BOUNDARY VALUE PROBLEMS FOR ANALYTIC FUNCTIONS IN THE CLASS OF CAUCHY-TYPE INTEGRALS WITH DENSITY IN $L^{p(\cdot)}(\Gamma)$

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We study the Riemann boundary value problem $\Phi^+(t) = G(t)\Phi^-(t) + g(t)$, for analytic functions in the class of analytic functions represented by the Cauchy-type integrals with density in the spaces $L^{p(\cdot)}(\Gamma)$ with variable exponent. We consider both the case when the coefficient G is piecewise continuous and the case when it may be of a more general nature, admitting its oscillation. The explicit formulas for solutions in the variable exponent setting are given. The related singular integral equations in the same setting are also investigated. As an application there is derived some extension of the Szegö-Helson theorem to the case of variable exponents.

1. Introduction

Let Γ be an oriented rectifiable closed simple curve in the complex plane $\mathbb{C}$. We denote by D^+ and D^- the bounded and unbounded component of $\mathbb{C} \setminus \Gamma$, respectively.

The main goal of the paper is to investigate the Riemann problem: find an analytic function Φ on the complex plane cut along Γ whose boundary values satisfy the conjugacy condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{1.1}$$

where G and g are the given functions on Γ, and Φ^+, and Φ^- are boundary values of Φ on Γ from inside and outside Γ, respectively. This problem is also known as the problem of linear conjugation.

We seek the solution of (1.1) in the class of analytic functions represented by the Cauchy-type integral with density in the spaces $L^{p(\cdot)}(\Gamma)$ with variable exponent assuming that g belongs to the same class. We consider the cases when the coefficient G is continuous or piecewise continuous as well as the case of oscillating coefficient. The solvability conditions are derived and in all the cases of solvability the explicit formulas are given. The related boundary singular integral equations in $L^{p(\cdot)}(\Gamma)$ are treated. The solution of the boundary value problem (BVP) (1.1) allows us to obtain the weight results for Cauchy singular integral operator in $L^{p(\cdot)}(\Gamma)$-spaces, among them some extension of the well-known Helson-Szegö theorem.

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The problem (1.1) is first encountered in Riemann [36]. Important results on which the posterior solution of problem (1.1) was based, were obtained by Yu. Sokhotski, D. Hilbert, I. Plemely, and T. Carleman. The complete solution of the Riemann problem was first given in the works of Gakhov [7, 8] and Muskhelishvili [27, 28]; we refer also to the works [14, 15, 16, 17] on investigation of the last decades of the Riemann problem in L_p-spaces (with constant p).

The generalized Lebesgue spaces, that is, Lebesgue spaces with variable exponent, have been intensively studied since 1970s. One may see an evident rise of interest in these spaces during the last decade, especially in the last years. The interest was aroused, apart from mathematical curiosity, by possible applications to models with the so-called non-standard growth in fluid mechanics, elasticity theory, in differential equations (see, e.g., [5, 37] and the references therein).

The development of the operator theory in the spaces $L^{p(\cdot)}$ encountered essential difficulties from the very beginning. For example, the translation operator and the convolution operators are not in general bounded in these spaces. The boundedness of the maximal operator was recently proved by Diening [4]. See further results in [2, 30]. There is also an evident progress in this direction for singular operators [5, 20].

As is known, for applications to singular integral equations and BVPs the weighted boundedness of singular operators is required. The weighted estimates in $L^{p(\cdot)}$-spaces with power weight were proved for the maximal operator on bounded domains in [21] and for singular operators in [20]. It is worthwhile mentioning that the Fredholmness criteria for singular integral equations with Cauchy kernel were proved in [19] for the spaces $L^{p(\cdot)}$ and in [12] for such spaces with power weight.

2. Preliminaries

Throughout the paper in all statements we suppose that $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell\}$, with an arc-length s, is a simple closed rectifiable curve. Let a measurable $p : \Gamma \to [1,\infty)$. The $L^{p(\cdot)}$-space on Γ may be introduced via the modular

$$I_p(f) = \int_\Gamma |f(t)|^{p(t)}|dt| = \int_0^\ell |f[t(s)]|^{p[t(s)]}ds. \tag{2.1}$$

By $L^{p(\cdot)} = L^{p(\cdot)}(\Gamma)$ we denote the set of all measurable complex-valued functions f on Γ such that $I_p(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$.

This set becomes a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{2.2}$$

Sometimes norm (2.2) is called Luxemburg norm because of a similar norm for Orlicz spaces [24]. However, just in the form (2.2), this norm for the spaces $L^{p(\cdot)}$ was introduced before Luxemburg by Nakano [29]. The spaces $L^{p(\cdot)}([0,1])$ probably first appeared in [29] as an example illustrating the theory of modular spaces developed by Nakano.

The spaces $L^{p(\cdot)}$ were studied by Orlicz [31] for the first time in 1931. They are the special cases of the Musielak-Orlicz spaces generated by Young functions with parameter, see [25, 26, 33, 34].

However, it was namely the specifics of the spaces $L^{p(\cdot)}$ which attracted an interest of many researchers and allowed to develop a rather rich basic theory of these spaces, this interest being also roused by applications in various areas.

Meanwhile, the norm of the type (2.2), as well as a similar norm for the Orlicz spaces is nothing else but the realization of a general norm for "normalizable" topological spaces provided by the famous Kolmogorov theorem. This theorem runs as follows, see [11, Chapter 4] and [22].

Theorem 2.1 (Kolmogorov theorem). *A Hausdorff linear topological space X admits a norm if and only if it has a convex bounded neighbourhood of the null-element and in this case Minkowsky functional of this neighbourhood is a norm.*

We remind that the Minkowsky functional of a set $U \subset X$ is the functional $M_U(x)$, $x \in X$, defined as

$$M_U(x) = \inf\left\{\lambda : \lambda > 0,\ \frac{1}{\lambda}x \in U\right\}, \quad x \in X, \tag{2.3}$$

so that the infinum of $I_p(f/\lambda)$ is nothing else but the Minkowsky functional of $f \in X = L^{p(\cdot)}$ related to the set $U = \{f : I_p(f) \leq 1\}$.

Therefore, there are many more reasons to call the norm (2.2) the *Kolmogorov-Minkowsky norm*.

If

$$1 < \underline{p} = \operatorname{ess\,inf} p(t), \qquad \overline{p} = \operatorname{ess\,sup} p(t) < \infty, \tag{2.4}$$

then the space $L^{p(\cdot)}$ is reflexive. Its associate space coincides, up to equivalence, with the space $L^{q(\cdot)}$, where $1/q(t) + 1/p(t) = 1$.

In the sequel, by $\mathcal{P}(\Gamma)$, or simply by $\mathcal{P}$, we denote the class of functions p measurable with respect to the arc measure and satisfying condition (2.4). Under this condition the space $L^{p(\cdot)}$ coincides with the space

$$\left\{f(t) : \left|\int_\Gamma f(t)g(t)dt\right| < \infty \ \forall g \in L^{q(\cdot)}(\Gamma)\right\} \tag{2.5}$$

up to equivalence of the norms

$$\|f\|_{p(\cdot)} \sim \sup_{\|g\|_{L^{q(\cdot)}(\Gamma)} \leq 1} \left|\int_\Gamma f(t)g(t)dt\right|, \tag{2.6}$$

see [23]. There holds the following generalization of the Hölder inequality:

$$\left|\int_\Gamma f(t)g(t)dt\right| < c_0\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}, \tag{2.7}$$

where $c_0 = 1 + 1/\underline{p} + 1/\overline{p}$. We refer also to [6, 23] for other properties of the spaces $L^{p(\cdot)}$.

Note that

$$\min\left(\ell^{1/\underline{p}}, \ell^{1/\overline{p}}\right) \leq \|1\|_{p(\cdot)} \leq \max\left(\ell^{1/\underline{p}}, \ell^{1/\overline{p}}\right). \tag{2.8}$$

If $p(t) \le p_1(t)$, then

$$\|f\|_{p(\cdot)} \le (1+\ell)\|f\|_{p_1(\cdot)}. \tag{2.9}$$

In the sequel we need the following condition on $p(t)$:

$$|p(t_1) - p(t_2)| \le \frac{A}{\ln 1/|t_1 - t_2|}, \quad |t_1 - t_2| \le \frac{1}{2},\ t_1, t_2 \in \Gamma, \tag{2.10}$$

where $A > 0$ does not depend on t_1 and t_2, or on the function $p_0(s) = p[t(s)]$, where

$$|p_0(s_1) - p_0(s_2)| \le \frac{A}{\ln 1/|s_1 - s_2|}, \quad |s_1 - s_2| \le \frac{1}{2},\ s_1, s_2 \in [0,\ell]. \tag{2.11}$$

Since $|t(s_1) - t(s_2)| \le |s_1 - s_2|$, condition (2.10) always implies (2.11). Inversely, (2.11) implies (2.10) if, for instance, there exists a $\gamma > 0$ such that $|s_1 - s_2| \le C|t_1 - t_2|^\gamma$ with some $C > 0$. Therefore, conditions (2.10) and (2.11) are equivalent, for example, on curves with the so-called chord condition.

Let ρ be a measurable, almost everywhere positive function on Γ. By $L_\rho^{p(\cdot)}(\Gamma)$ we denote the Banach space of functions f for which

$$\|f\|_{p(\cdot),\rho} = \|\rho f\|_{p(\cdot)} < \infty. \tag{2.12}$$

One of the main tools of our investigation is the Cauchy singular integral

$$(S_\Gamma f)(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)d\tau}{\tau - t}, \quad t \in \Gamma,\ f \in L^1(\Gamma). \tag{2.13}$$

In the case where the operator $S_\Gamma : f \to S_\Gamma f$ is bounded from the space $L^{p(\cdot)}(\Gamma)$ into the space $L^{p_1(\cdot)}(\Gamma)$ we denote its norm as $\|S_\Gamma\|_{p(\cdot)\to p_1(\cdot)}$ and as $\|S\|_{p(\cdot)}$ when $p(t) \equiv p_1(t)$.

Let

$$\mathcal{K}_\rho^p(\Gamma) = \left\{\Phi(z) : \Phi(z) = (K_\Gamma\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z},\ z \notin \Gamma \text{ with } \varphi \in L_\rho^{p(\cdot)}(\Gamma)\right\}, \tag{2.14}$$

and let

$$\widetilde{\mathcal{K}}_\rho^p(\Gamma) = \{\Phi(z) : \Phi(z) = \Phi_0(z) + \text{const},\ \Phi_0 \in \mathcal{K}_\rho^p(\Gamma)\}. \tag{2.15}$$

We write $\mathcal{K}^p(\Gamma) = \mathcal{K}_\rho^p(\Gamma)$ and $\widetilde{\mathcal{K}}_\rho^p(\Gamma) = \widetilde{\mathcal{K}}^p(\Gamma)$ in the case $\rho(t) \equiv 1$.

For a simply connected domain D, bounded by a rectifiable curve Γ, by $E^\delta(D)$, $\delta > 0$, we denote the Smirnov class of functions $\Phi(z)$ analytic in D for which

$$\sup_r \int_{\Gamma_r} |\Phi(z)|^\delta |dz| < \infty, \tag{2.16}$$

where Γ_r is the image of $\gamma_r = \{z : |z| = r\}$ under conformal mapping of $U = \{z : |z| < 1\}$ onto D. (When D is an infinite domain, then the conformal mapping means the one which transforms 0 into infinity.)

A function $\Phi \in E^\delta(D)$ possesses almost everywhere angular boundary values on Γ and the boundary function belongs to $L^\delta(\Gamma)$ (see [35, page 205]).

