligned} \tag{8.65}$$

From the definition of v it follows that

$$(v_t, h) + (\nabla v, \nabla h) = (f, h). \tag{8.66}$$

Therefore,

$$||f - f^\gamma||_{E^*} \leqslant M\Big(||v_t - v_t^\gamma||_{E^*} + ||v - v^\gamma||_{0,1}\Big). \tag{8.67}$$

Using this inequality, we have

$$\begin{aligned} ||f - f^{\gamma,n}||_{E^*} &\leqslant ||f - f^\gamma||_{E^*} + ||f^\gamma - f^{\gamma,n}||_{E^*} \\ &\leqslant M\Big(||v_t - v_t^\gamma||_{E^*} + ||v - v^\gamma||_{0,1} + ||v_t^\gamma - v_t^{n,\gamma}||_{E^*} + ||v^\gamma - v^{n,\gamma}||_{0,1}\Big). \end{aligned} \tag{8.68}$$

From (8.65), (8.67), and (8.68) it follows that choosing $\gamma > 0$ small enough and n large enough, it is possible to make $\| f - f^{\gamma,n}\|_{E^*}$ small enough.

The density of $R(\bar{L})$ in E^* for the star domain Q is established.

Let now Q not be a star domain. As the lateral side of Q is smooth, by means of sufficiently fine splitting $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_N \equiv T$ it is possible to get the decomposition of Q by the layers

$$Q_i = \{(t,x) : (t,x) \in Q,\ t \in [t_i, t_{i+1}]\}, \quad i = 0,\dots,N-1, \tag{8.69}$$

so that both Q_i and $Q_{ih} = \{(t,x) : (t,x) \in Q,\ t \in [t_i - h, t_{i+1} + h]\}$ are star domains for sufficient small $h > 0$. Let $\sum_{i=1}^N \varphi_i(t) \equiv 1$ be the smooth decomposition of $[0,T]$ so that $\operatorname{supp}\varphi_i(t) \in [t_i - h, t_{i+1} + h]$. Let $f \in D$ and $f^i = \varphi_i(t) f$. Obviously, $f^i \in E^*$ and $\operatorname{supp} f^i \in Q_{ih}$. Since Q_{ih} is a star domain, then reasoning as above, we get that the equation $\bar{L}u^i = f^i$ has a unique solution u^i and

$$||u_t^i||_{E^*} + ||u^i||_E \leq M||f^i||_{E^*}. \tag{8.70}$$

It follows from this that $u = \sum_{i=1}^N u^i$ is a solution to the equation $\bar{L}u = f$ for $f \in D$ and

$$||u_t||_{E^*} + \|u\|_E \leq M\| f\|_{E^*}. \tag{8.71}$$

Let now f be an arbitrary element from E^*. Since D is dense in E^*, we can choose a sequence f_n which converges to f in E^* by $n \to \infty$. Taking into account the estimate (8.71) for solutions u_n to the equation $\bar{L}u^i = f^i$ and passing to the limit by $n \to \infty$ we get that the equation $\bar{L}u = f$ has the solution for an arbitrary $f \in E^*$.

Theorem 3.8 is proved.

Remark 8.7. From Theorems 8.5-8.6 it follows that if v is a solution to the equation $\bar{L}v = f$ then there exists such a sequence $v^n \in W_2^{1,2}(Q)$ that $f^n = Lv^n$ converges to f in E^* and the identity holds:

$$\langle \bar{L}v, h\rangle = \langle v_t, h\rangle + \langle \nabla v, h\rangle, \quad h \in E. \tag{8.72}$$

Acknowledgment

The research was partially supported by Russian Foundation of Basical Researches (Grant 01-01-00425).

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