MULTIPLE POSITIVE SOLUTIONS OF SINGULAR $p$-LAPLACIAN PROBLEMS BY VARIATIONAL METHODS

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We obtain multiple positive solutions of singular $p$-Laplacian problems using variational methods. The techniques are applicable to other types of singular problems as well.

1. Introduction

We consider the singular quasilinear elliptic boundary value problem

$$
-\Delta_p u = a(x)u^{-\gamma} + \lambda f(x,u) \quad \text{in } \Omega,
$$
$$
u > 0 \quad \text{in } \Omega,
$$
$$
u = 0 \quad \text{on } \partial \Omega,
$$

(1.1)

where $\Omega$ is a bounded $C^2$ domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $1 < p < \infty$, $a \geq 0$ is a nontrivial measurable function, $\gamma > 0$ is a constant, $\lambda > 0$ is a parameter, and $f$ is a Carathéodory function on $\Omega \times [0, \infty)$ satisfying

$$
\sup_{(x,t) \in \Omega \times [0,T]} |f(x,t)| < \infty \quad \forall T > 0.
$$

(1.2)

The semilinear case $p = 2$ with $\gamma < 1$ and $f = 0$ has been studied extensively in both bounded and unbounded domains (see [5, 6, 7, 10, 11, 12, 14, 20] and their references). In particular, Lair and Shaker [11] showed the existence of a unique (weak) solution when $\Omega$ is bounded and $a \in L^2(\Omega)$. Their result was extended to the sublinear case $f(t) = t^\beta$, $0 < \beta \leq 1$ by Shi and Yao [15] and Wiegner [18]. In the superlinear case $1 < \beta < 2^* - 1$ and for small $\lambda$, Coclite and Palmieri [4] obtained a solution when $a = 1$ and Sun et al. [16] obtained two solutions using the Ekeland’s variational principle for more general $a$’s. Zhang [19] extended their multiplicity result to more general superlinear terms $f(t) \geq 0$ using critical point theory on closed convex sets. The ODE case $n = 1$ was studied by Agarwal and O’Regan [1] using fixed point theory and by Agarwal et al. [2] using variational methods. The purpose of the present paper is to treat the general quasilinear case $p \in (1, \infty)$, $\gamma \in (0, \infty)$, and $f$ is allowed to change sign. We use a simple cutoff argument and only the basic critical point theory. Our results seem to be new even for $p = 2$. 

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First we assume

\((H_1)\) \(\exists \varphi \geq 0\) in \(C_0^1(\Omega)\) and \(q > n\) such that \(a\varphi^{-\gamma} \in L^q(\Omega)\).

This does not require \(\gamma < 1\) as usually assumed in the literature. For example, when \(\Omega\) is the unit ball, \(a(x) = (1 - |x|^2)^\sigma, \sigma \geq 0,\) and \(\gamma < \sigma + 1/n,\) we can take \(\varphi(x) = 1 - |x|^2\) and \(q < 1/(\gamma - \sigma)\) (resp., \(q\) with no additional restrictions) if \(\gamma > \sigma\) (resp., \(\gamma \leq \sigma\)).

**Theorem 1.1.** If \((H_1)\) and (1.2) hold and \(f \geq 0\), then there exists \(\lambda_0 > 0\) such that problem (1.1) has a solution \(\forall \lambda \in (0, \lambda_0)\).

**Corollary 1.2.** Problem (1.1) with \(f = 0\) has a solution if \((H_1)\) holds.

Next we allow \(f\) to change sign, but strengthen \((H_1)\) to

\((H_2)\) \(a \in L^\infty(\Omega)\) with \(a_0 := \inf_{\Omega} a > 0\) and \(\gamma < 1/n\).

This implies that \(a\varphi^{-\gamma} \in L^q(\Omega)\) for any \(\varphi\) whose interior normal derivative \(\partial \varphi / \partial \nu > 0\) on \(\partial \Omega\) and \(q < 1/\gamma\).

**Theorem 1.3.** If \((H_2)\) and (1.2) hold, then \(\exists \lambda_0 > 0\) such that problem (1.1) has a solution \(\forall \lambda \in (0, \lambda_0)\).

Finally we assume that \(f\) is \(C^1\) in \(t\), satisfies

\[ |f_t(x,t)| \leq C(t^{r-2} + 1) \quad (1.3) \]

for some \(2 \leq r < p^*\), and \(p\)-superlinear:

\[ 0 < \theta F(x,t) \leq t f(x,t), \quad t \text{ large} \quad (1.4) \]

for some \(\theta > p\). Here \(p^* = np/(n - p)\) (resp., \(\infty\)) if \(p < n\) (resp., \(p \geq n\)) is the critical Sobolev exponent and \(C\) denotes a generic positive constant.