It is known that $E^1(D)$ coincides with the class of analytic functions represented by Cauchy integrals. Therefore for the function $\Phi(z)$, which is analytic on the plane cutting along closed curve Γ and belongs to $E^1(D^\pm)$,

$$\Phi(z) = K_\Gamma(\Phi^+ - \Phi^-) \tag{2.17}$$

(see, e.g., [16, page 98]).

We make use of the following notations:

$$\begin{aligned} \mathcal{R}^{p(\cdot)} &= \{\Gamma : S_\Gamma \text{ is bounded in } L^{p(\cdot)}(\Gamma)\}, \\ W^{p(\cdot)}(\Gamma) &= \left\{\rho : \rho S_\Gamma \frac{1}{\rho} \text{ is bounded in } L^{p(\cdot)}(\Gamma)\right\}. \end{aligned} \tag{2.18}$$

As shown in [20] the following statement is true.

Proposition 2.2. *Let Γ be a Lyapunov curve or a curve of bounded turning (Radon curve) without cusps. Assume that $p \in \mathcal{P}$ and condition (2.10) is satisfied. Then the weight*

$$w(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k}, \tag{2.19}$$

where t_k are distinct points of Γ, belongs to $W^{p(\cdot)}(\Gamma)$ if and only if

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)}. \tag{2.20}$$

3. Some properties of the Cauchy-type integrals with densities in $L^{p(\cdot)}(\Gamma)$

In this section, we present some auxiliary results which provide an extension of known properties of the Cauchy singular integrals in the Lebesgue spaces with constant p to the case of variable $p(\cdot)$.

Proposition 3.1. *Let $p \in \mathcal{P}$ and let Γ be a closed Jordan curve. Then the set of rational functions with a unique pole inside of Γ is dense in $L^{p(\cdot)}(\Gamma)$.*

The validity of this statement follows from the denseness in $L^{p(\cdot)}(\Gamma)$ of the set of continuous functions and the fact that any continuous function may be approximated in $C(\Gamma)$ by rational functions, whatsoever Jordan curve Γ we have according to the Walsh theorem (see, for instance, [40, Chapter II, Theorem 7]).

Proposition 3.2. *Let Γ be a rectifiable Jordan curve, let $p(t) \in \mathcal{P}$. If $1/\rho \in L^{q(\cdot)}(\Gamma)$, then the operator S_Γ is continuous in measure, that is, for any sequence f_n converging in $L_\rho^{p(\cdot)}(\Gamma)$ to function f_0 the sequence $S_\Gamma f_n$ converges in measure to $S_\Gamma f_0$.*

The validity of this statement may be obtained by word-for-word repetition from [14, proof of Theorem 2.1, page 21], since $L_\rho^{p(\cdot)}(\Gamma) \subset L^1(\Gamma)$ according to our assumption.

THEOREM 3.3. *Let Γ be a simple closed rectifiable curve bounding the domains D^+ and D^-. The following statements are valid.*

(i) *Let p and μ belong to $\mathcal{P}$ and let S_Γ map $L_\rho^{p(\cdot)}(\Gamma)$ to $L_\omega^{\mu(\cdot)}(\Gamma)$ for some weight functions ρ and ω. Then $1/\rho \in L^{q(\cdot)}(\Gamma)$ and S_Γ is bounded from $L_\rho^{p(\cdot)}(\Gamma)$ into $L_\omega^{\mu(\cdot)}(\Gamma)$.*

(ii) *Let S_Γ be bounded from $L_\rho^{p(\cdot)}(\Gamma)$ to $L_\omega^{\alpha}(\Gamma)$, $\alpha > 0$. Then for arbitrary $\varphi \in L_\rho^{p(\cdot)}(\Gamma)$ the Cauchy-type integral $(K_\Gamma\varphi)(z)$ belongs to $E^\alpha(D^\pm)$.*

(iii) *Let $p \in \mathcal{P}$ and let S_Γ be bounded in $L^{p(\cdot)}(\Gamma)$. Then for arbitrary $\varphi \in L^{p(\cdot)}(\Gamma)$,*

$$(K_\Gamma\varphi)(z) \in E^{\underline{p}}. \tag{3.1}$$

(iv) *For $\rho \in W^{p(\cdot)}(\Gamma)$ and $\varphi \in L^{p(\cdot)}(\Gamma)$ the function $K_\Gamma(\varphi/\rho)$ belongs to $E^1(D^\pm)$.*

Proof. (i) Since S_Γ is defined for any function in $L_\rho^{p(\cdot)}(\Gamma)$, we have the embedding $L_\rho^{p(\cdot)}(\Gamma) \subset L^1(\Gamma)$. Then for any $\varphi \in L^{p(\cdot)}(\Gamma)$ the function φ/ρ is integrable on Γ. Therefore, $1/\rho \in L^{q(\cdot)}(\Gamma)$. According to the Proposition 3.2 we conclude that for the sequence of functions φ_n converging to φ in $L^{p(\cdot)}(\Gamma)$ the sequence $S_\Gamma\varphi_n$ converges to $S_\Gamma\varphi$ in measure. Thus, if S_Γ maps $L_\rho^{p(\cdot)}(\Gamma)$ into $L_\omega^{\mu(\cdot)}(\Gamma)$, then S_Γ is a closed operator and by the closed graph theorem we conclude that it is bounded.

(ii) Let S_Γ be bounded from $L_\rho^{p(\cdot)}(\Gamma)$ into $L^\alpha(\Gamma)$, $\alpha > 0$. Let $\varphi \in L_\rho^{p(\cdot)}(\Gamma)$ and let φ_n be a sequence of rational functions (with a unique pole in D^+) such that φ_n converges to φ in $L_\rho^{p(\cdot)}(\Gamma)$ (see Proposition 3.1). Then for the functions $\Phi_n(z) = (K_\Gamma\varphi_n)(z)$ we have $\Phi_n(z) \in L^\alpha(D^\pm)$ and $\|\Phi_n^\pm\|_\alpha \le M\|\varphi_n\|_{p(\cdot),\rho}$ and by Proposition 3.2 $\Phi_n^\pm$ converges in measure to the function $\pm(1/2)\varphi + (1/2)S_\Gamma\varphi$. Applying Tumarkin's Theorem [35, page 269], we conclude that $\Phi(z) = \lim_{n\to\infty}\Phi_n(z)$ belongs to $E^\alpha(D^\pm)$. In our case $\Phi(z) = (K_\Gamma\varphi)(z)$.

(iii) From the embedding $L^{p(\cdot)}(\Gamma) \subseteq L^{\underline{p}}(\Gamma)$ and the boundedness of S_Γ in $L^{p(\cdot)}(\Gamma)$ it follows that S_Γ maps $L^{p(\cdot)}(\Gamma)$ into $L^{\underline{p}}(\Gamma)$. Then by (i) S_Γ is bounded from $L^{p(\cdot)}(\Gamma)$ into $L^{\underline{p}}(\Gamma)$. In view of (ii), then $K_\Gamma\varphi \in E^{\underline{p}}(D^\pm)$ for arbitrary $\varphi \in L^{p(\cdot)}(\Gamma)$.

(iv) Since $\rho \in W^{p(\cdot)}(\Gamma)$, we have $1/\rho \in L^{q(\cdot)}(\Gamma)$ and then $S_\Gamma(\varphi/\rho) \in L^1(\Gamma)$ for any $\varphi \in L^{p(\cdot)}(\Gamma)$. The last follows from the equality $S_\Gamma(\varphi/\rho) = (1/\rho)(\rho S_\Gamma(\varphi/\rho))$. Therefore, the operator $S_\Gamma(1/\rho)$ is defined on $L^{p(\cdot)}(\Gamma)$ and acts into $L^1(\Gamma)$. Then it is continuous in measure and consequently is a closed operator and therefore, it is bounded from $L^{p(\cdot)}(\Gamma)$ to $L^1(\Gamma)$. Applying (ii) when $L_\omega^\alpha(\Gamma) \subset L^1(\Gamma)$ we conclude that $K_\Gamma(\varphi/\rho)(z) \in E^1(D^\pm)$. □

COROLLARY 3.4. *If $\Gamma \in \mathcal{R}^{p(\cdot)}$ and $p \in \mathcal{P}(\Gamma)$, then Γ is a Smirnov curve.*

Indeed, since $\Gamma \in \mathcal{R}^{p(\cdot)}$ and $L^{\overline{p}}(\Gamma) \subset L^{p(\cdot)}(\Gamma) \subset L^{\underline{p}}(\Gamma)$, it follows that S_Γ maps $L^{\overline{p}}(\Gamma)$ into $L^{\underline{p}}(\Gamma)$. Then Γ is a Smirnov curve (see [10] and [14, page 22]).

COROLLARY 3.5. *Let $\Gamma \in \mathcal{R}^{p(\cdot)}$ and $p \in \mathcal{P}(\Gamma)$. Then for arbitrary bounded function φ, it holds that $(K_\Gamma\varphi)(z) \in \bigcap_{\beta>1} E^\beta(D^\pm)$.*

Proof. Since $\varphi \in \bigcap_{\alpha>1} L^{\alpha p(\cdot)}$ according to the statement (iii) from Theorem 3.3 we obtain that $(K_\Gamma\varphi)(z) \in \bigcap_{\alpha>1} E^{\alpha\underline{p}}(D^\pm)$. Therefore $(K_\Gamma\varphi)(z) \in \bigcap_{\beta>\underline{p}} E^\beta(D^\pm)$, that is, $(K_\Gamma\varphi)(z) \in \bigcap_{\beta>1} E^\beta(D^\pm)$. □

THEOREM 3.6. *Let $p \in \mathcal{P}$ and let S_Γ be bounded in the space $L^{p(\cdot)}(\Gamma)$. Then S_Γ is also bounded in the space $L^{\alpha p(\cdot)}(\Gamma)$ for any $\alpha > 1$ and the inequality*

$$||S_\Gamma||_{\alpha p(\cdot)} \leq \operatorname{ctg}\frac{\pi}{4\alpha}||S_\Gamma||_{p(\cdot)} \tag{3.2}$$

holds.

Proof. We follow Cotlar's idea [1] and [18]. We make use of the well-known relation

$$(S_\Gamma\varphi)^2 = -\varphi^2 + 2S_\Gamma(\varphi S_\Gamma\varphi), \tag{3.3}$$

see, for instance, [14, page 33], which follows also as a particular case from the Poincaré-Bertrand formula (see, e.g., [8, Section 7.2] or [16, page 96])

$$\frac{1}{\pi i}\int_\Gamma \frac{d\tau}{\tau - t}\frac{1}{\pi i}\int_\Gamma \frac{a(\tau,\tau_1)}{\tau_1 - \tau}d\tau_1 = a(t,t) + \frac{1}{\pi i}\int_\Gamma d\tau_1 \frac{1}{\pi i}\int_\Gamma \frac{a(\tau,\tau_1)}{(\tau - t)(\tau_1 - t)}d\tau \tag{3.4}$$

under the choice $a(t,\tau) = \varphi(t)\varphi(\tau)$; we take φ a rational function.

We observe that

$$||\varphi^2||_{p(\cdot)} = \|\varphi\|^2_{2p(\cdot)}, \tag{3.5}$$

and obtain from (3.3)

$$||S_\Gamma\varphi||^2_{2p(\cdot)} \leq \|\varphi\|^2_{2p(\cdot)} + 2||S_\Gamma||_{p(\cdot)}||\varphi S_\Gamma\varphi||_{p(\cdot)}. \tag{3.6}$$

By the usual Hölder inequality we have $\|\varphi S_\Gamma\varphi\|_{p(\cdot)} \leq \|\varphi\|_{2p(\cdot)} \cdot \|S_\Gamma\varphi\|_{2p(\cdot)}$ and then from (3.6),

$$||S_\Gamma\varphi||^2_{2p(\cdot)} - 2||S_\Gamma||_{p(\cdot)}||S_\Gamma\varphi||_{2p(\cdot)}\|\varphi\|_{2p(\cdot)} - \|\varphi\|^2_{2p(\cdot)} \leq 0 \tag{3.7}$$

whence the estimate

$$||S_\Gamma\varphi||_{2p(\cdot)} \leq \left(||S_\Gamma||_{p(\cdot)} + \sqrt{||S_\Gamma||^2_{p(\cdot)} + 1}\right)\|\varphi\|_{2p(\cdot)} \tag{3.8}$$

follows for any rational function φ. By denseness of rational functions in $L^{2p(\cdot)}$, this estimate is extended to the whole space $L^{2p(\cdot)}$.

Further by induction we prove that

$$||S_\Gamma||_{2^{k-1}p(\cdot)} \leq \operatorname{ctg}\frac{\pi}{2^{k+1}}||S_\Gamma||_{p(\cdot)}, \quad k \in \mathbb{N}. \tag{3.9}$$

Indeed, from (3.8) we obtain that

$$\begin{aligned} ||S_\Gamma||_{2^k p(\cdot)} &\leq ||S_\Gamma||_{p(\cdot)}\left(\operatorname{ctg}\frac{\pi}{2^{k+1}} + \sqrt{1 + \operatorname{ctg}^2\frac{\pi}{2^{k+1}}}\right) \\ &\leq ||S_\Gamma||_{p(\cdot)}\left(\operatorname{ctg}\frac{\pi}{2^{k+1}} + \frac{1}{\sin\pi/2^{k+1}}\right) \\ &= ||S_\Gamma||_{p(\cdot)}\operatorname{ctg}\frac{\pi}{2^{k+2}}. \end{aligned} \tag{3.10}$$

Now we apply the Riesz-type interpolation theorem known for the spaces $L^{p(\cdot)}$ (see [25, Theorem 14.16]) in the following form: *if a linear operator A is bounded in the spaces $L^{2^k p(\cdot)}(\Gamma)$ and $L^{2^{k+1}p(\cdot)}(\Gamma)$, then it is also bounded in the space $L^{\alpha p(\cdot)}(\Gamma)$ with $\alpha \in [2^k, 2^{k+1})$, $1/\alpha = \theta 2^{-k} + (1-\theta)2^{-k-1}$, and*

$$\|A\|_{\alpha p(\cdot)} \leq \|A\|^{\theta}_{2^k p(\cdot)} \|A\|^{1-\theta}_{2^{k+1}p(\cdot)}. \tag{3.11}$$

Then from (3.9) and (3.10) we get

$$\|S_\Gamma\|_{\alpha p(\cdot)} \leq \|S_\Gamma\|_{p(\cdot)} \left\{\operatorname{ctg}\left(\frac{\pi}{2^{k+1}}\right)\right\}^{\theta} \left\{\operatorname{ctg}\left(\frac{\pi}{2^{k+2}}\right)\right\}^{1-\theta}. \tag{3.12}$$

Obviously,

$$\left\{\operatorname{ctg}\left(\frac{\pi}{2^{k+1}}\right)\right\}^{\theta} \left\{\operatorname{ctg}\left(\frac{\pi}{2^{k+2}}\right)\right\}^{1-\theta} \leq \operatorname{ctg}\left(\frac{\pi}{2^{k+2}}\right) = \operatorname{ctg}\left(\frac{\pi}{4}\cdot\frac{1}{2^k}\right). \tag{3.13}$$

But $\alpha \geq 2^k$. Therefore,

$$\left\{\operatorname{ctg}\left(\frac{\pi}{2^{k+1}}\right)\right\}^{\theta} \left\{\operatorname{ctg}\left(\frac{\pi}{2^{k+2}}\right)\right\}^{1-\theta} \leq \operatorname{ctg}\left(\frac{\pi}{4}\cdot\frac{1}{\alpha}\right) = \operatorname{ctg}\left(\frac{\pi}{4\alpha}\right). \tag{3.14}$$

Consequently,

$$\|S_\Gamma\|_{\alpha p(\cdot)} \leq \operatorname{ctg}\frac{\pi}{4\alpha}\|S_\Gamma\|_{p(\cdot)}. \tag{3.15}$$

□

4. On belongingness of $\exp(K_\Gamma\varphi)$ to the Smirnov classes when $\Gamma \in \mathcal{R}^{p(\cdot)}$

THEOREM 4.1. *Let a closed curve $\Gamma \in \mathcal{R}^{p(\cdot)}$ and $p \in \mathcal{P}(\Gamma)$. Let φ be a bounded measurable function on Γ. Assume that $z_0 \in D^+$. Then*

(i) *there exists an integer $k \geq 0$ such that*

$$\exp\{(K_\Gamma\varphi)(z)\} =: X(z) \in E^\delta(D^+), \qquad \frac{X(z)-1}{(z-z_0)^k} \in E^\delta(D^-), \tag{4.1}$$

where

$$0 < \delta < \frac{\pi \underline{p}}{2(1+\ell)eM\|S_\Gamma\|_{p(\cdot)}}, \quad M = \sup_{t\in\Gamma}|\varphi(t)|; \tag{4.2}$$

(ii) *in case $\varphi \in C(\Gamma)$*

$$X(z) \in \bigcap_{\delta>1} E^\delta(D^+), \qquad X(z)-1 \in \bigcap_{\delta>1} E^\delta(D^-). \tag{4.3}$$

Proof. We use an idea developed in [18]. Let Γ_r be the image of $\gamma_r = \{z : |z| = r\}$, $r < 1$, under the conformal mapping of $U = \{z : |z| < 1\}$ onto D^+. We have

$$\int_{\Gamma_r} |X(z)|^\delta |dz| \leq \int_{\Gamma_r} \sum_{n=0}^{\infty} \frac{1}{n!} |\delta\Phi(z)|^n |dz|, \quad \text{where } \Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}. \tag{4.4}$$

According to Corollary 3.5 we have $\Phi(z) \in E^n(D^+)$ for any $n \geq 1$. Then by the known property of the class E^p (see [35, Chapter III]), we have

$$\int_{\Gamma_r} |\Phi(z)|^n |dz| \leq \int_{\Gamma} |\Phi^+(t)|^n |dt|, \tag{4.5}$$

and then from (4.4) we obtain

$$\begin{aligned}\int_{\Gamma_r} |X(z)|^{\delta} |dz| &\leq \sum_{n=0}^{\infty} \int_{\Gamma} |\delta\Phi^+(t)|^n |dt| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma} \left| \frac{\delta\varphi(t)}{2} + \frac{\delta}{2}(S_\Gamma\varphi(t)) \right|^n |dt| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma} |\delta\varphi(t)|^n |dt| + \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma} |\delta(S_\Gamma\varphi(t))|^n |dt|.\end{aligned} \tag{4.6}$$

Hence

$$\int_{\Gamma_r} |X(z)|^{\delta} |dz| \leq \ell e^{\delta M} + \left(\sum_{n=0}^{n_0-1} + \sum_{n=n_0}^{\infty} \right) \frac{1}{n!} \int_{\Gamma} |\delta(S_\Gamma\varphi(t))|^n |dt|, \tag{4.7}$$

where we take any $n_0 > \underline{p}$. It remains to show that the series $\sum_{n=n_0}^{\infty}$ converges. Let $\alpha_n = n/\underline{p} > 1$. Then $n = \alpha_n \underline{p} \leq \alpha_n p(t)$ and by (2.9) we have

$$||S_\Gamma\varphi||_n \leq (1+\ell)||S_\Gamma\varphi||_{\alpha_n p(\cdot)}. \tag{4.8}$$

Then by (3.2) we obtain

$$||S_\Gamma\varphi||_n \leq (1+\ell)\operatorname{ctg}\frac{\pi}{4\alpha_n}||S_\Gamma||_{p(\cdot)}\|\varphi\|_{\alpha_n p(\cdot)}. \tag{4.9}$$

Taking (2.8) into account, we see that $\|\varphi\|_{\alpha_n p(\cdot)} \leq M \max(1, \ell^{1/n})$ and then

$$||S_\Gamma\varphi||_n \leq c_0 n \max(1, \ell^{1/n})||S_\Gamma||_{p(\cdot)}, \quad c_0 = \frac{4}{\pi\underline{p}}(1+\ell)M. \tag{4.10}$$

Therefore,

$$\sum_{n=n_0}^{\infty} \frac{1}{n!} \int_{\Gamma} |\delta(S_\Gamma\varphi(t))|^n |dt| \leq \sum_{n=n_0}^{\infty} \frac{\delta^n}{n!}||S_\Gamma\varphi||_n^n \leq \max(1,\ell) \sum_{n=n_0}^{\infty} \frac{(c_0\delta)^n n^n}{n!}||S_\Gamma||_{p(\cdot)}^n, \tag{4.11}$$

where the series on the right-hand side converges if $c_0\delta\|S_\Gamma\|_{p(\cdot)}e < 1$.

Thus it was proved that $X(z) \in E^{\delta}(D^+)$ when

$$0 < \delta < \delta_0 = \frac{\pi\underline{p}}{4(1+\ell)eM||S_\Gamma||_{p(\cdot)}}. \tag{4.12}$$

In the case D^- and for $1 < \delta < \delta_0$ and arbitrary r we are able to obtain similar estimates by the same way as in case D^+. As to $\delta \le 1$ it is necessary to consider two cases: $0 < r < r_0$ and $r_0 < r < 1$ for some fixed r_0. In the last case the appropriate estimates can be proved as in the case $\delta > 1$. As to the case $0 < r < r_0$ the needed inequalities are obtained by means of choice of number $k > [1/\delta]$.

Now we are able to get a stronger result, namely, that $X(z) \in E^\delta(D^+)$ and $(X(z)-1)/(z-z_0)^k \in E^\delta(D^-)$ for $\delta < 2\delta_0$.

Indeed,

$$\int_\Gamma |X^\pm(t)|^\delta |dt| = \int_\Gamma |e^{\pm\delta\varphi(t)/2}|\,|e^{\delta/2}(S_\Gamma\varphi)(t)|\,|dt| \le e^{\delta M/2}\sum_{n=0}^{\infty}\frac{1}{n!}\int_\Gamma\left|\frac{\delta}{2}(S_\Gamma\varphi)(t)\right|^n |dt|$$
$$\le e^{\delta M/2}\left(\sum_{n=1}^{n_0-1}\frac{1}{n!}\int_\Gamma\left|\frac{\delta}{2}(S_\Gamma\varphi)(t)\right|^n|dt| + \sum_{n=n_0}^{\infty}\frac{1}{n!}\int_\Gamma\left|\frac{\delta}{2}(S_\Gamma\varphi)(t)\right|^n|dt|\right), \tag{4.13}$$

where $n_0 > \underline{p}$.

From the previous proof it is clear that last series converges when $\delta < 2\delta_0$.

Now apply Smirnov's following theorem (see, e.g., [35, Chapter III]): *let* $\Phi \in E^{\gamma_1}(D)$ *and* $\Phi^+ \in L^{\gamma_2}(\Gamma)$ *where* $\gamma_2 > \gamma_1$, *then* $\Phi \in E^{\gamma_2}(D)$. According to this statement in our case we have $X(z) \in E^\delta(D^+)$ and $(X(z)-1)/(z-z_0)^k \in E^\delta(D^-)$ when $\delta < 2\delta_0$ with δ_0 from (4.12).

By this (i) is proved.