**Theorem 1.4.** If \(p \geq 2, \ (H_1), \ (1.3), \) and (1.4) hold, and \(f \geq 0\), then \(\exists \lambda_0 > 0\) such that problem (1.1) has two solutions \(\forall \lambda \in (0, \lambda_0)\).

**Theorem 1.5.** If \(p \geq 2\) and \((H_2), \ (1.3), \) and (1.4) hold, then \(\exists \lambda_0 > 0\) such that problem (1.1) has two solutions \(\forall \lambda \in (0, \lambda_0)\).

2. Preliminaries on the \(p\)-Laplacian

Consider the problem

\[-\Delta_p u = g(x) \quad \text{in} \ \Omega, \]
\[ u = 0 \quad \text{on} \ \partial \Omega. \quad (2.1)\]

**Proposition 2.1.** If \(g \in L^q(\Omega)\) for some \(q > n\), then (2.1) has a unique weak solution \(u \in C_0^1(\Omega)\). If, in addition, \(g \geq 0\) is nontrivial, then

\[ u > 0 \quad \text{in} \ \Omega, \quad \partial u / \partial \nu > 0 \quad \text{on} \ \partial \Omega. \quad (2.2)\]
Proof. The existence of a unique solution $u \in W^{1,p}_0(\Omega)$ is well-known. The problem

\[-\Delta v = g(x) \quad \text{in } \Omega,
\]
\[v = 0 \quad \text{on } \partial \Omega
\]

has a unique solution $v \in W^{2,q}(\Omega) \hookrightarrow C^{1,a}(\overline{\Omega})$, $a = 1 - n/q$. Then $u$ satisfies

\[
\text{div} \left( |\nabla u|^{p-2} \nabla u - G(x) \right) = 0 \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial \Omega,
\]

where $G = \nabla v \in C^{a}(\overline{\Omega})$, and $u$ is bounded by Guedda and Véron [8] since $q > n/p$ if $p \leq n$, so $u \in C^{1}(\overline{\Omega})$ by Lieberman [13]. The rest now follows from Vázquez [17]. \qed

3. Proofs of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Since $a \in L^q(\Omega)$ by (H1), the problem

\[-\Delta_p v = a(x) \quad \text{in } \Omega,
\]
\[v = 0 \quad \text{on } \partial \Omega
\]

has a unique positive solution $v \in C^{1}(\Omega)$ with $\partial v/\partial v > 0$ on $\partial \Omega$ by Proposition 2.1. Then $\inf_{\Omega}(v/\varphi) > 0$ and hence $a v^{-\gamma} \in L^q(\Omega)$. Fix $0 < \varepsilon \leq 1$ so small that $u := \varepsilon^{1/(p-1)} v \leq 1$. Then

\[-\Delta_p u - a(x) u^{-\gamma} - \lambda f(x,u) \leq -(1 - \varepsilon) a(x) \leq 0,
\]

so $u$ is a subsolution of (1.1).

Since $a u^{-\gamma} \in L^q(\Omega)$, the problem

\[-\Delta_p u = a(x) u(x)^{-\gamma} + 1 \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \partial \Omega
\]

has a unique solution $\overline{u} \in C^{1}(\Omega)$ by Proposition 2.1, and $\overline{u} \geq u$ since

\[-\Delta_p \overline{u} \geq a(x) \geq \varepsilon a(x) = -\Delta_p u.
\]

Then

\[-\Delta_p \overline{u} - a(x) \overline{u}^{-\gamma} - \lambda f(x,\overline{u}) \geq 1 - \lambda \sup_{x \in \Omega, t \leq \max_{\Omega} \overline{u}} f(x,t),
\]

so $\exists \lambda_0 > 0$ such that \( \overline{u} \) is a supersolution of (1.1) $\forall \lambda \in (0,\lambda_0)$ by (1.2).
Let
\[ g_{λ,π}(x,t) = \begin{cases} a(x)\bar{u}(x)^{-γ} + λ f(x,\bar{u}(x)), & t > \bar{u}(x) \\ a(x)t^{-γ} + λ f(x,t), & u(x) \leq t \leq \bar{u}(x) \\ a(x)u(x)^{-γ} + λ f(x,u(x)), & t < u(x) \end{cases} \]
(3.6)

\[ G_{λ,π}(x,t) = \int_0^t g_{λ,π}(x,s)ds, \]

\[ Φ_{λ,π}(u) = \int_{Ω} |∇u|^p - pG_{λ,π}(x,u), \quad u ∈ W_0^{1,p}(Ω). \]