Now we prove (ii). For arbitrary $\varepsilon > 0$ we can find a Hölder function ψ on Γ such that

$$\operatorname*{ess\,sup}_{t\in\Gamma} |\varphi(t)-\psi(t)| < \varepsilon. \tag{4.14}$$

On the other hand, for the Hölder function $\psi(t)$ there exist positive numbers a_1 and a_2 such that $0 < a_1 \le |\exp(K_\Gamma\psi)(z)| \le a_2 < \infty$.

Thus from (i) and (4.12) we conclude (ii). □

Remark 4.2. As it follows from the proof of the final part of previous theorem the number M in formula (4.12) can be replaced by $\nu(\varphi) = \inf\|\varphi-\psi\|_C$, where the infinum is taken over all rational functions ψ.

5. The problem of linear conjugation with continuous coefficients

In the present paper, we proceed to the solution of problem (1.1) in the class $\mathcal{K}_p^{p(\cdot)}(\Gamma)$ under various assumptions with respect to the data.

We begin with the case when $p \in \mathcal{P}$, $\Gamma \in \mathcal{R}^{p(\cdot)}(\Gamma)$, and G is a nonvanishing continuous function on Γ. The function g is assumed to be in $L^{p(\cdot)}(\Gamma)$. We look for a function $\Phi \in \mathcal{K}^{p(\cdot)}(\Gamma)$ whose boundary values $\Phi^\pm$ satisfy relation (1.1) almost everywhere on Γ.

Let $\varkappa = (1/2\pi)[\arg G(t)]_\Gamma$ be the index of G on Γ. Below we will show that for the above formulated problem all the statements for its solvability known for constant p remain valid in the general case of variable exponent; namely, the following statement is valid.

Theorem 5.1. *Let $p \in \mathcal{P}$, $\Gamma \in \mathcal{R}^{p(\cdot)}$, and let $g \in L^{p(\cdot)}(\Gamma)$. Assume that $G \in C(\Gamma)$ and $G(t) \neq 0$, $t \in \Gamma$. Then for problem (1.1) the following statements hold:*

(i) *for $\varkappa \geq 0$, problem (1.1) is unconditionally solvable in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$ and all its solutions are given by*

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_\Gamma \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - z} + X(z)Q_{\varkappa-1}(z) \tag{5.1}$$

with

$$X(z) = \begin{cases} \exp h(z), & z \in D^+, \\ (z - z_0)^{-\varkappa}\exp h(z), & z \in D^-,\ z_0 \in D^+, \end{cases} \tag{5.2}$$

where

$$h(z) = K_\Gamma\big(\ln G(t)(t - z_0)^{-\varkappa}\big)(z), \tag{5.3}$$

and $Q_{\varkappa-1}(z)$ is an arbitrary polynomial of degree $\varkappa - 1(Q_{-1}(z) \equiv 0)$;

(ii) *for $\varkappa < 0$, problem (1.1) is solvable in this class if and only if*

$$\int_\Gamma \frac{g(t)}{X^+(t)}t^k dt = 0, \quad k = 0, 1, \dots, |\varkappa| - 1, \tag{5.4}$$

and under these conditions problem (1.1) has the unique solution given by (5.1) with $Q_{\varkappa-1} = 0$.

Proof. Consider first the case $\varkappa = 0$. We choose a rational function $\widetilde{G}(t)$ such that

$$\sup_{t\in\Gamma}\left|\frac{G(t)}{\widetilde{G}(t)} - 1\right| < \frac{1}{2}\Big(1 + \|S_\Gamma\|_{p(\cdot)}\Big)^{-1}. \tag{5.5}$$

Obviously $\operatorname{ind}\widetilde{G} = 0$ and therefore the function $\widetilde{X}(z) = \exp(K_\Gamma(\ln\widetilde{G}))(z)$ is continuous in the domains $D^\pm$. Now recall that if $\Phi \in \mathcal{K}^{p(\cdot)}(\Gamma)$, then according to Theorem 3.3(iii) we have $\Phi \in E^{\underline{p}}(D^\pm)$. Since $\underline{p} > 1$, the equality $\Phi(z) = (K_\Gamma(\Phi^+ - \Phi_-))(z)$ holds (see (2.17)). Now we have

$$\left(\frac{\Phi}{\widetilde{X}}\right)^+ = \frac{G}{\widetilde{G}}\left(\frac{\Phi}{\widetilde{X}}\right)^- + \frac{g}{\widetilde{X}^+}, \tag{5.6}$$

where $\widetilde{X}(z) = \exp\{K_\Gamma(\ln\widetilde{G})(z)\}$. We show that $\Phi/\widetilde{X} \in \mathcal{K}^{p(\cdot)}(\Gamma)$. To this end, we observe that $\Phi \in E^{p(\cdot)}(D^\pm)$ and $1/X$ is bounded so that $\Phi/\widetilde{X} \in E^{\underline{p}}(D^\pm)$ and therefore

$$\frac{\Phi}{\widetilde{X}} = K_\Gamma\left(\frac{\Phi^+}{\widetilde{X}^+} - \frac{\Phi^-}{\widetilde{X}^-}\right). \tag{5.7}$$

From the Sokhotski-Plemelj formula and from the condition $\Gamma \in \mathcal{R}^{p(\cdot)}$ it follows that $\Phi^\pm \in L^{p(\cdot)}(\Gamma)$ and hence $\Phi/\widetilde{X} \in \mathcal{K}^{p(\cdot)}(\Gamma)$.

Let

$$\frac{\Phi(z)}{\widetilde{X}(z)} = (K_\Gamma\psi)(z), \quad \psi \in L^{p(\cdot)}(\Gamma). \tag{5.8}$$

Then equality (5.6) yields

$$\psi(t) = \left(\frac{G(t)}{\widetilde{G}(t)} - 1\right)\left(-\frac{1}{2}\psi(t) + \frac{1}{2}S_\Gamma\psi(t)\right) + \frac{g(t)}{\widetilde{X}(t)}, \tag{5.9}$$

that is, the function ψ is a solution of the equation of the type $\psi = K\psi$ in the space $L^{p(\cdot)}(\Gamma)$, where K is a contractive operator. Therefore, (5.9) and consequently problem (1.1) have the unique solution in $\mathcal{K}^{p(\cdot)}(\Gamma)$. Basing on Theorem 4.1 we construct the solution. Let

$$X(z) = \exp\left(K_\Gamma(\ln G)\right)(z). \tag{5.10}$$

As far as $\varkappa = 0$, we find that

$$\ln G(t) = \ln|G(t)| + i\arg G(t) \tag{5.11}$$

is a continuous function, and by Theorem 4.1

$$\frac{1}{X(z)} - 1 \in \bigcap_{\delta>1} E^\delta(D^\pm). \tag{5.12}$$

If Φ is a solution of problem (1.1), then $\Phi \in \mathcal{K}^{\underline{p}}(\Gamma)$ and therefore, $\Phi \in E^{\underline{p}}(D^\pm)$. Moreover, $\Phi/X \in E^{\underline{p}-\varepsilon}(D^\pm)$ for arbitrary $\varepsilon \in (0, 1/\underline{p})$. Therefore $\Phi/X \in \mathcal{K}^{p-\varepsilon}(\Gamma)$. So $\Phi/X \in \mathcal{K}^1(\Gamma)$. At the same time

$$\left(\frac{\Phi}{X}\right)^+ - \left(\frac{\Phi}{X}\right)^- = \frac{g}{X^+}. \tag{5.13}$$

Since this problem has a unique solution in $\mathcal{K}^1(\Gamma)$, then the function

$$\Phi(z) = X(z)K_\Gamma\left(\frac{g}{X^+}\right)(z) \tag{5.14}$$

is the solution of (1.1) in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$.

Let now $\varkappa > 0$. We choose a point $z_0 \in D^+$ and rewrite (1.1) in the form

$$\Phi^+(t) = G_1(t)(t - z_0)^\varkappa\Phi^-(t) + g(t), \tag{5.15}$$

where $G_1(t) = (t - z_0)^{-\varkappa}G(t)$ is a continuous function with zero index. We introduce a new unknown function

$$F(z) = \begin{cases} \Phi(z), & z \in D^+, \\ (z - z_0)^\varkappa\Phi(z), & z \in D^-. \end{cases} \tag{5.16}$$

For $F(z)$ there exists a polynomial $Q_{\varkappa-1}(z)$ such that

$$\Psi(z) = F(z) - Q_{\varkappa-1}(z) \in E^1(D^-). \tag{5.17}$$

Then $\Psi(z) = K_\Gamma(\Psi^+ - \Psi^-)$. But

$$\Psi^+(t) - \Psi^-(t) = F^+(t) - F^-(t) = \Phi^+(t) - (t-z_0)^{\varkappa}\Phi^-(t) \in L^{p(\cdot)}(\Gamma) \tag{5.18}$$

so that $\Psi \in \mathcal{K}^{p(\cdot)}(\Gamma)$. Moreover,

$$\Psi^+(t) = G_1(t)\Psi^-(t) + g_1(t), \tag{5.19}$$

where $g_1(t) = g(t) - Q_{\varkappa-1}(t) + G_1(t)Q_{\varkappa-1}(t)$. Since $\operatorname{ind} G_1 = 0$, according to what was proved above,

$$\Psi(z) = X_1(z)K_\Gamma\left(\frac{g_1}{X_1^+}\right)(z), \quad \text{where } X_1(z) = \exp\{(K_\Gamma \ln G_1)(z)\}. \tag{5.20}$$

Here

$$K_\Gamma\left(\frac{g_1}{X_1^+}\right)(z) = K_\Gamma\left(\frac{g}{X_1^+}\right)(z) - \frac{1}{2\pi i}\int_\Gamma \frac{Q_{\varkappa-1}(t)}{X_1^+(t)}\frac{dt}{t-z} + \frac{1}{2\pi i}\int_\Gamma \frac{Q_{\varkappa-1}(t)}{X_1^-(t)}\frac{dt}{t-z}. \tag{5.21}$$

But

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{Q_{\varkappa-1}(t)}{X_1^+(t)}\frac{dt}{t-z} &= \begin{cases} \dfrac{Q_{\varkappa-1}(z)}{X_1(z)}, & z \in D^+, \\ 0, & z \in D^-, \end{cases} \\
\frac{1}{2\pi i}\int_\Gamma \frac{Q_{\varkappa-1}(t)}{X_1^-(t)}\frac{dt}{t-z} &= \frac{1}{2\pi i}\int_\Gamma \left[\frac{Q_{\varkappa-1}(t)}{X_1^-(t)} - Q_{\varkappa-1}(t)\right]\frac{dt}{t-z} + \frac{1}{2\pi i}\int_\Gamma \frac{Q_{\varkappa-1}(t)}{t-z}dt \\
&= \begin{cases} Q_{\varkappa-1}(z), & z \in D^+, \\ -\dfrac{Q_{\varkappa-1}(z)}{X_1(z)} + Q_{\varkappa-1}(z), & z \in D^-. \end{cases}
\end{aligned} \tag{5.22}$$

Therefore,

$$\Psi(z) = X_1(z)K_\Gamma\left(\frac{g_1}{X_1^+}\right)(z) = X_1(z)K_\Gamma\left(\frac{g}{X_1^+}\right)(z) + X_1(z)Q_{\varkappa-1}(z) - Q_{\varkappa-1}(z). \tag{5.23}$$

Then by (5.16) and (5.17) we arrive at formula (5.1).

It can be easily verified that the latter provides the solution of problem (1.1) for an arbitrary polynomial $Q_{\varkappa-1}(z)$ which does not depend on the choice of the point z_0.

Finally, we consider the case $\varkappa < 0$. This time the function F given by (5.16) is in $\mathcal{K}^{p(\cdot)}(\Gamma)$. Moreover, $F^+ = G_1F^- + g$, whence

$$F(z) = X_1(z)K_\Gamma\left(\frac{g}{X_1^+}\right)(z) \tag{5.24}$$

and the condition $\Phi(z) = (z - z_0)^{-\varkappa} F \in E^1(D^-)$ is fulfilled if and only if the conditions (5.4) are satisfied. □

Via the solution of (1.1) in $\mathcal{K}^{p(\cdot)}(\Gamma)$ with nonvanishing $G \in C(\Gamma)$ and $g \in L^{p(\cdot)}(\Gamma)$ we are now able to derive the following weight result for Cauchy singular integrals.

Theorem 5.2. *Let $\Gamma \in \mathcal{R}^{p(\cdot)}$ and $p \in \mathcal{P}(\Gamma)$. Let φ be a real-valued function in $C(\Gamma)$. Then the function*

$$\rho(t) = \left| \exp\left(\frac{1}{2\pi} \int_\Gamma \frac{\varphi(\tau) d\tau}{\tau - t} \right) \right| \tag{5.25}$$

belongs to the class $W^{p(\cdot)}(\Gamma)$.

Proof. Consider the problem (1.1) in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$ with $G(t) = \exp(i\varphi(t))$ and $g \in L^{p(\cdot)}(\Gamma)$. Obviously, $G \in C(\Gamma)$ and $\operatorname{ind} G = 0$. Consequently, the function

$$\Phi(z) = X(z) K_\Gamma\left(\frac{g}{X^+}\right)(z) \tag{5.26}$$

belongs to $\mathcal{K}^{p(\cdot)}(\Gamma)$ for arbitrary $g \in L^{p(\cdot)}(\Gamma)$. This implies that the function

$$\Phi^+(t) = X^+(t)\left(S_\Gamma \frac{g}{X^+}\right)(t) \tag{5.27}$$

belongs to $L^{p(\cdot)}(\Gamma)$, that is,

$$e^{i \arg X^+(t)} \frac{\rho(t)}{\pi i} \int_\Gamma \frac{g(\tau) e^{-i \arg X^+(\tau)}}{\rho(\tau)} \frac{d\tau}{\tau - t} \in L^{p(\cdot)}(\Gamma). \tag{5.28}$$

Since $|e^{\pm i \arg X^+(\tau)}| = 1$, by Theorem 3.3(i) we immediately conclude that $\rho \in W^{p(\cdot)}(\Gamma)$. □

6. The problem of linear conjugation with continuous coefficients in the weighted class $\mathcal{K}_\rho^{p(\cdot)}$

If we assume that $\rho \in W^{p(\cdot)}(\Gamma)$ and choose the function $\widetilde{G}$ in the proof of Theorem 5.1 such that instead of condition (5.5), the condition

$$\sup_{t \in \Gamma} \left| \frac{G(t)}{\widetilde{G}(t)} - 1 \right| < \frac{1}{2}\left(1 + \|S_\Gamma\|_{L_\rho^{p(\cdot)}}\right)^{-1} \tag{6.1}$$

is fulfilled, then for $\varkappa = 0$ we conclude that problem (1.1) is uniquely solvable in $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ for arbitrary $g \in L_\rho^{p(\cdot)}(\Gamma)$. If, in addition, we assume that $1/\rho \in L^{q(\cdot)+\varepsilon}(\Gamma)$, then formula (5.1) remains valid because in this case $\Phi/X \in E^{1+\delta}(D^\pm)$, $\delta > 0$. The last inclusion is valid since $\Phi \in E^{1+\varepsilon}(D^\pm)$, $1/X \in \bigcap_{\delta>1} E^\delta(D^\pm)$, and

$$\left(\frac{\Phi}{X}\right)^+ - \left(\frac{\Phi}{X}\right)^- = \frac{g}{X^+}. \tag{6.2}$$

Consequently, we arrive at the following statement.

Theorem 6.1. *Let the exponent $p \in \mathcal{P}$ satisfy condition (2.10), $\rho \in W^{p(\cdot)}(\Gamma)$, and $1/\rho \in L^{q(\cdot)+\varepsilon}$ $(\varepsilon > 0)$. Assume that $G \in C(\Gamma)$, $G(t) \neq 0$, $t \in \Gamma$, $g \in L_\rho^{p(\cdot)}(\Gamma)$. Then for problem (1.1) in the class $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ all the statements of Theorem 5.1 remain valid.*

Corollary 6.2. *Let the exponent $p \in \mathcal{P}$ satisfy condition (2.10), $\rho \in W^{p(\cdot)}(\Gamma)$, and $1/\rho \in L^{q(\cdot)+\varepsilon}(\Gamma)$ $(\varepsilon > 0)$. Then the function*

$$r(t) = \rho(t) \exp \left| \frac{1}{2\pi} \int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t} \right| \tag{6.3}$$

with real continuous φ, belongs to $W^{p(\cdot)}(\Gamma)$.

7. The problem of linear conjugation with a piecewise continuous coefficient

Proposition 2.2 allows us to investigate problem (1.1) in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$ in the case of piecewise continuous coefficient G and $g \in L^{p(\cdot)}(\Gamma)$.

Thus let $G \in C(\Gamma, t_1, t_2, \ldots, t_m)$, that is, G be continuous on the arcs $[t_k, t_{k+1}]$, $k = 1, \ldots, m-1$. Assume that $\inf_{t\in\Gamma} |G(t)| > 0$. We will follow [8, 15, 28].

Let

$$\frac{G(t_k^-)}{G(t_k^+)} = e^{2\pi i \lambda_k}, \quad k = 1, 2, \ldots, m, \tag{7.1}$$

where the real part of the complex numbers $\lambda_k = \alpha_k + i\beta_k$ is defined up to an arbitrary additive integer. We assume that

$$\alpha_k \neq \frac{1}{q_k} \bmod(1), \qquad q_k = q(t_k), \qquad q(t) = \frac{p(t)}{p(t)-1}, \tag{7.2}$$

and choose α_k in the interval

$$-\frac{1}{p_k} < \alpha_k < \frac{1}{q_k}, \quad p_k = p(t_k). \tag{7.3}$$

In D^+ we choose an arbitrary point z_0. Let γ_k be a simple smooth curve (a cut) from the point z_0 to ∞ which crosses Γ only at the point t_k so that the function $(z - z_0)^{\lambda_k}$ is analytic on the complex plane cut along γ_k and

$$(t_k - z_0)_-^{\lambda_k} = (t_k - z_0)_+^{\lambda_k} \exp(2\pi i \lambda_k), \tag{7.4}$$

where $(t_k - z_0)_\pm^{\lambda_k} = \lim_{\substack{t \to t_k \pm \\ t \in \Gamma}} (t - z_0)^{\lambda_k}$. Then the function

$$G_1(t) = \frac{G(t)}{\prod_{k=1}^m (t - z_0)^{\lambda_k}} \tag{7.5}$$

is continuous [8, page 432] and $G_1(t) \neq 0$, $t \in \Gamma$.

Consider now the function

$$\rho(z)=\prod_{k=1}^{m}\rho_{t_k}(z)\quad\text{with } \rho_{t_k}(z)=\begin{cases}\left(z-t_k\right)^{\lambda_k}, & z\in D^+,\\ \left(\dfrac{z-t_k}{z-z_0}\right)^{\lambda_k}=\dfrac{\left(z-t_k\right)^{\lambda_k}}{\left(z-z_0\right)^{\lambda_k}}, & z\in D^-,\end{cases}\tag{7.6}$$

where the branch for the function $((z-t_k)/(z-z_0))^{\lambda_k}$ is chosen so that it tends to 1 as $z\to\infty$, $z\in D^-$, so that $\rho_{t_k}(z)$ is analytic in $D^\pm$.

Assuming that the curve Γ at the points t_k has at least one-sided tangents, and taking into account inequalities (7.3), we conclude from the equality

$$\rho_{t_k}(z)=e^{\alpha_k\ln|z-t_k|-\beta_k\arg(z-t_k)}e^{i(\beta_k\ln|z-t_k|+\alpha_k\arg(z-t_k))}\tag{7.7}$$

that there exists an $\varepsilon>0$ such that

$$\rho_{t_k}(z)\in E^{p_k+\varepsilon}(D^\pm),\qquad \frac{1}{\rho_{t_k}(z)}\in E^{q_k+\varepsilon}(D^\pm).\tag{7.8}$$

Therefore, we have

$$\rho(z)\in E^{p_0}(D^\pm),\qquad \frac{1}{\rho(z)}\in E^{q_0}(D^\pm),\tag{7.9}$$

where $p_0=\min p_k$, $q_0=\min q_k$.

Let

$$X(z)=\rho(z)X_1(z),\tag{7.10}$$

where

$$X_1(z)=\exp\left\{\mathcal{K}_\Gamma\left(\ln G_1(t)\right)\right\},\tag{7.11}$$

and introduce a new unknown function

$$\Phi_1(z)=\frac{\Phi(z)}{\rho(z)}.\tag{7.12}$$

As far as $\Phi\in E^{\underline{p}}(D^\pm)$ and (7.9) hold, we have that $\Phi_1(z)\in E^\delta(D^\pm)$, $\delta>0$. By Proposition 2.2 we have $\rho\in W^{p(\cdot)}(\Gamma)$ and therefore $1/\rho(t)\in L^{q(\cdot)+\varepsilon}$, $\varepsilon>0$. So $\Phi_1^\pm\in L^{1+\eta}(\Gamma)$ for some $\eta>0$. According to Smirnov's theorem we conclude that $\Phi_1\in E^{1+\eta}(D^\pm)$. Now by (2.17) the equality $\Phi_1=K_\Gamma(\Phi_1^+-\Phi_1^-)$ holds and therefore $\Phi_1\in\mathcal{K}^{1+\eta}(\Gamma)$. From (7.1) and (7.12) we derive that

$$\Phi_1^+(t)=G_1(t)\Phi_1^-(t)+g(t)\rho(t).\tag{7.13}$$

Having resolved (7.13), we find that all the possible solutions of problem (1.1) in the case $\varkappa = \operatorname{ind} G_1(t) \geq 0$ are given by

$$\Phi(z) = \rho(z)X_1(z)K_\Gamma\left(\frac{g}{\rho^+X_1^+}\right)(z) + \rho(z)X(z)Q_{\varkappa-1}(z), \tag{7.14}$$

where $Q_{\varkappa-1}(z)$ is an arbitrary polynomial of degree $\varkappa$.

By Proposition 2.2 we have $\rho \in W^{p(\cdot)}(\Gamma)$ and $1/\rho \in L^{q(\cdot)+\varepsilon}(\Gamma)$. Thus all the conditions of Corollary 6.2 are satisfied. Therefore, for the function Φ from (7.14) we get $\Phi^\pm \in L^{p(\cdot)}(\Gamma)$. Consequently, (7.14) with an arbitrary polynomial $Q_{\varkappa-1}$ provides us with the solution of (1.1) in $\mathcal{K}^{p(\cdot)}(\Gamma)$.

The case of negative index is considered in the standard way.

As a result, we arrive at the following theorem.

Theorem 7.1. *Let Γ be a closed Lyapunov curve or a curve of turning without cusps. Let the exponent $p \in \mathcal{P}$ satisfy condition (2.10) and*

$$G \in C(\Gamma, t_1, t_2, \dots, t_m), \qquad \inf_{t\in\Gamma} |G(t)| > 0, \tag{7.15}$$

and suppose that the curve Γ has at least one-sided tangent lines at the points t_k, $k = 1,2,\dots,m$. Let $\varkappa = \operatorname{ind} G_1(t)$, where G_1 is given by (7.5). Under assumptions (7.1), (7.2), and (7.3) the statements of Theorem 5.1 hold for problem (1.1), if $X(z)$ is replaced by $X_1(z)$ and formula (5.1) is replaced by formula (7.14).