Since
\[ 0 ≤ g_{λ,π}(x,t) ≤ a(x)u(x)^{-γ} + λ \sup_{x ∈ Ω, t ≤ \max u} f(x,t), \quad ∀ (x,t) ∈ Ω × ℝ, \] (3.7)

and \( au^{-γ} ∈ L^q(Ω) \), \( Φ_{λ,π} \) is bounded from below and has a global minimizer \( u_0 \), which then is a solution of (1.1) in the order interval \([u,\bar{u}]\). □

Proof of Theorem 1.3. The problem
\[ -Δ_p v = a_0 \quad \text{in } Ω, \]
\[ v = 0 \quad \text{on } ∂Ω \]
(3.8)

has a unique positive solution \( v ∈ C^1_0(Ω) \) with \( ∂v/∂ν > 0 \) on \( ∂Ω \). Fix \( 0 < ε < 1 \) so small that \( u := ε^{1/(p-1)} v ≤ 1 \). Then
\[ -Δ_p u - a(x)u^{-γ} - λ f(x,u) ≤ -(1 - ε)a_0 + λ \sup_{x ∈ Ω, t ≤ \max u} |f(x,t)|, \] (3.9)

so \( ∃ λ_0 > 0 \) such that \( u \) is a subsolution of (1.1) \( ∀ λ ∈ (0,λ_0) \). The rest of the proof now proceeds as above. □

4. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Define a Carathéodory function on \( Ω × ℝ \) by
\[ g_{λ}(x,t) = \begin{cases} a(x)t^{-γ} + λ f(x,t), & t ≥ u(x) \\ a(x)u(x)^{-γ} + λ f(x,u(x)), & t < u(x) \end{cases} \]
(4.1)

and consider the problem
\[ -Δ_p u = g_{λ}(x,u) \quad \text{in } Ω, \]
\[ u = 0 \quad \text{on } ∂Ω. \]
(4.2)

Every solution of (4.2) is \( ≥ u \) and hence also a solution of (1.1). By (1.3),
\[ 0 ≤ g_{λ}(x,t) ≤ a(x)u(x)^{-γ} + λC((t^+)^{p-1} + 1), \quad ∀ (x,t) ∈ Ω × ℝ \] (4.3)
so solutions of (4.2) are the critical points of the $C^1$ functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - pG_\lambda(x,u), \quad u \in W^{1,p}_0(\Omega),$$  

(4.4)

where

$$G_\lambda(x,t) = \int_0^t g_\lambda(x,s)ds.$$  

Since $u_0$ solves

$$-\Delta_p u = g_{\lambda,\pi}(x,u_0(x)) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega$$  

(4.5)

and $g_{\lambda,\pi}(\cdot, u_0(\cdot)) \in L^q(\Omega)$ by (3.7), $u_0 \in C^1_0(\Omega)$ by Proposition 2.1. Note that, with a possibly smaller $\lambda_0$, $2\overline{u}$ is also a supersolution of (1.1) $\forall \lambda \in (0, \lambda_0)$. We assume that $u_0$ is the global minimizer of the corresponding functional $\Phi_{\lambda,2\overline{u}}$ also, for otherwise we are done. Since

$$u_0 \leq \overline{u} < 2\overline{u} \quad \text{in } \Omega, \quad \partial u_0/\partial \nu \leq \partial \overline{u}/\partial \nu < \partial (2\overline{u})/\partial \nu \quad \text{on } \partial \Omega,$$  

(4.6)

$\Phi_{\lambda,2\overline{u}} = \Phi_\lambda$ in a $C^1_0(\Omega)$-neighborhood of $u_0$, so $u_0$ is a local minimizer of $\Phi_\lambda|_{C^1_0(\Omega)}$, and hence also of $\Phi_\lambda$ by Brezis and Nirenberg [3] for $p = 2$ and by Guo and Zhang [9] for $p > 2$. The mountain pass lemma now gives a second critical point as (1.4) implies that $\Phi_\lambda$ satisfies the (PS) condition and $\Phi_\lambda(tu) \to -\infty$ as $t \to \infty$. □

Proof of Theorem 1.5 is similar and therefore omitted.

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