8. The problem of linear conjugation with bounded measurable coefficient

On the basis of various approaches to investigate problem (1.1), we are able to study this problem in the classes $\mathcal{K}^{p(\cdot)}(\Gamma)$ either when

(I) Γ belongs to a rather narrow class of curves (Lyapunov curves) and G is in a sufficiently wide class of bounded measurable functions, or

(II) Γ belongs to a wide class of curves but G is in a more narrow class than in the previous case.

In situation (I), we assume that $p \in \mathcal{P}$ and satisfies the logarithmic condition (2.10); Γ is a Lyapunov curve. We assume that $G \in \mathcal{A}(\lambda)$, $\lambda > 1$, where $\mathcal{A}(\lambda)$ is the Simonenko class. We recall its definition.

Definition 8.1. A measurable function G on Γ is said to be in $\mathcal{A}(\lambda)$, $\lambda > 1$, if it satisfies the following conditions:

(1) $0 < \operatorname{ess\,inf}_{t\in\Gamma} |G(t)|$, $\operatorname{ess\,sup}_{t\in\Gamma} |G(t)| < \infty$,

(2) for any $t_0 \in \Gamma$ there exists on Γ a neighborhood γ_{t_0} of t_0 such that all the values of $G(t)$, $t \in \gamma_{t_0}$, lie in the sector centered at the origin and of the angle $a(\lambda) = (2\pi - \delta)/\max(\lambda, \lambda')$, where $\lambda' = \lambda/(\lambda - 1)$ and $\delta > 0$ is constant on γ_{t_0}, see [38, pages 278–279] and [39], on several versions of this class.

In [38, 39], for every $G \in \mathcal{A}(\lambda)$ an argument $\alpha(t)$ of $G(t)$ at every point $t \in \Gamma$ and its increment $2\pi\varkappa_\lambda(G)$ was introduced and it was shown that there exists a function $G_0(t) = \exp\{i\alpha_0(t)\}$ satisfying the Lipschitz condition ($G_0 \in \operatorname{Lip} 1$) such that

$$\operatorname*{ess\,sup}_{t\in\Gamma} |\alpha(t) - \alpha_0(t)| < \frac{a(\lambda)}{2}, \qquad \varkappa(G_0) = \varkappa_\lambda(G), \tag{8.1}$$

and thus

$$G(t) = |G(t)| e^{i\alpha_0(t)} e^{i[\alpha(t)-\alpha_0(t)]}. \tag{8.2}$$

In the sequel the index $\varkappa = \varkappa(G)$ of a function $G \in \mathcal{A}_\lambda$ is interpreted, by definition, as

$$\varkappa := \varkappa(G) = \varkappa_\lambda(G). \tag{8.3}$$

In situation (II), it is known that in the general case of $\Gamma \in \mathcal{R}^{p(\cdot)}$—even when p is constant, $1 < p < \infty$—the statements of Theorem 5.1 may become invalid for problem (1.1) with real-valued function G under the only condition $0 < m_1 \le |G(t)| \le m_2 < \infty$ (see [3]). In this general case of $\Gamma \in \mathcal{R}^{p(\cdot)}$, we restrict ourselves to oscillatory coefficients of the form

$$G(t) = e^{i\alpha(t)} \tag{8.4}$$

with a real-valued function $\alpha(t)$.

Theorem 8.2. *Let p belong to $\mathcal{P}$ and satisfy condition (2.10). Suppose that either*

(I) Γ *is a Lyapunov curve and $G \in \mathcal{A}(\lambda)$, where*

$$\lambda = \max\left(\frac{\pi}{2\operatorname{arcctg}\|S_{\Gamma_0}\|_{p(\cdot)}}, \overline{q}\right), \tag{8.5}$$

or

(II) $\Gamma \in \mathcal{R}^{p(\cdot)}$ *and G is an oscillating function having form (8.4) with real-valued $\alpha(t)$, $e^{i\alpha(t)} \in \mathcal{A}(\lambda)$, where*

$$\lambda = \max\left(\frac{\pi}{2\operatorname{arcctg}\|S_\Gamma\|_{p(\cdot)}}, \frac{2(1+\ell)e\|S_\Gamma\|_{p(\cdot)}}{\underline{p}}\right). \tag{8.6}$$

Then for problem (1.1) in $\mathcal{K}^{p(\cdot)}(\Gamma)$ all the conclusions of Theorem 5.1 hold, whereas in (5.2) we mean that

$$h(z) = K_\Gamma(\ln|G(t)| + i\alpha(t) - \varkappa\ln(t-z_0))(z). \tag{8.7}$$

Proof. (I) Let Φ be a solution of problem (1.1) in $\mathcal{K}^{p(\cdot)}(\Gamma)$ and let $Q_{\varkappa-1}(z)$ be such a polynomial that the function $(z-z_0)^\varkappa\Phi(z) - Q_{\varkappa-1}(z)$ belongs to $\mathcal{K}^{p(\cdot)}(\Gamma)$ ($Q_{\varkappa-1}(z) \equiv 0$ when $\varkappa < 0$). We rewrite problem (1.1) as

$$\begin{aligned}\Phi^+(t) = (t-z_0)^{-\varkappa}G(t)[\Phi^-(t)(t-z_0)^\varkappa - Q_{\varkappa-1}(t)]\\ + g(t) + (t-z_0)^{-\varkappa}G(t)Q_{\varkappa-1}(t).\end{aligned} \tag{8.8}$$

By (8.2) we have

$$G(t) = e^{i[\alpha(t)-\alpha_0(t)]} \frac{X_{\alpha_0}^{+}(t)}{X_{\alpha_0}^{-}(t)}, \tag{8.9}$$

where

$$\begin{aligned} X_{\alpha_0}(z) &= \exp\left(\frac{1}{2\pi i}\int_\Gamma \frac{\ln\left[(t-z_0)^{-\varkappa}\exp\left(i\alpha_0(t)\right)\right]dt}{t-z}\right)\exp\left(\frac{1}{2\pi i}\int_\Gamma \frac{\ln|G(t)|}{t-z}dt\right) \\ &= X_1(z)X_2(z). \end{aligned} \tag{8.10}$$

Since Γ is a Lyapunov curve, $X_2^{\pm}(z)$ are bounded functions (see [9] or [35, pages 253 and 260]). The function $G_1(t) = (t-z_0)^{-\varkappa}\exp(i\alpha_0(t))$ belongs to Lip 1 and $\operatorname{ind} G_1 = 0$. Hence $X_1^{\pm}$ are continuous in $D^{\pm}$ and $X_1^{\pm} \neq 0$. Therefore, $X_{\alpha_0}(z)$ and $1/X_{\alpha_0}(z)$ are bounded analytic functions in $D^{\pm}$. By (8.9) our boundary problem takes the form

$$\frac{\Phi^+(t)}{X_{\alpha_0}^{+}(t)} = \exp\left[i(\alpha(t)-\alpha_0(t))\right]\frac{\Phi^-(t)(t-z_0)^{\varkappa} - Q_{\varkappa-1}(t)}{X_{\alpha_0}^{-}(t)} + g_1(t), \tag{8.11}$$

where

$$g_1(t) = \frac{g(t) + (t-z_0)^{-\varkappa}G(t)Q_{\varkappa-1}(t)}{X_{\alpha_0}^{+}(t)}. \tag{8.12}$$

Put

$$\Psi(z) = \begin{cases} \dfrac{\Phi(z)}{X_{\alpha_0}(z)}, & z \in D^+, \\ \dfrac{\Phi(z)(z-z_0)^{-\varkappa} - Q_{\varkappa-1}(z)}{X_{\alpha_0}(z)}, & z \in D^-. \end{cases} \tag{8.13}$$

Then $\Psi(z) \in \mathcal{K}^{p(\cdot)}(\Gamma)$ and

$$\Psi^+(t) = \exp\left(i(\alpha(t)-\alpha_0(t))\right)\Psi^-(t) + g_1(t). \tag{8.14}$$

Since $\Psi(z) \in \mathcal{K}^{p(\cdot)}(\Gamma)$, we put $\Psi(z) = (K\psi)(z)$ with $\psi \in L^{p(\cdot)}(\Gamma)$ and rewrite (8.14) as

$$(G_1+1)\psi = (G_1-1)S_\Gamma\psi + 2g_1, \tag{8.15}$$

where

$$G_1(t) = \exp\left\{i(\alpha(t)-\alpha_0(t))\right\} =: \exp\left(i\beta(t)\right). \tag{8.16}$$

Equation (8.15) yields

$$\psi(t) = itg\frac{\beta(t)}{2}(S_\Gamma\psi)(t) + \frac{2g_1(t)}{1+G_1(t)}. \tag{8.17}$$

Let $M\psi = itg(\beta(t)/2)(S_\Gamma\psi)(t) + 2g_1(t)/(1+G_1(t))$ so that $\psi = M\psi$, where M will be a contraction operator in $L^{p(\cdot)}(\Gamma)$ when $|\beta(t)| = |\alpha(t) - \alpha_0(t)| < 2\,\mathrm{arcctg}\,\|S_\Gamma\|_{p(\cdot)}$, that is,

$$e^{i\alpha(t)} \in \mathcal{A}(\lambda) \quad \text{with } \lambda = \frac{\pi}{2\,\mathrm{arcctg}\,||S_\Gamma||_{p(\cdot)}}. \tag{8.18}$$

Thus under condition (8.18), BVP (1.1) has the unique solution in the case $\varkappa = 0$.

Now we need that the condition

$$Y_\alpha(z) := \exp\left\{ -\frac{1}{2\pi i}\int_\Gamma \frac{\alpha(t)dt}{t-z} \right\} - 1 \in \mathcal{K}^{q(\cdot)}(\Gamma) \tag{8.19}$$

be satisfied. The last inclusion holds if, for example,

$$e^{i\alpha(t)} \in \mathcal{A}\big(\max(2,\overline{q})\big), \quad \overline{q} = \operatorname*{ess\,sup}_{t\in\Gamma} q(t) \tag{8.20}$$

(see [38]).

Assuming that conditions (8.18) and (8.19) (or (8.20)) are fulfilled and taking into account that in this case (8.14) is uniquely solvable, just in the same way as in Section 6 we establish that the function given by (5.1) and (5.2) provides us again with the solution of BVP (1.1) in $\mathcal{K}^{p(\cdot)}(\Gamma)$ when $\varkappa \geq 0$, while in the case $\varkappa < 0$ for the solvability, it is necessary and sufficient that conditions (5.4) are fulfilled.

When the curve is such that the operator

$$(T\varphi)(t) = \frac{1}{\pi i}\int_0^\ell \psi(t(\sigma))\left(\frac{t'(\sigma)}{t(\sigma)-t(s)} - \frac{i\gamma e^{i\gamma\sigma}}{e^{i\gamma\sigma}-e^{i\gamma s}}\right)d\sigma, \quad \gamma = \frac{2\pi}{\ell}, \tag{8.21}$$

is compact in the space $L^{p(t(\cdot))}([0,\ell])$, then in the same manner as for constant p in [14, page 101], we can conclude that for such curves statements (i) and (ii) of Theorem 5.1 are valid under the condition

$$\big|\beta(t)\big| < 2\,\mathrm{arcctg}\,||S_{\Gamma_0}||_{p(\cdot)}. \tag{8.22}$$

From the compactness of the operators with a weak singularity in the spaces $L^{p(\cdot)}$ (see [21, Theorem C]) it follows that when Γ is a Lyapunov curve, then T is a compact operator in $L^{p(\cdot)}$.

Therefore, when $p \in \mathcal{P}$ and p satisfies condition (2.10), then in the case of Lyapunov boundary Γ all the statements of Theorem 5.1 remain valid under conditions (8.19) and (8.22) (in particular, when (8.20) and (8.22) are fulfilled).

(II) Condition (8.20) being fulfilled, condition (8.19) is also satisfied when

$$\operatorname*{ess\,sup}_{t\in\Gamma} \big|\alpha(t)\big| < \frac{\pi \underline{p}}{4(1+\ell)e\overline{q}||S_\Gamma||_{p(\cdot)}} \tag{8.23}$$

which can be easily verified by means of Theorem 4.1.

Then following the arguments in the proof of part (I), we again obtain an analogous statement. □

Remark 8.3. In the case $p(t) = p = \text{const}$ one has $\|S_{\Gamma_0}\|_p = \operatorname{ctg} \pi/2 \max(p,q)$, see [9, Section 13.3], so that Theorem 8.2 is a generalization to the spaces $L^{p(\cdot)}(\Gamma)$ of the well-known Simonenko results [38, 39].

9. On the boundedness of the singular operator in weighted $L^{p(\cdot)}_\rho$-spaces

As is well known, when investigating the problem of linear conjugation in $\mathcal{K}^p(\Gamma)$ by the method of factorization, the most important fact is that singular operator S_Γ is bounded in Lebesgue weighted spaces, see, for instance, [16, pages 113–114].

In Section 7, it was shown that basing on the boundedness of S_Γ in $L^{p(\cdot)}_\rho(\Gamma)$ when ρ is a power weight, we can solve the problem of linear conjugation in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$, if the coefficient G is piecewise continuous.

Another approach is known, which is opposite in a sense. When solving the problem of linear conjugation with a measurable bounded coefficient G in the explicit form, in this or other way, not making use of the boundedness results, one is able to conclude that the singular operator is bounded in the Lebesgue space with weight generated by the coefficient G. This approach was developed by Simonenko [38] in the case of constant p.

Based on the solution of BVP with continuous coefficient in Section 5, we proved the weighted inequality for the singular integral operator, see Theorem 5.2. Now we utilize the solution of problem (1.1) with oscillating coefficient in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$ given in Section 8 avoiding weighted boundedness results, and deduce the boundedness statements for the operator S_Γ in the weighted spaces $L^{p(\cdot)}_\rho(\Gamma)$ with weights more general than the power ones.

Theorem 9.1. *Let $p \in \mathcal{P}$ and satisfy condition (2.10) and let Γ be a Lyapunov curve or a curve of bounded turning. Assume that*

$$\operatorname*{ess\,sup}_{t\in\Gamma} |\alpha(t)| < \min\left(2\operatorname{arcctg}\|S_{\Gamma_0}\|_{p[t(\cdot)]}, \frac{\pi}{q}\right). \tag{9.1}$$

Then the function

$$\rho(t) = \left|\exp\left(\frac{1}{2\pi}\int_\Gamma \frac{\alpha(\tau)}{\tau - t}d\tau\right)\right| \tag{9.2}$$

belongs to $W^{p(\cdot)}(\Gamma)$.

Proof. Taking into account that the function $G(t) = \exp(i\alpha(t))$ under condition (9.1) satisfies conditions (8.18) and (8.20), we can state that for arbitrary $g \in L^{p(\cdot)}(\Gamma)$ the function

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_\Gamma \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - z} \tag{9.3}$$

with

$$X(z) = \exp\left(\frac{1}{2\pi i}\int_\Gamma \frac{\ln G(\tau)}{\tau - z}d\tau\right) = \exp\left(\frac{1}{2\pi}\int_\Gamma \frac{\alpha(\tau)}{\tau - t}d\tau\right) \tag{9.4}$$

belongs to $\mathcal{K}^{p(\cdot)}(\Gamma)$. Consequently, the function Φ^+ and thus

$$Y(t) = \frac{X^+(t)}{2\pi i} \int_\Gamma \frac{g(\tau)d\tau}{X^+(\tau)(\tau - t)} \tag{9.5}$$

belong to $L^{p(\cdot)}(\Gamma)$.

Then by virtue of statement (i) of Theorem 3.3 we conclude that $\rho \in W^{p(\cdot)}(\Gamma)$ (see also the proof of Theorem 5.2). □

Remark 9.2. According to Remark 4.2, the number $\operatorname{ess\,sup}_{t\in\Gamma} |\alpha(t)|$ in (9.1) can be replaced by $\nu(\alpha) = \inf \|\alpha - \psi\|_C$ in the case of bounded $\alpha(t)$, where the infinum is taken over all rational functions ψ.

The following example given on the basis of Theorem 9.1 is of interest. Let t_k ($k = 1,2,\dots$) be arbitrary distinct points on Γ. Then the function

$$\rho(t) = \prod_{k=1}^{\infty} |t - t_k|^{\beta_k} \tag{9.6}$$

belongs to $W^{p(\cdot)}(\Gamma)$ under the conditions

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{q(t_k)}, \quad k = 1,2,\dots, \quad \left|\sum_{k=1}^{\infty} \beta_k\right| < \infty. \tag{9.7}$$

10. The problem of linear conjugation in weighted spaces $L_\rho^{p(\cdot)}(\Gamma)$

Let Γ be a simple rectifiable curve bounding the domains $D^\pm$ and $\rho \in W^{p(\cdot)}(\Gamma)$. We consider the following problem: find functions $\Phi \in \mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ whose boundary conditions satisfy almost everywhere on Γ the condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \tag{10.1}$$

where $g \in L_\rho^{p(\cdot)}(\Gamma)$ and G is a bounded measurable function on Γ such that $\operatorname{ess\,inf}_{t\in\Gamma} |G(t)| > 0$.

We show that under certain assumptions on the curve Γ, the exponent $p(t)$ and the weight function $\rho(t)$ in this general setting is reduced to the problem of conjugation without weight. We consider weight functions of the form

$$\rho(t) = \exp\left(\frac{i}{2} S_\Gamma \mu\right) \in W^{p(\cdot)}(\Gamma), \tag{10.2}$$

where μ is a real-valued function on Γ, and assume that

$$\Gamma \in \mathcal{R}^{p(\cdot)}, \qquad p \in \mathcal{P}. \tag{10.3}$$

We need the following auxiliary statements well known for constant p and easily proved for the variable exponent $p(\cdot)$ under assumptions (10.2) and (10.3).

Lemma 10.1. *If $\rho \in W^{p(\cdot)}(\Gamma)$, then $1/\rho \in W^{q(\cdot)}(\Gamma)$ and for all $\varphi \in L^{p(\cdot)}_{\rho}(\Gamma)$ and $\psi \in L^{q(\cdot)}_{1/\rho}(\Gamma)$ the equality*

$$\int_\Gamma \varphi(t)(S_\Gamma\psi)(t)dt = -\int_\Gamma \psi(t)(S_\Gamma\varphi)(t)dt \tag{10.4}$$

holds.

Proof. We observe that equality (10.4) is well known, for instance, on rational functions. Let Q be the set of rational functions on Γ. For functions $\psi \in Q$ according to (2.6) we have

$$\begin{aligned}
\|S_\Gamma\psi\|_{L^{q(\cdot)}_{1/\rho}(\Gamma)} &\sim \sup_{\|\varphi\|_{L^{p(\cdot)}_{\rho}(\Gamma)}\le 1} \left|\int_\Gamma \varphi(t)(S_\Gamma\psi)(t)dt\right| \\
&= \sup_{\substack{\varphi\in Q \\ \|\varphi\|_{L^{p(\cdot)}_{\rho}(\Gamma)}\le 1}} \left|\int_\Gamma \varphi(t)(S_\Gamma\psi)(t)dt\right| \\
&= \sup_{\substack{\varphi\in Q \\ \|\varphi\|_{L^{p(\cdot)}_{\rho}(\Gamma)}\le 1}} \left|\int_\Gamma \psi(t)(S_\Gamma\varphi)(t)dt\right| \\
&\le c\|\psi\|_{L^{q(\cdot)}_{1/\rho}(\Gamma)}\|S_\Gamma\|_{L^{p(\cdot)}_{\rho}}.
\end{aligned} \tag{10.5}$$

The obtained estimate $\|S_\Gamma\psi\|_{L^{q(\cdot)}_{1/\rho}(\Gamma)} \le c\|\psi\|_{L^{q(\cdot)}_{1/\rho}(\Gamma)}$ is extended to all $\psi \in L^{q(\cdot)}_{1/\rho}(\Gamma)$ by denseness of Q in weighted $L^{p(\cdot)}(\Gamma)$-spaces, see [21, Theorem 2.3]. Therefore, $1/\rho \in W^{q(\cdot)}(\Gamma)$. The validity of equality (10.4) on the whole range $L^{p(\cdot)}_{\rho} \times L^{q(\cdot)}_{1/\rho}$ follows in the same way since both the left-hand side and the right-hand side of (10.4) are bounded bilinear functionals in $L^{p(\cdot)}_{\rho} \times L^{q(\cdot)}_{1/\rho}$. □

Lemma 10.2. *If $\Phi \in \widetilde{\mathcal{K}}^{p(\cdot)}_{\rho}(\Gamma)$ and $\Psi \in \widetilde{\mathcal{K}}^{q(\cdot)}_{1/\rho}(\Gamma)$, then $\Phi\Psi \in \widetilde{\mathcal{K}}^{1}(\Gamma)$.*

Lemma 10.1 having been proved, the proof of this lemma is obtained in the same way as in the case of constant p, see [13] or [16, pages 98-99].

Now we get back to problem (10.1). The following statement is valid.

Theorem 10.3. *Let $p \in \mathcal{P}$ and $\Gamma \in \mathcal{R}^{p(\cdot)}(\Gamma)$ and assume that condition (10.2) is fulfilled. If $\Phi(z)$ is a solution of problem (10.1) in the class $\mathcal{K}^{p(\cdot)}_{\rho}(\Gamma)$ and*

$$Y(z) = \exp\left[-i(\mathcal{K}_\Gamma\mu)(z)\right], \tag{10.6}$$

then the function

$$\Psi(z) = \frac{\Phi(z)}{Y(z)} \tag{10.7}$$

is a solution of the problem

$$\Psi^+(t) = G(t)e^{i\mu(t)}\Psi^-(t) + g_1(t) \tag{10.8}$$

in the class $\mathcal{K}^{p(\cdot)}(\Gamma)$, *where* $g_1(t) = g(t)/Y^+(t)$.

Conversely, if $\Psi(z)$ *is a solution of problem (10.8) in the class* $\mathcal{K}^{p(\cdot)}(\Gamma)$, *then the function* $\Phi(z) = \Psi(z)Y(z)$ *is a solution of problem (10.1) in the class* $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$.

Proof. We follow the papers [18, 17] (see also [14, pages 119–120]). Since $\Gamma \in \mathcal{R}^{p(\cdot)}$ and μ is bounded, by Theorem 4.1 there exists a number $\delta > 0$ such that $1/Y \in E^\delta(D^+)$ and $(z - z_0)^k(1/Y - 1) \in E^\delta(D^-)$ for some nonnegative integer k. Moreover, the functions

$$\left(\frac{1}{Y}\right)^{\pm} = \exp\left(\pm\frac{i\mu}{2} + \frac{i}{2}S_\Gamma\mu\right) = \rho\exp\left(\pm\frac{i\mu}{2}\right) \tag{10.9}$$

belong to $L^{p(\cdot)}(\Gamma)$. Therefore, $(1/Y)^\pm \in L^{\underline{p}}(\Gamma)$. Since Γ is a Smirnov curve (see Corollary 3.4), we can take $\delta = \underline{p}$ and $k = 0$. But then $1/Y - 1 \in E^1(D^\pm)$ and hence $1/Y - 1 \in \mathcal{K}^1(\Gamma)$. In addition, $(1/Y)^\pm \in L_{1/\rho}^{q(\cdot)}(\Gamma)$ and consequently, $1/Y \in \mathcal{K}_{1/\rho}^{q(\cdot)}(\Gamma)$. By virtue of Lemma 10.2, from the equality

$$\psi = \Phi\cdot\frac{1}{Y} \tag{10.10}$$

we conclude that $\Psi \in \widetilde{\mathcal{K}}^1(\Gamma)$. But $\Psi(\infty) = 0$ so that $\Psi \in \mathcal{K}^1(\Gamma)$. As far as

$$\Psi^\pm(t) = \frac{\Phi^\pm(t)}{Y^\pm(t)} = \frac{\Phi^\pm(t)}{\rho(t)}e^{\pm i(\mu(t)/2)}, \tag{10.11}$$

we have $\Psi^+ - \Psi_- \in L^{p(\cdot)}(\Gamma)$. Then from the equality $\Psi = K_\Gamma(\Psi^+ - \Psi^-)$ the inclusion $\Psi \in \mathcal{K}^{p(\cdot)}(\Gamma)$ follows.

The inverse statement can be analogously proved.

We have taken advantage of the fact that if $\Psi \in \mathcal{K}^{p(\cdot)}(\Gamma)$, then $\Psi^+ - \Psi^- \in L^{p(\cdot)}(\Gamma)$ and $\Psi(z) = \mathcal{K}_\Gamma(\Psi^+ - \Psi^-)(z)$. Indeed, since $\Psi \in \mathcal{K}^{p(\cdot)}(\Gamma)$, we have $\Psi(z) = (K_\Gamma\psi)(z)$ with $\psi \in L^{p(\cdot)}(\Gamma)$. This implies that

$$\Psi^\pm(t) = \pm\frac{1}{2}\psi(t) + \frac{1}{2}(S_\Gamma\psi)(t), \tag{10.12}$$

and hence $\Psi^+ - \Psi^- = \psi$. □

Relying on the results of Sections 5, 7, and 8, we may use Theorem 10.3 to get a picture of solvability of the problem of linear conjugation in the weighted class $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ under certain assumptions on the coefficient G.

Theorem 10.4. *Let* $p \in \mathcal{P}$, $\Gamma \in \mathcal{R}^{p(\cdot)}$, *and let*

$$\rho \in W^{p(\cdot)}(\Gamma) \text{ and has the form } \rho = \exp\left(\frac{i}{2}S_\Gamma\mu\right), \tag{10.13}$$

where μ is a bounded real measurable function. Suppose that the curve Γ and the function

$$G_\mu(t) = G(t)\exp\left(i\mu(t)\right) \tag{10.14}$$

satisfy the appropriate conditions of one of Theorems 5.1, 7.1, 8.2 with $G(t)$ replaced by $G_\mu(t)$, and put $\varkappa = \operatorname{ind} G_\mu(t)$. Then for problem (10.1) in the weighted class $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ all the statements of Theorem 5.1 are valid.

The following remarks give us a good reason to consider the assumptions $\rho \in W^{p(\cdot)}(\Gamma)$ and $\rho = \exp(i/2)S_\Gamma\mu$ of Theorem 10.4 as natural and rather general.

Remark 10.5. The latter condition in (10.13) is fulfilled automatically in the case when $p(t) = \text{const}$ and Γ is a Lyapunov curve: in this case all the weight functions of the class $W^p(\Gamma)$ have the form $\rho = \exp(i/2)S_\Gamma\mu$ with a real-valued bounded function μ (see [32] and [14, page 64]). With respect to the former condition of (10.13), the following is a necessary condition for the elementary BVP:

$$\Phi^+(t) + \Phi^-(t) = g(t) \tag{10.15}$$

to have a solution in $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ for any $g \in L_\rho^{p(\cdot)}(\Gamma)$. From the boundary condition it follows that $S_\Gamma\varphi = g$. Together with this, by the conditions $p \in \mathcal{P}$ and $\varphi \in L_\rho^{p(\cdot)}(\Gamma)$ we have $\varphi = \varphi_1/\rho$ with $\varphi_1 \in L^{p(\cdot)}(\Gamma)$. Consequently, by the statement (iv) of Theorem 3.3 we derive that $\mathcal{K}_\Gamma(\varphi_1/\rho) \in E^1(D^+)$. Therefore, $S_\Gamma^2(\varphi_1/\rho) = Sg$ (see, for instance, [16, page 103]), that is, $Sg = \varphi_1/\rho$ so that $\rho S_\Gamma g \in L^{p(\cdot)}(\Gamma)$ and then $S_\Gamma g \in L_\rho^{p(\cdot)}(\Gamma)$ for any $g \in L_\rho^{p(\cdot)}(\Gamma)$. Thus, the equation $S_\Gamma\varphi = g$ being uniquely solvable in $L_\rho^{p(\cdot)}(\Gamma)$ for any $g \in L_\rho^{p(\cdot)}(\Gamma)$, the operator S_Γ maps the space $L_\rho^{p(\cdot)}(\Gamma)$ onto itself. Applying the statement (ii) of Theorem 3.3, we conclude that the operator S_Γ is bounded in $L_\rho^{p(\cdot)}(\Gamma)$, that is, $\rho \in W^{p(\cdot)}(\Gamma)$.

Remark 10.6. It should be noted that it is impossible to derive Theorem 6.1 from Theorem 10.4. By this reason it was separately presented.

11. On singular integral equations in $L_\rho^{p(\cdot)}(\Gamma)$

We apply now the above results to the singular integral equation

$$a(t)\varphi(t) + b(t)\left(S_\Gamma\varphi\right)(t) + V\varphi = f(t) \tag{11.1}$$

in the space $L_\rho^{p(\cdot)}(\Gamma)$, where V is any operator compact in the space $L_\rho^{p(\cdot)}(\Gamma)$.

We assume that the assumptions of Section 10 are fulfilled, that is, Γ is a rectifiable curve bounding the domains $D^\pm$ and conditions (10.2) and (10.3) are satisfied. The coefficients $a(t)$ and $b(t)$ are assumed to be bounded measurable functions on Γ.

Theorem 11.1. *Let $\Gamma \in \mathcal{R}^{p(\cdot)}$, the weight function ρ satisfy the assumptions in (10.2) and* $\operatorname{ess\,inf}_{t\in\Gamma} |a(t) + b(t)| > 0$. *Assume that for the function*

$$G_\mu(t) = \frac{a(t) - b(t)}{a(t) + b(t)} e^{i\mu(t)} \tag{11.2}$$

the conditions of Theorem 8.2 with G replaced by G_μ are fulfilled. Then for (11.1) the Noether theorems are valid and its index in the space $L_\rho^{p(\cdot)}(\Gamma)$ is equal to $\varkappa = \varkappa(G_\mu)$, where $\varkappa$ is interpreted in accordance with (8.3). In the case $V = 0$ the solutions of (11.1) in the space $L_\rho^{p(\cdot)}(\Gamma)$ are given by the formula $\varphi = \Phi^+ - \Phi^-$, where $\Phi(z)$ is the solution of the following BVP:

$$\Phi^+(t) = G_\mu(t)\Phi^-(t) + g_1(t), \quad g_1(t) = \frac{f(t)}{a(t)+b(t)} \exp\left\{\frac{i}{2}\mu(t) - \frac{i}{2}(S_\Gamma\mu)(t)\right\}. \tag{11.3}$$

Proof. It suffices to give the proof for the case $V = 0$. In this case (11.1) in the space $L_\rho^{p(\cdot)}(\Gamma)$ is equivalent to a BVP of type (10.1) in the class $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$, which is established in the usual way via the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z} \in \mathcal{K}_\rho^{p(\cdot)}(\Gamma) \tag{11.4}$$

so that (11.1) may be rewritten in the form

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \tag{11.5}$$

with $G(t) = (a(t) - b(t))/(a(t) + b(t))$ and $g(t) = f(t)/(a(t) + b(t))$ under the assumption that

$$\operatorname*{ess\,inf}_{t\in\Gamma} |a(t) + b(t)| \neq 0. \tag{11.6}$$

Thus any solution $\varphi \in L_\rho^{p(\cdot)}(\Gamma)$ of (11.1) generates a solution of (11.5) in $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$ of form (11.4). Conversely, if $\Phi(z)$ is a solution of problem (11.5) in the class $\mathcal{K}_\rho^{p(\cdot)}(\Gamma)$, then the function $\varphi(t) = \Phi^+(t) - \Phi^-(t)$ is a solution of (11.1).

Let

$$Y(z) = \exp\{-i(K_\Gamma\mu)(z)\}. \tag{11.7}$$

By Theorem 10.3 the function

$$\Psi(z) = \frac{\Phi(z)}{Y(z)} = \Phi(z)\exp\{i(K_\Gamma\mu)(z)\} \tag{11.8}$$

is a solution in $\mathcal{K}^{p(\cdot)}(\Gamma)$ of the BVP

$$\Psi^+(t) = G(t)\exp\{i\mu(t)\}\Psi^-(t) + g_1(t), \tag{11.9}$$

where $g_1(t) = g(t)/Y^+(t) = f(t)/[a(t) + b(t)]Y^+(t)$. Via $\Psi(z) = (K_\Gamma\psi)(z)$ this is equivalent in $L^{p(\cdot)}(\Gamma)$ to the equation

$$a_1(t)\psi(t) + b_1(t)(S_\Gamma\psi)(t) = g_1(t), \tag{11.10}$$

where

$$a_1(t) = \frac{1}{2}\left[a(t)\left(1 - e^{i\mu(t)}\right) + b(t)\left(1 + e^{i\mu(t)}\right)\right],$$
$$b_1(t) = \frac{1}{2}\left[a(t)\left(1 + e^{i\mu(t)}\right) + b(t)\left(1 - e^{i\mu(t)}\right)\right]. \tag{11.11}$$

Thus any solution $\Psi \in \mathcal{K}^{p(\cdot)}(\Gamma)$ of problem (11.9) generates the solution ψ of (11.9) via the equality

$$\psi = \Psi^+ - \Psi^- = \frac{\Phi^+}{Y^+} - \frac{\Phi^-}{Y^-} = \frac{\rho}{2}\left[\left(e^{i(\mu/2)} + e^{-i(\mu/2)}\right)\varphi + \left(e^{i(\mu/2)} + e^{-i(\mu/2)}\right)S_\Gamma\varphi\right], \tag{11.12}$$

where $\rho = \exp(i/2)S_\Gamma\mu$. Conversely, if $\operatorname{ess\,inf}_{t\in\Gamma}|a_1(t) + b_1(t)| = 2\operatorname{ess\,inf}_{t\in\Gamma}|a(t) + b(t)| > 0$, then for the solution $\psi \in L^{p(\cdot)}(\Gamma)$ of (11.10) the function

$$\begin{aligned}\varphi &= \Phi^+ - \Phi^- = \Psi^+Y^+ - \Psi^-Y^- \\ &= \frac{1}{2\rho}\left[\left(e^{i(\mu/2)} + e^{-i(\mu/2)}\right)\psi + \left(e^{i(\mu/2)} - e^{-i(\mu/2)}\right)S_\Gamma\psi\right]\end{aligned} \tag{11.13}$$

is a solution of (11.1) in the space $L_\rho^{p(\cdot)}(\Gamma)$. □

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